Correction of the exercises from the book
A Wavelet Tour of Signal Processing

Gabriel Peyré
Ceremade
Université Paris-Dauphine

Stéphane Mallat
CMAP
École Polytechnique

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Abstract

These corrections refer to the 3rd edition of the book *A Wavelet Tour of Signal Processing – The Sparse Way* by Stéphane Mallat, published in December 2008 by Elsevier. If you find mistakes or imprecisions in these corrections, please send an email to Gabriel Peyré (`gabriel.peyre@ceremade.dauphine.fr`). More information about the book, including how to order it, numerical simulations, and much more, can be find online at `wavelet-tour.com`.

1 Chapter 2

Exercise 2.2. If $\int |h| = +\infty$, for all $A > 0$ there exists $B > 0$ such that $\int_{-B}^{B} |h| > A$. Taking $f(x) = 1_{[-A,A]}\,\mathrm{sign}(h(-x))$ which is integrable and bounded by 1 shows that

$$f \star h(0) = \int_{-B}^{B} \mathrm{sign}(h(t))h(t)\mathrm{d}t > A.$$

This shows that the operator $f \mapsto f \star h$ is not bounded on L^∞, and thus the filter h is unstable.

Exercise 2.4. One has

$$f_r(t) = \mathrm{Re}[f(t)] = [f(t) + f^*(t)]/2 \quad \text{and} \quad f_i(t) = \mathrm{Ima}[f(t)] = [f(t) - f^*(t)]/2$$

so that

$$\begin{aligned}\hat{f}_r(\omega) &= \int \frac{f(t) + f^*(t)}{2} e^{-i\omega t}\mathrm{d}t = \hat{f}(\omega)/2 + \mathrm{Conj}\left(\int f(t)e^{i\omega t}\mathrm{d}t\right)/2 \\ &= [\hat{f}(\omega) + \hat{f}^*(-\omega)]/2,\end{aligned}$$

where $\mathrm{Conj}(a) = a^*$ is the complex conjugate. The same computation leads to

$$\hat{f}_i(\omega) = [\hat{f}(\omega) - \hat{f}^*(-\omega)]/2.$$

Exercise 2.6. If $f = 1_{[-\pi,\pi]}$ then one can verify that

$$\hat{f}(\omega) = \frac{2\sin(\pi\omega)}{\omega}.$$

It result that

$$\int \frac{\sin(\pi\omega)}{\pi\omega} = \frac{1}{2\pi}\int \hat{f}(\omega)\mathrm{d}\omega = f(0) = 1.$$

If $g = 1_{[-1,1]}$ then $\hat{g}(\omega)/2 = \sin(\omega)/\omega$. The inverse Fourier transform of $\hat{g}(\omega)^3$ is $g \star g \star g(t)$ so

$$\int \frac{\sin^3(\omega)}{\omega^3}\mathrm{d}\omega = \frac{1}{8}\int \hat{g}(\omega)^3\mathrm{d}\omega = \frac{2\pi}{8} g\star g\star g(0) = \frac{3\pi}{4},$$

where we used the fact that

$$g\star g\star g(0) = \int_{-1}^{1} h(t)\mathrm{d}t = 3$$

where h is a piecewise linear hat function with $h(0) = 2$.

Exercise 2.8. If f is $\mathbf{C}^1$ with a compact support, with an integration by parts we get

$$\hat{f}(\omega) = \frac{1}{i\omega}\int f'(t)e^{-i\omega t}\mathrm{d}t$$

so that

$$|\hat{f}(\omega)| \leqslant \frac{C}{\omega} \quad \text{with} \quad C = \int |f'(t)|\mathrm{d}t < +\infty,$$

which proves that $f(\omega) \to 0$ when $|\omega| \to +\infty$.

Let $f \in \mathbf{L}^1(\mathbb{R})$ and $\varepsilon > 0$. Since $\mathbf{C}^1$ functions are dense in $\mathbf{L^1}(\mathbb{R})$, one can find g such that $\int |f - g| \leqslant \varepsilon/2$. Since $\hat{g}(\omega) \to 0$ when $|\omega| \to +\infty$, there exists A such that $|\hat{g}(\omega)| \leqslant \varepsilon/2$ when $|\omega| > A$. Moreover, the Fourier integral definition implies that

$$|\hat{f}(\omega) - \hat{g}(\omega)| \leqslant \int |f(t) - g(t)|\, dt$$

so for all $|\omega| > A$ we have $|\hat{f}(\omega)| \leqslant \varepsilon$ which proves that $f(\omega) \to 0$ when $|\omega| \to +\infty$.

Exercise 2.10. For $a > 0$ and $u > 0$ and g a Gaussian function, define

$$f_{a,u}(t) = e^{iat}g(t-u) + e^{-iat}g(t+u).$$

We verify that $\sigma_\omega(f_{a,u})$ increases proportionally to u. Its Fourier transform is

$$\hat{f}_{a,u}(\omega) = e^{-iu\omega}\hat{g}(\omega - a) + e^{iu\omega}\hat{g}(\omega + a)$$

so $\sigma_\omega(f_{a,u})$ increases proportionally to a. For a and u sufficiently large we get the the result.

Exercise 2.12. a) Denoting $u(t) = |\sin(t)|$, one has $g(t) = a(t)u(\omega_0 t)$ so that

$$\hat{g}(\omega) = \frac{1}{2\pi}\hat{a}(\omega)\star\hat{u}(\omega/\omega_0)$$

where $\hat{u}(\omega)$ is a distribution

$$\hat{u}(\omega) = \sum_n c_n\delta(\omega - n)$$

and c_n is the Fourier coefficient

$$c_n = \int_{-\pi}^{\pi} |\sin(t)|e^{-int}\mathrm{d}t = -\int_{-\pi}^{0}\sin(t)e^{-int}\mathrm{d}t + \int_0^{\pi}\sin(t)e^{-int}\mathrm{d}t.$$

The change of variable $t \to t+\pi$ in the first integral shows that $c_{2k+1} = 0$ and for $n = 2k$,

$$c_{2k} = 2 \int_0^{\pi} \sin(t) e^{-i2kt} \mathrm{d}t = \frac{4}{1-4k^2}.$$

One thus has

$$\hat{u}(\omega) = \frac{1}{2\pi} \sum_n c_n \hat{a}(\omega - n\omega_0) = \frac{2}{\pi} \sum_k \frac{\hat{a}(\omega - 2k\omega_0)}{1-4k^2}.$$

b) If $\hat{a}(\omega) = 0$ for $|\omega| > \omega_0$, then h defined by $\hat{h}(\omega) = \frac{\pi}{2} 1_{[-\omega_0, \omega_0]}$ guarantees that $\hat{g}\hat{h} = \hat{a}$ and hence $a = g \star h$.

Exercise 2.14. The function $\phi(t) = \sin(\pi t)/(\pi t)$ is monotone on $[-3/2, 0]$ and $[0, 3/2]$ on which is variation is $1 + \frac{2}{3\pi}$. For each $k \in \mathbb{N}^*$, it is also monotone on each interval $[k+1/2, k+3/2]$ on which the variation is $\frac{1}{\pi}[(k+1/2)^{-1} + (k+3/2)^{-1}]$. One thus has

$$\|\phi\|_V = 2(1 + \frac{2}{3\pi}) + \frac{2}{\pi} \sum_{k \geqslant 1} [(k+1/2)^{-1} + (k+3/2)^{-1}] = +\infty.$$

For $\phi = \lambda 1_{[a,b]}$, $|\phi'| = \lambda\delta_a + \lambda\delta_b$ and hence $\|\phi\|_V = 2\lambda$.

Exercise 2.16. Let

$$f(x) = 1_{[0,1]^2}(x_1, x_2) = f_0(x_1) f_0(x_2) \quad \text{where} \quad f_0(x_1) = 1_{[0,1]}(x_1).$$

One has

$$\hat{f}(\omega_1, \omega_2) = \hat{f}_0(\omega_1)\hat{f}_0(\omega_2) = \frac{(e^{i\omega_1} - 1)(e^{i\omega_2} - 1)}{\omega_1 \omega_2}.$$

Let

$$f(x) = e^{-x_1^2 - x_2^2} = f_0(x_1) f_0(x_2) \quad \text{where} \quad f_0(x_1) = e^{-x_1^2}.$$

One has

$$\hat{f}(\omega_1, \omega_2) = \hat{f}_0(\omega_1)\hat{f}_0(\omega_2) = \pi e^{-(\omega_1^2 + \omega_2^2)/4}.$$

Exercise 2.18. We prove that the Gibbs oscillation amplitude is independent of the angle θ and is equal to a one-dimensional Gibbs oscillation. Let us decompose $f(x)$ into a continuous part $f_0(x)$ and a discontinuity of constant amplitude A:

$$f(x) = f_0(x) + A\, u(\cos(\theta) x_1 + \sin(\theta) x_2)$$

where $u(t) = 1_{[0,+\infty)}(t)$ is the one-dimensional Heaviside function. The filter satisfies $h_\xi(x_1, x_2) = g_\xi(x_1)\, g_\xi(x_2)$ with $g_\xi(t) = \sin(\xi t)/(\pi t)$. The Gibbs phenomena is produced by the discontinuity corresponding to the Heaviside function so we can consider that $f_0 = 0$. Let us suppose that $|\theta| \leqslant \pi/4$, with no loss of generality. We first prove that

$$f \star h_\xi(x) = f \star g_\xi(x) \tag{1}$$

where $\hat{g}_\xi(\omega_1, \omega_2) = 1_{[-\xi, \xi]}(\omega_2)$. Indeed $f(x)$ is constant along any line of angle θ, one can thus verify that its Fourier transform has a support located on the line in the Fourier plane, of angle $\theta + \pi/2$ which goes through 0. It results that $\hat{f}(\omega)\hat{h}_\xi(\omega) = \hat{f}(\omega)\hat{g}_\xi(\omega)$ because the filtering limits the support of $\hat{f}$ to $|\omega_2| \leqslant \xi$. But $g_\xi(x_1, x_2) = \delta(x_1)\, \sin(\xi x_2)/(\pi x_2)$. The convolution (1) is thus a one-dimensional convolution along the x_2 variable, which is computed in the Gibbs Theorem 2.8. The resulting one-dimensional Gibbs oscillations are of the order of $A \times 0.045$.

2 Chapter 3

Exercise 3.2. If $\text{Supp}(\hat{f}) \subset [-\pi/s, \pi/s]$, then

$$\hat{f}(\omega) = \hat{f}(\omega) 1_{[-\pi/s,\pi/s]}(\omega) = \frac{1}{s}\hat{f}(\omega)\hat{\phi}_s(\omega)$$

and hence using the Fourier convolution theorem and the fact that ϕ_s is symmetric,

$$f(u) = \frac{1}{s} f \star \phi_s(u) = \frac{1}{s}\langle f(t),\, \phi_s(t-u)\rangle.$$

Exercise 3.4. Let $A(t) = \sum_n f(t-n)$, then in the sense of distribution

$$\hat{A}(\omega) = \hat{f}(\omega)\sum_n e^{-in\omega} = \hat{f}(\omega)\sum_n \delta(\omega - 2n\pi) = \sum_n \hat{f}(2n\pi)\delta(\omega - 2n\pi).$$

Taking the inverse Fourier transform leads to

$$\sum_n f(t-n) = \sum_n \hat{f}(2n\pi)e^{2in\pi\omega}.$$

Exercise 3.6. The sufficient condition comes from

$$\|Lf\|_\infty \leqslant \|f\|_\infty \|h\|_1.$$

If $\|h\|_\infty = +\infty$, then for any $A > 0$ it exists B such that $\sum_{|k|<B} |h[k]| > A$. Taking $f[k] = \text{sign}(h[-k]) 1_{[-B,B]}[-k]$ that satisfies $\|f\|_\infty \leqslant 1$ shows that

$$Lf[0] = \sum_{|k|\leqslant B} |h[k]| > A.$$

The operator $f \mapsto f \star h$ is not bounded on ℓ^∞, so that the filter is unbounded.

Exercise 3.8. Let $e_k[n] = e^{\frac{2i\pi}{N}kn} = \omega^{kn}$ where $\omega = e^{\frac{2i\pi}{N}}$. If $k \neq k'$, one has a geometrical sum

$$\langle e_k,\, e_{k'}\rangle = \sum_n \omega^{kn}\omega^{-k'n} = \sum_n (\omega^{k-k'})^n = \frac{1 - \omega^{N(k-k')}}{1 - \omega^{k-k'}} = 0$$

because $\omega^N = 1$. Since $\|e_k\| = \sqrt{N}$, the family $\{e_k/\sqrt{N}\}_k$ is an orthonormal basis of $\mathbb{C}^N$.

Exercise 3.10. a) One has $\tilde{f}(ns) = f \star \phi_s$ where $\phi_s = 1_{[-s/2,s/2]}$.
b) Since $\hat{\tilde{f}}(\omega) = \hat{f}(\omega)\hat{\phi}_s(\omega)$, $\text{supp}(\hat{\tilde{f}}) \subset [-\pi/s, \pi/s]$, and hence $\tilde{f}$ is recovered using Shannon interpolation formula.
c) One has

$$\hat{\tilde{f}}(\omega) = \hat{f}(\omega)\frac{\sin(\omega s/2)}{\omega s/2}$$

and thus

$$f = \tilde{f} \star \psi_s \quad \text{with} \quad \hat{\psi}_s(\omega) = \frac{\omega s/2}{\sin(\omega s/2)}.$$

d) For $\omega \in [-\pi/s, \pi/s]$, one has

$$\hat{\phi}_s(\omega) > 2/\pi$$

and hence $\|f\| \leqslant \pi/2\|\tilde{f}\|$ which shows that the reconstruction is stable.

Exercise 3.12. One has

$$|\hat{f}[k]| = |\sum_n f[n]e^{-\frac{2i\pi}{N}nk}| \leqslant \sum_n |f[n]||e^{-\frac{2i\pi}{N}nk}| = \sum_n |f[n]|.$$

Exercise 3.14. Since $(-1)^n = e^{-i\pi n}$, one has

$$\hat{g}(\omega) = \sum_n h[n]e^{i\pi n}e^{in\omega} = \hat{h}(\omega + \pi).$$

If h is a low-pass filter, then g is a high-pass filter.

Exercise 3.16. a) Since $\hat{h}(\omega)\hat{h}^{-1}(\omega) = \hat{\delta}(\omega) = 1$, one has $\hat{h}^{-1} = 1/\hat{h}$.
b) Up to translation we suppose that h is causal. The filter h^{-1} and h have finite support if and only if $\hat{h}(\omega) = P(e^{-i\omega})$ where $P(z)$ and $z^k/P(z)$ are polynomial for some $k \in \mathbb{N}$. This can only happens if $P(z)$ is a monomial $P(z) = az^p$, so that $h[n] = a\delta[n-p]$.

Exercise 3.18. a) For $h_0[n] = a^n 1_{[0,+\infty)}[n]$, one has the following geometrical sum

$$\hat{h}_0(\omega) = \sum_n (ae^{-i\omega})^n = \frac{1}{1 - ae^{-i\omega}}.$$

Let $h_p(\omega) = (1 - ae^{-i\omega})^{-p}$. Observe that

$$h_p'(\omega) = \frac{-paie^{-i\omega}}{(1 - ae^{-i\omega})^{p+1}} = \sum_n h_p[n](-in)e^{-in\omega} \ .$$

The inverse Fourier transform of $h_{p+1}(\omega) = (1 - ae^{-i\omega})^{-p-1}$ thus satisfies

$$h_{p+1}[n] = \frac{(n+1)\, h_p[n+1]}{a\, p} \ .$$

Iterating on this relation with $h_1[n] = a^n 1_{[0,+\infty)}[n]$ gives

$$h_p[n] = \frac{h_1[n+p-1]\,, \prod_{k=1}^{p-1}(n+k)}{a^{p-1}\,(p-1)!} \ .$$

b) Taking the Fourier transform of the recursion formula shows that

$$\hat{g}(\omega) = \hat{h}(\omega)\hat{f}(\omega) \quad \text{where} \quad \hat{h}(\omega) = \frac{\sum_{k=0}^{K} a_k e^{-ik\omega}}{\sum_{k=0}^{M} b_k e^{-ik\omega}}.$$

c) Let a_k be the roots of the polynomial $\sum_{k=0}^{M} b_k z^{-k}$ with multiplicity p_k. Then

$$\hat{h}(\omega) = P(e^{-i\omega}) \prod_k (1 - a_k e^{-i\omega})^{-p_k},$$

where P is a polynomial. If one has $|a_k| < 1$ for all k, then using question a), each $(1 - a_k e^{-i\omega})^{-p_k}$ is the Fourier transform of a stable causal filter, and so is h.

If there exists l such that $|a_l| = 1$, then it can be written $a_l = e^{i\alpha}$ so $|\hat{h}(\alpha)| = +\infty$ and the filter can therefore not be stable.

If, there exists l with $|a_l| > 1$ then let $h_l(\omega) = (1 - a_l e^{-i\omega})^{-p_l}$. Observe that

$$h_l(-\omega) = (1 - a_l e^{i\omega})^{-p_l} = a_l^{-p_l} e^{-i\omega p_l} (a_l^{-1} e^{-i\omega} - 1)^{-p_1} .$$

Its inverse Fourier transform is $h_l[-n]$ and with question (a) we verify that it is not a causal filter. To prove that h is then not a causal filter, one can decompose

$$\hat{h}(\omega) = \sum_{k=0}^{L} \frac{Q_k(e^{-i\omega})}{(1 - a_k e^{-i\omega})^{n_k}}$$

where each Q_k is a polynomial of degree strictly smaller than n_k and observe that the component for $k = l$ is not causal so h can not be causal.

Exercise 3.20. a) One has

$$\hat{x}(\omega) = \sum_n y[Mn] e^{-i\omega n} = \sum_n (y \cdot h)[n] e^{-i\omega n/M} = \frac{1}{2\pi} \hat{y} \star \hat{h}(\omega/M)$$

where we have use the convolution result of Exercise 3.7 and where

$$h[n] = \sum_k \delta[n - kM]$$

which satisfies, in the sense of distributions

$$\hat{h}(\omega) = \frac{2\pi}{M} \sum_k \delta(\omega - 2\pi k/M).$$

This shows that

$$\hat{x}(\omega) = \frac{1}{M} \int_{-\pi}^{\pi} y(\xi) \sum_k \delta(\omega/M - \xi - 2\pi k/M) \mathrm{d}\xi = \frac{1}{M} \sum_{k=0}^{M-1} \hat{y}((\omega - 2k\pi)/M).$$

b) If for $\omega \in [-\pi, \pi]$, $\hat{y}(\omega) = 0$ for $\omega \notin [-\pi/M, \pi/M]$, then

$$\hat{y}(\omega) = \hat{x}(M\omega) \hat{\phi}_M(\omega) \quad \text{where} \quad \hat{\phi}_M(\omega) = M 1_{[-\pi/M, \pi/M]}(\omega).$$

Let $(x \uparrow M)[Mn] = x[n]$ and $(x \uparrow M)[k] = 0$ if $k \mod M \neq 0$ be the up-sampled signal. The Fourier transform of $x \uparrow M$ is $\hat{x}(M\omega)$, so that

$$y = (x \uparrow M) \star \phi_m \quad \text{where} \quad \phi_M[n] = \frac{\sin(n\pi/M)}{n\pi/M}.$$

Exercise 3.22. a) One has

$$\hat{f}[\ell] = \frac{N}{2} \left(\delta[\ell - k] + \delta[\ell + k] \right).$$

So

$$f_a[n] = e^{\frac{2i\pi}{N} kn}.$$

b) One has $g[n] = (f[n] + f^*[n])/2$, and using a change of variable $n \to -n$ in the summation gives

$$\hat{g}[k] = \hat{f}[k]/2 + \sum_n f^*[n] e^{-\frac{2i\pi}{N} kn} = \hat{f}[k]/2 + \hat{f}[-k]^*/2.$$

c) The definition of $\hat{f}_a$ shows that $g = \text{Re}(f_a)$ satisfies

$$\hat{g}[k] = (\hat{f}_a[k] + \hat{f}_a[-k]^*)/2 = \hat{f}[k],$$

and hence $g = f$.

Exercise 3.24. We define the sub-images

$$\forall\, 0 \leqslant k_1, k_2 < L, \quad \forall\, 0 \leqslant n_1, n_2 < M, \quad f_k[n] = f[n + kM].$$

One has

$$f[n] = \sum_k f_k[n - kM],$$

so that

$$f \star h[n] = \sum_k f_k[n - kM] \star h[n] = \sum_k (\tilde{f}_k \circledast h)[n - kM]$$

where we have denoted by $\tilde{f}_k$ the image of size $(2M-1) \times (2M-1)$ obtained by zero-padding from f_k. This allows one to compute $f \star h$ using L^2 circular FFT of size $(2M-1)^2$, followed by $4N$ additions to reconstruct the full image. The overall complexity of this overlap add method is thus approximately $8KN \log(M)$.

The complexity of a direct evaluation of the convolution is $K'NM^2$, where $K' \sim 2$ (1 addition and 1 multiplication).

For $K = 6$ and $K' = 2$, using the overlap-add algorithm is better in the range of M such that

$$48N \log_2(M) \leqslant 2NM^2 \quad \Leftrightarrow \quad M^2/\log_2(M) \geqslant 24$$

which we found numerically to be $M \geqslant 9$.

3 Chapter 4

Exercise 4.2. The formula (2.32) for the Fourier transform of the Gaussian implies

$$\begin{aligned} Ag(\tau, \gamma) &= \frac{1}{\sqrt{\pi\sigma^2}} \int \exp\left(\frac{1}{2\sigma^2}\left[(v + \tau/2)^2 + (v - \tau/2)^2\right]\right) e^{-i\gamma v} \mathrm{d}v \\ &= \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{\tau^2}{4\sigma^2}} \int e^{-v^2} e^{-i\gamma v} \mathrm{d}v = \exp\left(-\frac{\tau^2}{4\sigma^2} - \frac{\gamma^2\sigma^2}{4}\right) \end{aligned}$$

Exercise 4.4. One can write the discrete Fourier transform in (4.27)

$$Sf[m, l] = e^{-i2\pi lm/N}\, f \star g_l[m] \quad \text{with} \quad g_l[n] = g[-n]\, e^{-i2\pi ln/N}\,.$$

It is thus computed with N convolutions. Since g_l has a support of size L, the fast overlapp-add convolution algorithm of Section 3.3.4 computes each convolution with $O(N \log L)$ operations. The windowed Fourier transform is thus computed with $O(N^2 \log L)$ operations. **Exercise 4.6.** One has

$$b(t) = \frac{1}{C_\psi} \int_0^{s_0} f \star \psi_s^* \star \psi_s(t) \frac{\mathrm{d}s}{s^2} + \frac{1}{C_\psi s_0} f \star \phi_{s_0}^* \star \phi_{s_0}.$$

Taking the Fourier transform leads to

$$\hat{b}(\omega) = \frac{\hat{f}(\omega)}{C_\psi} \left[\int_0^{s_0} |\hat{\psi}_s(\omega)|^2 \frac{\mathrm{d}s}{s^2} + \frac{1}{s_0} |\hat{\phi}_{s_0}|^2\right].$$

Using the fact that $|\hat{\psi}_s(\omega)|^2 = s|\hat{\psi}(s\omega)|^2$ and

$$|\hat{\phi}_{s_0}|^2 = \int_1^{+\infty} |\hat{\psi}(s_0 s\omega)|^2 \frac{\mathrm{d}s}{s} = \int_{s_0}^{+\infty} |\hat{\psi}(s\omega)|^2 \frac{\mathrm{d}s}{s}$$

leads to the result.

Exercise 4.8. Let

$$b(t) = \frac{1}{C} \int_0^{+\infty} f \star \psi_s(t) \frac{\mathrm{d}s}{s^{3/2}}.$$

Its Fourier transform reads, after a change of variable $\xi = s\omega$,

$$\hat{b}(\omega) = \frac{\hat{f}(\omega)}{C} \int_0^{+\infty} \sqrt{s}\hat{\psi}(s\omega) \frac{\mathrm{d}s}{s^{3/2}} = \frac{\hat{f}(\omega)}{C} \int_0^{+\infty} \hat{\psi}(\xi) \frac{\mathrm{d}s}{s} = \hat{f}(\omega).$$

Exercise 4.10. For $f(t) = \cos(\theta(t))$ with $\theta(t) = a\cos(bt)$, the width s of the window is small enough if

$$s^2|\theta''(t)| = s^2 ab^2 |\cos(bt)| \ll 1$$

and thus $s^2 ab^2 \ll 1$.

For $\theta(t) = \cos(\theta_1(t)) + \cos(\theta_2(t))$ with $\theta_1(t) = a\cos(bt)$ and $\theta_2(t) = a\cos(bt) + ct$, there is enough frequency resolution if

$$|\theta_1'(t) - \theta_2'(t)| = |c| \geqslant \frac{\Delta_\omega}{s}$$

and enough spacial resolution if

$$s^2|\theta_1'(t)| = s^2|\theta_2''(t)| = s^2 ab^2 \ll 1.$$

This shows that one needs

$$\sqrt{\frac{c}{ab^2}} \ll \Delta_\omega \simeq 1.$$

Exercise 4.16. Example 4.20 shows that $P_\theta f$ is a spetrogram with a Gaussian window g_μ if

$$\theta(u,\xi) = g_\sigma(u) g_\beta(\xi) = P_V g_\mu(u,\xi) = g_{\mu/\sqrt{2}}(u) g_{\sqrt{2}\mu}(\xi),$$

where we have used (4.125) for the last equality. If $\sigma\beta = 1/2$, then one can take $\mu = \sqrt{2}\sigma$. Otherwise, if $\sigma\beta > 1/2$, one decomposes $\theta = \theta_0 \star \theta_1$ where

$$\theta_0(u,\xi) = g_{\mu/\sqrt{2}}(u) g_{\sqrt{2}\mu}(\xi) \quad \text{and} \quad \theta_1(u,\xi) = g_{\sigma-\mu/\sqrt{2}}(u) g_{\beta-\sqrt{2}\mu}(\xi)$$

where μ is chosen so that $\sigma - \mu/\sqrt{2} > 0$ and $\beta - \sqrt{2}\mu > 0$. This shows that P_θ is a smoothing with θ_1 of P_{θ_0} which is a spectrogram and hence positive.

Exercise 4.18. One has

$$A(t) = \int (\xi - \phi'(t))^2 P_V f(t\xi) \mathrm{d}\xi = \int h(\tau) \left(\int (\xi - \phi'(t))^2 e^{-i\xi\tau} \mathrm{d}\xi \right) \mathrm{d}\tau$$

where

$$h(\tau) = a(t + \tau/2) a(t - \tau/2) e^{i[\phi(t+\tau/2) - \phi(t-\tau/2)]}.$$

In the sense of distribution, one has the following equality

$$\int (\xi - \phi'(t))^2 e^{-i\xi\tau} \mathrm{d}\xi = 2\pi[-\delta''(\tau) + 2i\phi'(t)\delta'(\tau) + \phi'(t)^2\delta(\tau)]$$

where $\delta, \delta', \delta''$ are the Dirac distribution and its first and second derivatives. This shows that

$$A(t) = 2\pi[-h''(0) + 2i\phi'(t)h'(0) + \phi'(t)^2 h(0)].$$

After computing the derivatives of h and simplification, one finds

$$A(t) = -\pi(a(t)a''(t) - a'(t)^2),$$

which is the result.

4 Chapter 5

Exercise 5.2. A change of variable $t \to Kt$ gives

$$\langle f,\, \phi_p \rangle = \int_0^1 f(t) e^{2i\pi pt/K} \mathrm{d}t = K \int_0^{1/K} f(Kt) e^{2i\pi pt} = \int_0^1 f_K(t) e_p(t) \mathrm{d}t,$$

where $\{e_p(t) = e^{-2i\pi t}\}_p$ is an orthonormal basis of $\mathrm{L}^2[0,1]$. Let us define $f_K(t) = f(Kt)$ for $t \in [0, 1/K]$ and $f_K(t) = 0$ otherwise. For $K \geqslant 1$, we get

$$\sum_p |\langle f,\, \phi_p \rangle|^2 = \sum_p |\langle f_K,\, e_p \rangle|^2 = \|f_K\|^2 = K\|f\|^2$$

So $\{\phi_p\}_p$ is a tight frame of $\mathrm{L}^2[0,1]$ of frame bound K.

Exercise 5.4. One has

$$\mathrm{tr}(U_1 U_2) = \sum_{i,j} U_1[i,j] U_2[j,i] = \mathrm{tr}(U_2 U_1).$$

Exercise 5.6. If one does not constrain $\|\phi_p\|$, one can choose $\{\delta_0/N, N\delta_1, \delta_2 \ldots, \delta_{N-1}\}$. If one constrains $\|\phi_p\| = 1$, one can choose $\phi_p[n] = \phi[n + p \mod N]$ so that $\Phi f = f * \phi$ and thus one should impose

$$\sum_\omega |\hat{\phi}[\omega]|^2 = N \quad \text{and} \quad |\hat{\phi}[\omega]| > 0.$$

This can be achieved by setting $\hat{\phi}[\omega] = 1/N$ for $\omega \neq 0$ and $\hat{\phi}[0] = \sqrt{N - (N-1)/N^2}$. This implies

$$A = \min_\omega |\hat{\phi}[\omega]|^2 = 1/N^2 \to 0$$

and

$$B = \max_\omega |\hat{\phi}[\omega]|^2 = N - (N-1)/N^2 \to +\infty.$$

Exercise 5.8. One has

$$\Phi_m f[p] = \langle f,\, \phi_{m+p} \rangle = f \circledast \phi_m[p] \quad \Longrightarrow \quad \Phi^* \Phi = \sum_m \Phi_m^* \Phi_m$$

where Φ_m is a convolution operator, whose eigenvalues are $\hat{\phi}_m[\omega]$, and eigenvectors are the discrete Fourier vectors. The eigenvalues of $\Phi^*\Phi$ are thus $\sum_m |\hat{\phi}_m[\omega]|^2$.

Exercise 5.10. a) Since $\{1/\sqrt{K}e^{\frac{2i\pi}{K}nk}\}_{0\leqslant k<K}$ is an orthogonal basis of the signal supported in $I = [-K/2, K/2-1] + mM$, and $g[n-mM]f[n]$ is supported in I, one has

$$\sum_n |g[n-mM]|^2 |f[n]|^2 = \sum_{k=0}^{K-1} |\langle f, \frac{1}{\sqrt{K}} g_{mnk}\rangle|^2.$$

b) Summing over m leads to

$$K \sum_n |f[n]|^2 \sum_{m=0}^{N/M-1} |g[n-mM]|^2 = \sum_{k,m} |\langle f, g_{m,k}\rangle|^2,$$

and hence the result of Theorem 5.18.

Exercise 5.12. For $m = 3$, one has

$$\begin{aligned} \hat{h}(\omega)/\sqrt{2} &= \cos^4(\omega/2)^4 = P(z) = (z/2 + z^{-1}/2)^4 \\ &= (z^2 + 4z + 6 + 3z^{-1} + z^{-2})/2^4 \end{aligned}$$

and one can choose

$$\hat{g}(\omega)/\sqrt{2} = \sin^2(\omega/2) = Q(z) = (z - 2 + z^{-1})/4.$$

If $\tilde{h} = h$, then

$$\hat{\tilde{g}}(\omega) = \frac{2 - |\hat{h}(\omega)|^2}{\hat{g}^*(\omega)} = 2\frac{1-\cos^4(\omega/2)}{\sqrt{2}\sin^2(\omega/2)} = \sqrt{2}[1 + \cos^2(\omega/2)] = \sqrt{2}\tilde{Q}(z)$$

where $\tilde{Q}(z) = 1 + (z + 2 + z^{-1})/4$, so $\tilde{g}/\sqrt{2} = [1/4, 3/2, 1/4]$.

Exercise 5.14. As $\beta\eta \to 2\pi$, the frame becomes less orthogonal ($A/B \ \to +\infty$), so the dual vectors become more and more different from the primal ones, and hence the windows g and $\tilde{g}$ differ more and more.

Exercise 5.16. a) Each $\{\phi_s(t - ns - ks/K)\}_n$ is an orthogonal basis of U_s, the set of signals with Fourier support included in $[-\pi/s, \pi/s]$. This proves that $\{\phi_{s-n/Ks}\}_n$ is a tight frame with frame bound K.
b) One has

$$Pa[n] = \sum_{n'} a[n'] H[n, n'] \quad \text{where} \quad H[n,n'] = \langle \phi_s(t - n/Ks), \frac{1}{K}\phi_s(t - n'/Ks)\rangle,$$

and, using Plancherel formula,

$$\begin{aligned} H[n,n'] &= \frac{1}{2K\pi}\langle s^{1/2} 1_{[-\pi/s,\pi/s]}(\omega) e^{-in/Ks\omega}, s^{1/2} 1_{[-\pi/s,\pi/s]}(\omega) e^{-in'/Ks\omega}\rangle \\ &= \frac{s}{2K\pi}\int_{-\pi/s}^{\pi/s} e^{i(n-n')/Ks\omega} \mathrm{d}\omega = h[n-n'] \quad \text{where} \quad h[n] = \frac{\sin(\pi n/K)}{\pi n}. \end{aligned}$$

c) Since $\hat{h} = 1_{[-\pi/K,\pi/K]}$, one has

$$\mathrm{Im}(P) = \{a \setminus h \star a = a\} = \{a \setminus \mathrm{supp}(\hat{a}) \subset [-\pi/K, \pi/K]\}\,.$$

d) Theorem 3.5 proves that $s^{-1/2} f \star \phi_s(ns) = f(ns)$.
e) Let $a[n] = f \star \phi_s(ns_0)$. For $\omega \in [-\pi, \pi]$, one has

$$\hat{a}(\omega) = \frac{1}{s_0}\mathcal{F}(f \star \phi_s)(\omega/s_0) \quad \Longrightarrow \quad \mathrm{supp}(\hat{a}) \subset [-\pi/K, \pi/K]$$

where $\mathcal{F}$ is the Fourier transform. This shows that $a \in \mathrm{Im}(\Phi)$ and hence $Pa = a$.
One has

$$PY[Kn] = a[Kn] + P(W)[Kn] \quad \Longrightarrow \quad |PY[Kn] - s^{-1/2} f(ns)|^2 = |W \star h[Kn]|^2.$$

One then has

$$E(|W \star h[Kn]|^2) = \sigma^2 \sum_k |h[k]|^2 = \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi} |\hat{h}(\omega)|^2 = \frac{\sigma^2}{K} \to 0 \quad \text{when} \quad K \to +\infty.$$

Exercise 5.18. We prove the right hand side of the inequality

$$\begin{aligned}\sum_{j=\alpha}^{\beta} |\hat{\psi}(2^j\omega)|^2 &= \frac{1}{2}\sum_{j=\alpha}^{\beta} |\hat{g}(2^{j-1}\omega)|^2 |\hat{\phi}(2^{j-1}\omega)|^2 \\ &\leqslant B \sum_{j=\alpha}^{\beta} (1 - |\hat{h}(2^{j-1}\omega)|^2)|\hat{\phi}(2^{j-1}\omega)|^2 \\ &\leqslant B \sum_{j=\alpha}^{\beta} (|\hat{\phi}(2^{j-1}\omega)|^2 - |\hat{\phi}(2^j\omega)|^2) = B(|\hat{\phi}(2^{\alpha-1}\omega)|^2 - |\hat{\phi}(2^\beta\omega)|^2).\end{aligned}$$

Since $\hat{\phi}(0) = 1$ and $\phi(\omega)$ tends to zero as ω increases, letting α and β go to $-\infty$ and $+\infty$ proves that $\sum_j |\hat{\psi}(2^j\omega)|^2 \leqslant B$. A similar proof applies to the left hand side.

Exercise 5.21. We use the change of variables

$$x = \rho\cos(\alpha) - u\sin(\alpha) \quad \text{and} \quad y = \rho\sin(\alpha) + u\cos(\alpha).$$

For $k + \ell < p$, one can expand the monomial $P(x, y) = x^k y^\ell$ as a polynomial in u of degree less than p

$$P(x, y) = \sum_{t<p} A(\rho)u^t.$$

This shows that

$$\langle P,\, \psi\rangle = \sum_{t<p} \int A(\rho)\left(\int u^t \psi(\rho\cos(\alpha) - u\sin(\alpha), \rho\sin(\alpha) + u\cos(\alpha))\mathrm{d}u\right)\mathrm{d}\rho = 0.$$