Jean-François Aujol

CMLA, UMR CNRS 8536

A TV-Hilbert model for image denoising and decomposition

Joint work with Guy Gilboa (UCLA)

Workshop : *An interdisciplinary approach to Textures and Natural Images Processing*

8 January 2007

Overview

1) Model

2) Image restoration

3) Image decomposition

Rudin-Osher-Fatemi model

(Physica D. 1992)

Problem of image restoration (f degraded image, u restored image) :

$$f = \mathcal{R}u + n$$

A way to reconstruct u :

$$\inf_u \underbrace{\frac{1}{2\lambda}\|f - \mathcal{R}u\|_2^2}_{\text{data term}} + \underbrace{J(u)}_{\text{regularization}}$$

We will assume that $\mathcal{R} = Id$.

In the ROF model, one uses :

$$J(u) = \int |Du|$$

The problem can be rewritten (denoting $v = f - u$) :

$$\inf_{(u,v)\in BV\times L^2 / f=u+v} \left(J(u) + \frac{1}{2\lambda}\|v\|_2^2 \right)$$

Meyer's model

Y. Meyer (2001) has proposed the following model :

$$\inf_{(u,v)\in BV\times G/f=u+v} \left(J(u)+\alpha\|v\|_G\right)$$

The Banach space G contains signals with strong oscillations, and thus in particular textures and noise.

Definition : G is the Banach space composed of generalized functions v which can be written

$$v = \partial_1 g_1 + \partial_2 g_2 = \operatorname{div}(g)$$

with g_1 and g_2 in L^∞.

$$\|v\|_G = \inf\left\{\|g\|_\infty / v = \operatorname{div}(g), g=(g_1,g_2),\ g_1 \text{ et } g_2 \in L^\infty, |g(x)| = \sqrt{|g_1|^2+|g_2|^2}(x)\right\}$$

Osher-Sole-Vese model

Osher-Sole-Vese have introduced the following functional (MMS 2003) :

$$\inf_u \left(J(u) + \frac{1}{2\lambda} \|f - u\|^2_{H^{-1}} \right)$$

where

$$H = W^{1,2} = \{f \in L^2 \ / \ \nabla f \in L^2\}$$

and

$$H^{-1} = W^{-1,2} = \left(W_0^{1,2}\right)'$$

TV-Hilbert

Aujol-Gilboa 2004 :

$$\inf_u \left(J(u) + \frac{\lambda}{2} \|f-u\|^2_{\mathcal{H}} \right)$$

K is a linear positive symmetric operator, and

$$\langle f, g \rangle_{\mathcal{H}} = \langle f, Kg \rangle_{L^2}$$

1. When $K = Id$, then $\mathcal{H} = L^2 \Longrightarrow$ ROF model
2. When $K = -\Delta^{-1}$, then $\mathcal{H} = H^{-1} \Longrightarrow$ OSV model

Duality

$$\inf_u \left(J(u) + \frac{\lambda}{2} \|f - u\|_{\mathcal{H}}^2 \right) \tag{1}$$

K is a linear positive symmetric operator, and

$$\langle f, g \rangle_{\mathcal{H}} = \langle f, Kg \rangle_{L^2}$$

Proposition : Problem (1) has a unique solution $\hat{u}$, which is given by :

$$\hat{u} = f - P^{\mathcal{H}}_{K^{-1}G_{1/\lambda}}(f)$$

where $P^{\mathcal{H}}_{K^{-1}G_{1/\lambda}}(f)$ stands for the orthogonal projection (with respect to the $\mathcal{H}$ inner product) of f on $K^{-1}G_{1/\lambda}$.

We recall that

$$G_{1/\lambda} = \left\{ v \in G \ / \ \|v\|_G \leq \frac{1}{\lambda} \right\}$$

Projection algorithm

$$\inf_u \left(J(u) + \frac{\lambda}{2} \|f - u\|^2_{\mathcal{H}} \right) \tag{1}$$

It is possible to adapt Chambolle's projection algorithm (JMIV 2004) to this functional (Aujol-Gilboa, JMIV 2006).

Algorithm :

$$p^0 = 0$$

and

$$p^{n+1}_{i,j} = \frac{p^n_{i,j} + \tau(\nabla(K^{-1}\mathrm{div}\,(p^n) - \lambda f))_{i,j}}{1 + \tau|(\nabla(K^{-1}\mathrm{div}\,(p^n) - \lambda f))_{i,j}|}$$

Theorem : If $\tau \leq \frac{1}{8\|K^{-1}\|_{L^2}}$, then $\frac{1}{\lambda}K^{-1}\mathrm{div}\,p^n \to \hat{v}$ as $n \to \infty$, and $f - \frac{1}{\lambda}K^{-1}\mathrm{div}\,p^n \to \hat{u}$ as $n \to \infty$, where $\hat{u}$ is the solution of problem (1) and $\hat{v} = f - \hat{u}$.

Remark : Similar result in Almansa et al (MMS 2006).

Overview

1) Model

2) Image restoration

3) Image decomposition

Denoising

Model :

$$\inf_u \left(J(u) + \frac{\lambda}{2} \|f - u\|_{\mathcal{H}}^2 \right)$$

Constrained problem :

$$\min_u \left\{ J(u) \ / \ \|u - f\|_{\mathcal{H}}^2 = N^2 \rho^2 \right\}$$

where $N^2\rho^2$ is the $\mathcal{H}$ norm of an image (of size N^2) of a white Gaussian noise n with standard deviation σ.

Proposition :

$$N^2\rho^2 = E(\|n\|_{\mathcal{H}}^2) = C_{\mathcal{H}} E(\|n\|_{L^2}^2) = C_{\mathcal{H}} N^2 \sigma^2$$

Problem : If σ is known, how to choose the best (in some sens) parameter λ ?

Notations

Definitions : normalized, zero-mean *inner-product* and normalized, zero-mean square of a *norm*

$$\mathcal{I}(p,q) \doteq \frac{1}{|\Omega|} < p-\bar{p}, q-\bar{q} >_{\mathcal{H}}$$

$$\mathcal{N}(p) \doteq \mathcal{I}(p,p) = \frac{1}{|\Omega|} \|p-\bar{p}\|_{\mathcal{H}}^2$$

($\bar{p}$ stands for the mean of p)

$\Longrightarrow$ standard notions of empirical *covariance* and *variance*, respectively, for $\mathcal{H} = L^2$.

We have $f = s + n$ (signal+noise), and our problem can be written as

$$\inf_{(u,v)\ /\ f=u+v} \left(J(u) + \frac{\lambda}{2} \mathcal{N}(v) \right)$$

The solution is denoted by (u_λ, v_λ)

Notations

Definition : $\mathcal{H}$ Signal-to-Noise Ratio

$$SNR^{\mathcal{H}}(u) \doteq 10 \log \frac{\mathcal{N}(s)}{\mathcal{N}(u-s)} = 10 \log \frac{\mathcal{N}(s)}{\mathcal{N}(n-v)},$$

(we usually omit the $\mathcal{H}$ superscript)

The (square) norm of the noise is $\mathcal{N}(n) = \rho^2$. For $\mathcal{H} = L^2$ we have $\rho^2 = \sigma^2$.

Let us define the optimal SNR of a certain process applied to an input image f as :

$$SNR_{opt} \doteq \max_{\lambda} SNR(u_\lambda)$$

Optimality condition

We regard the SNR as a function $SNR(\mathcal{N}(v))$ and assume that it is smooth. A necessary condition for the maximum in the range $\mathcal{N}(v) \in (0, \mathcal{N}(f))$ is :

$$\frac{\partial SNR}{\partial \mathcal{N}(v)} = 0$$

Rewriting $\mathcal{N}(n - v)$ as $\mathcal{N}(n) + \mathcal{N}(v) - 2\mathcal{I}(n, v)$, one gets :

$$\frac{\partial \mathcal{I}(n, v)}{\partial \mathcal{N}(v)} = \frac{1}{2}$$

More specifically, as long as $\frac{\partial \mathcal{I}(n,v)}{\partial \mathcal{N}(v)} > \frac{1}{2}$, the SNR increases. When the optimality condition is reached, noise and signal are equally filtered and the SNR is at a local maximum. If filtering is continued, more signal than noise is filtered (in the $\mathcal{H}$-norm sense) and the SNR decreases.

Algorithm

$$\inf_{u,v}\left(J(u)+\frac{\lambda}{2}\mathcal{N}(v)\right), \text{ subject to } f=u+v$$

Provided that $\mathcal{I}(n,v)$ can be estimated, our basic numerical algorithm should be as follows :

1. Set $\mathcal{I}^0(n,v)=0$, $\mathcal{N}^0(v)=0$, $i=1$.
2. $\mathcal{N}^i(v)\leftarrow\mathcal{N}^{i-1}(v)+d\mathcal{N}(v)$. Compute $\mathcal{I}^i(n,v)$.
3. If $\frac{\mathcal{I}^i(n,v)-\mathcal{I}^{i-1}(n,v)}{d\mathcal{N}(v)}\leq\frac{1}{2}$ then stop.
4. $i\leftarrow i+1$. Goto step 2.

Estimation

$f = s + n$ (signal+noise)

Observation : The extent of filtering of additive noise n, with respect to λ, is not affected much by the underlying image s.

What mainly affects the denoising performance is the extent of filtering of s.

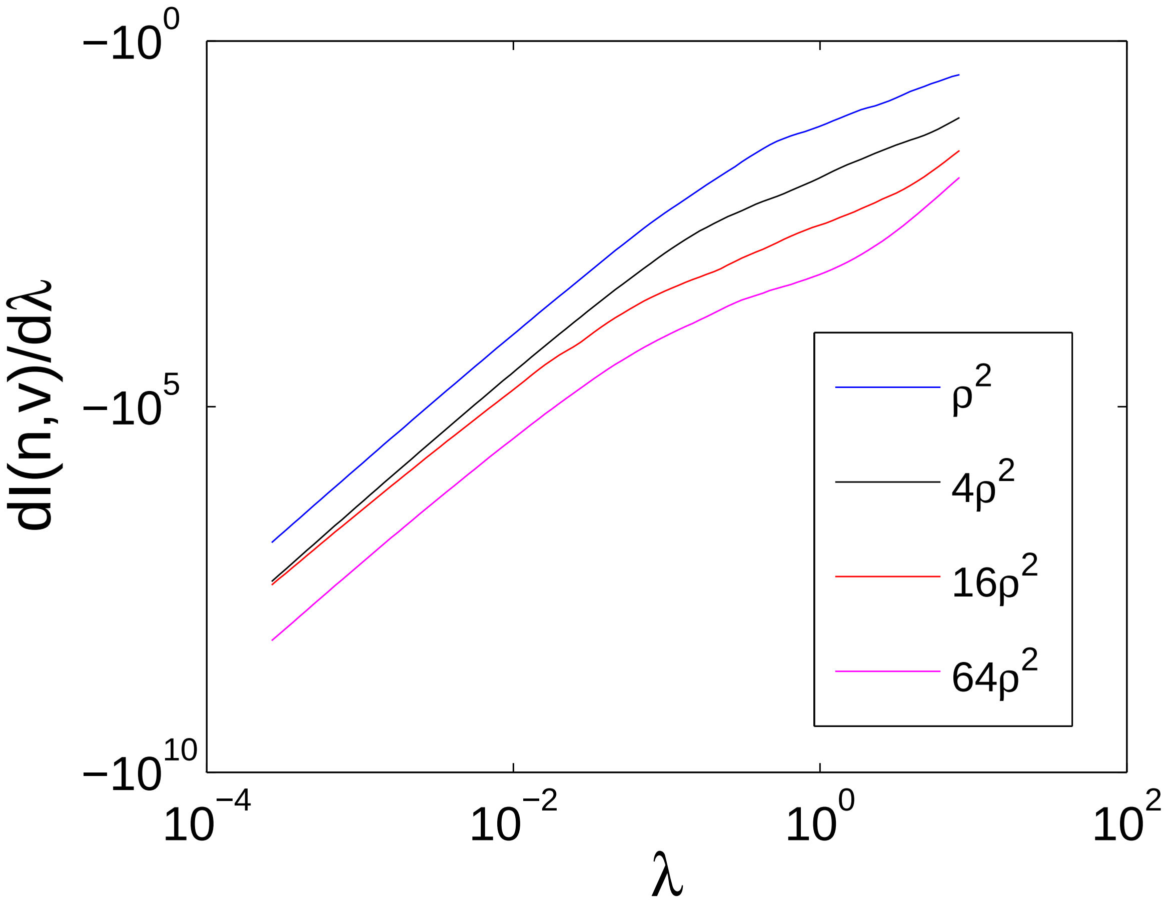
$\Longrightarrow$ We measure in advance the extent of filtering of noise with the same variance.

We compute the "statistics" by processing a patch of pure noise $\tilde{n}$.

Thanks to the chain-rule for differentiation, we then have :

$$\begin{aligned} \frac{\partial \mathcal{I}(n,v)}{\partial \mathcal{N}(v)} &= \frac{\partial \mathcal{I}(n,v)}{\partial \lambda} \frac{\partial \lambda}{\partial \mathcal{N}(v)} \\ &\approx \frac{\partial \mathcal{I}(\tilde{n},v)}{\partial \lambda}\Big|_{f=\tilde{n}} \frac{\partial \lambda}{\partial \mathcal{N}(v)}\Big|_{f=s+n} \end{aligned}$$

Look-up table



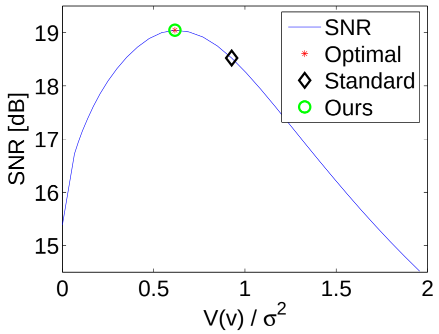
Precomputed term $\partial\mathcal{I}(\tilde{n}, v)/\partial\lambda$ as a function of λ (log scale), $\mathcal{H} = H^{-1}$. Graphs depict plots for different proportion of ρ^2 - $1 : 4 : 16 : 64$, from upper curve to lower curve, respectively.

$TV + L^2$ case

Gilboa et al, IEEE TIP 2006

f

u, SNR based

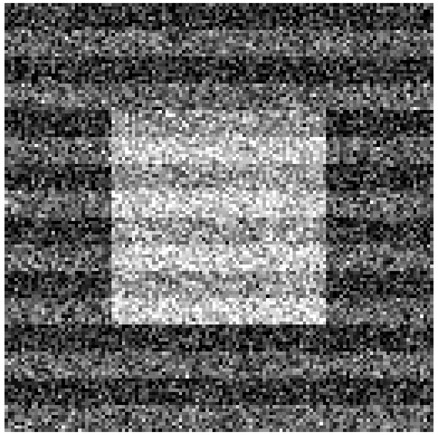


s

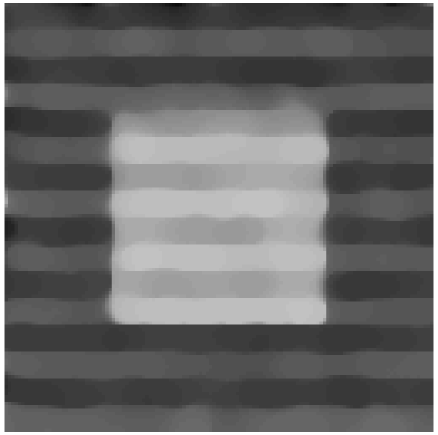
u, constrained

$TV + H^{-1}$ case

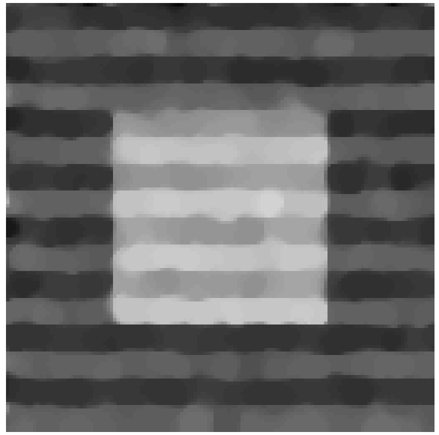
f



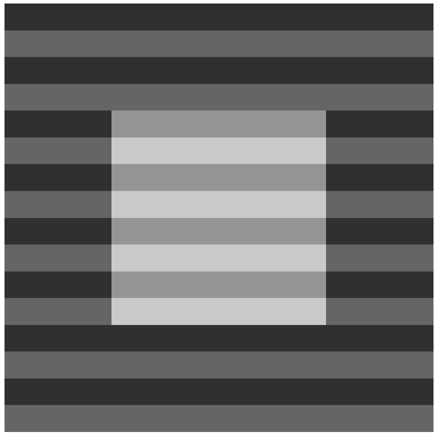
u, constrained



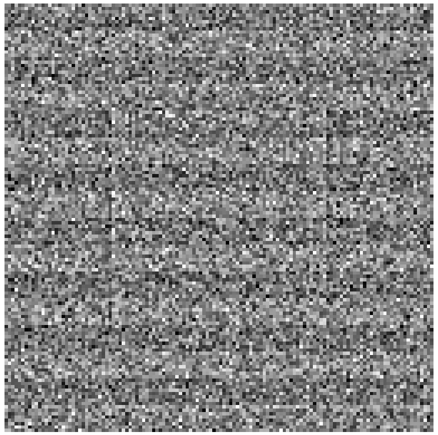
u, SNR-based



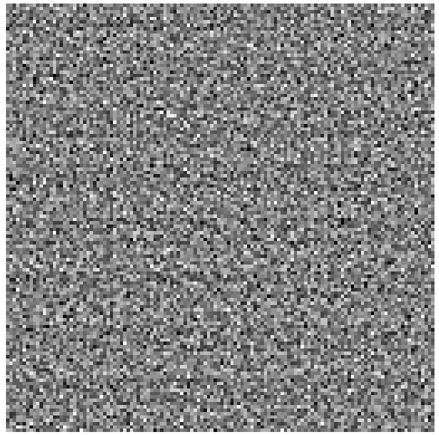
s



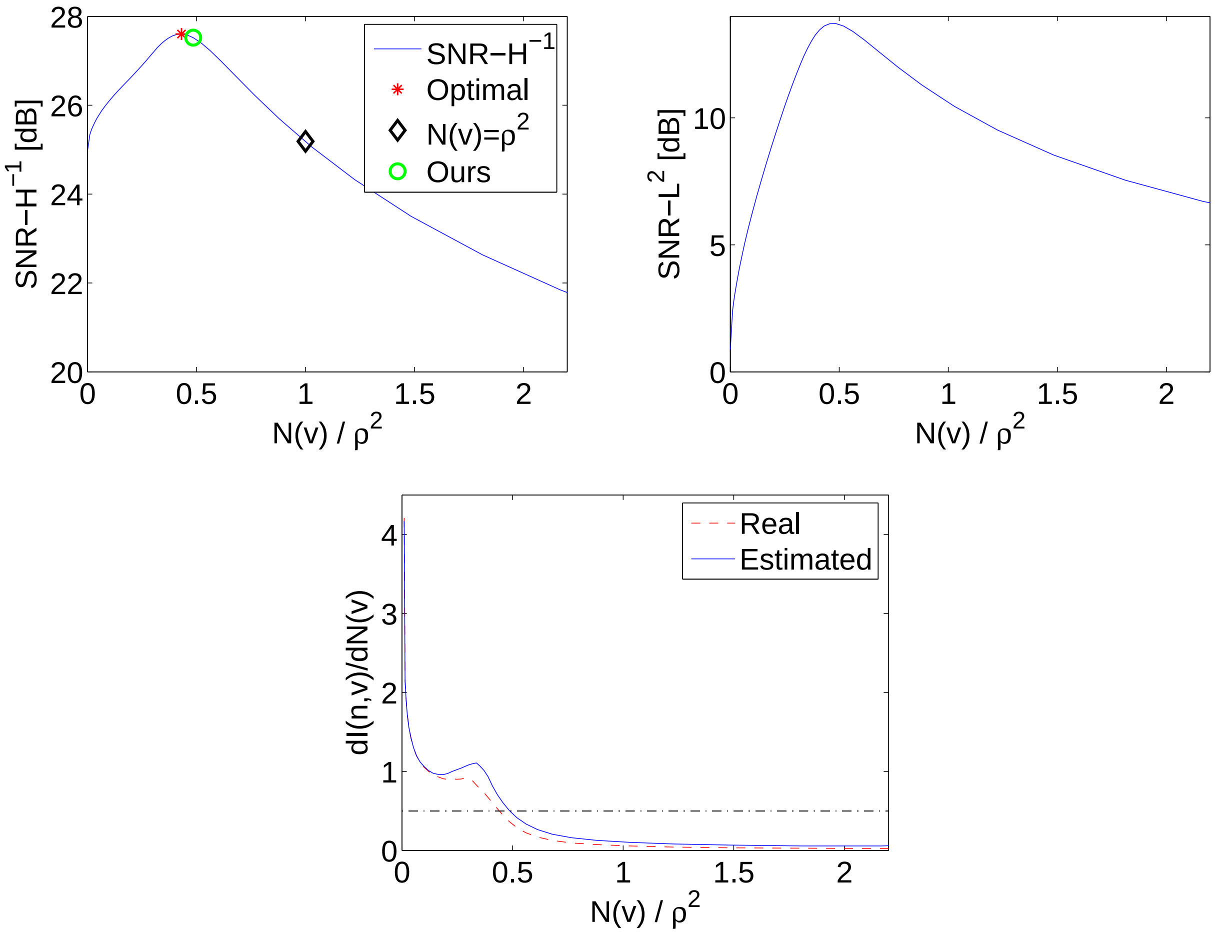
v, constrained



v, SNR-based



$TV + H^{-1}$ case

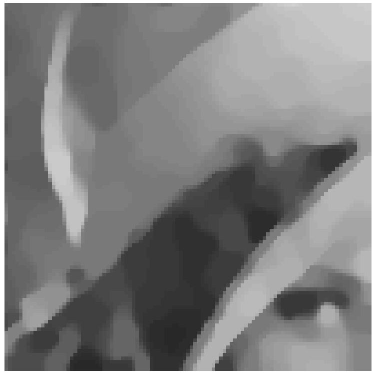


Top (left) : $SNR^{H^{-1}}$ as a function of $\mathcal{N}(v)$. Top (right) : a plot of the standard SNR^{L^2}. Bottom row : estimated $\partial\mathcal{I}(n,v)/\partial\mathcal{N}(v)$ vs. the ground truth.

Numerical experiment (2)

f

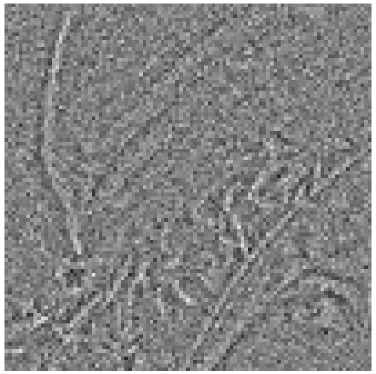
u, constrained



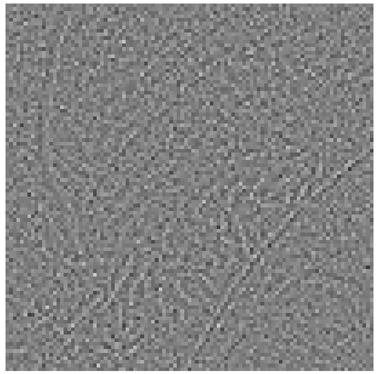
u, SNR-based

s

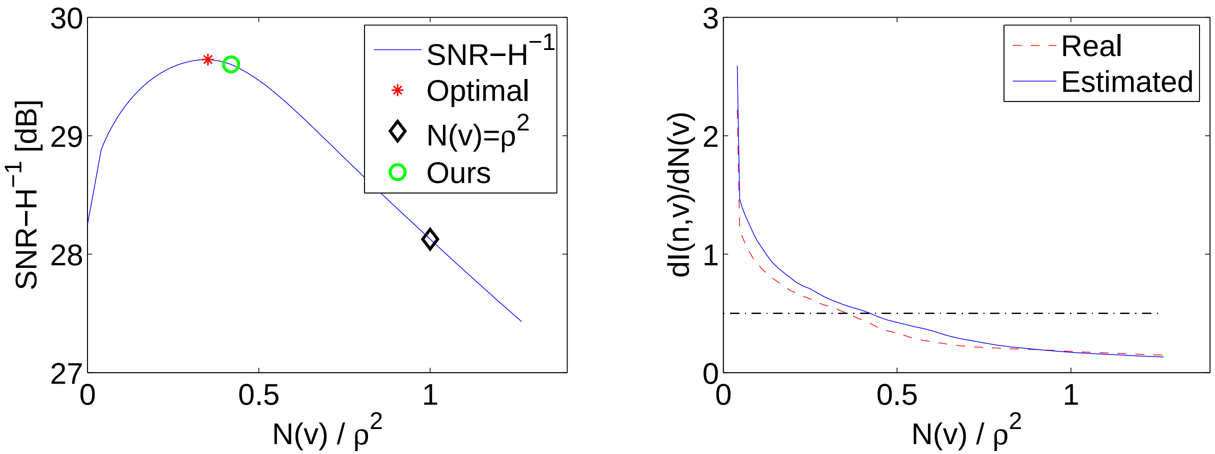
v, constrained



v, SNR-based



Numerical experiment (2)



Numerical experiment (3)

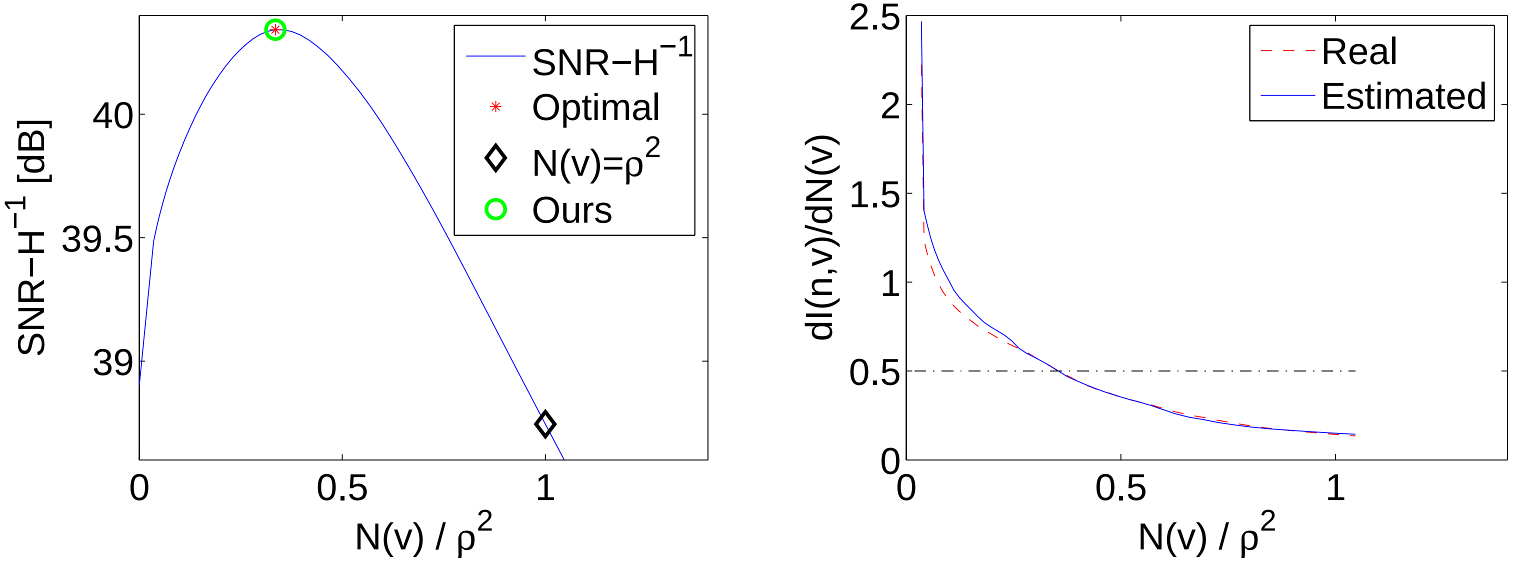
s

f

u, constrained

u, SNR-based

Numerical experiment (3)



Overview

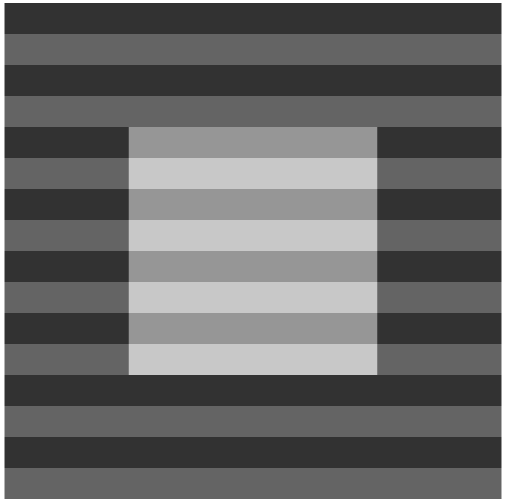
1) Model

2) Image restoration

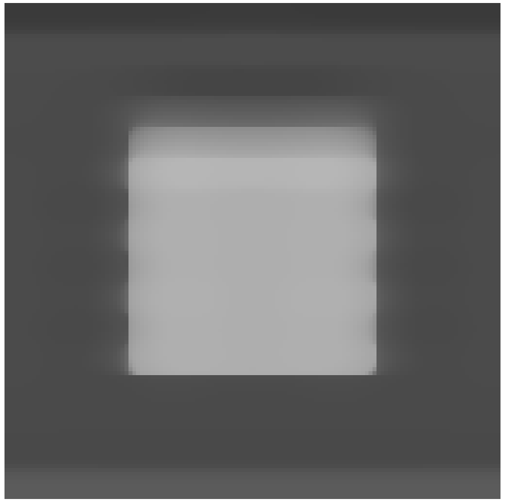
3) Image decomposition

Image decomposition

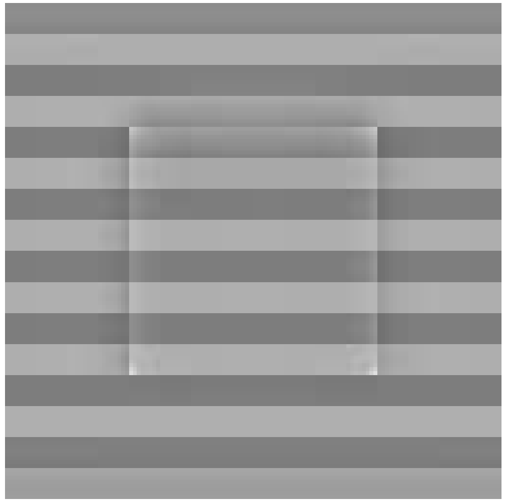
Original image f



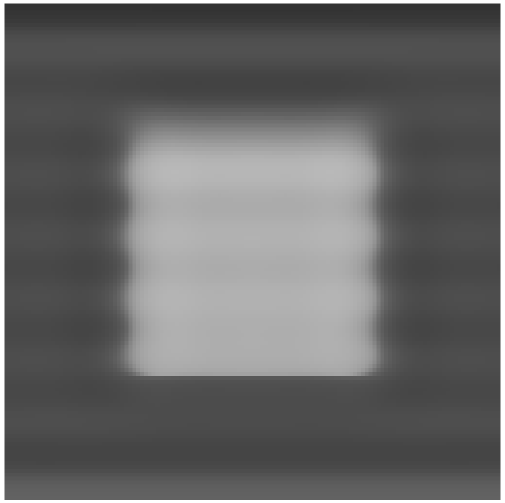
$u\ (G)$



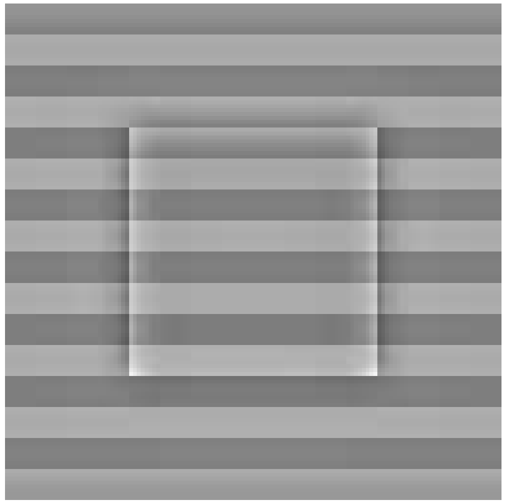
$v\ (G)$



u_{OSV}



v_{OSV}



TV-Hilbert

Aujol-Gilboa 2004 :

$$\inf_u \left(J(u) + \frac{\lambda}{2} \|f - u\|_{\mathcal{H}}^2 \right)$$

K is a linear positive symmetric operator, and

$$\langle f, g \rangle_{\mathcal{H}} = \langle f, Kg \rangle_{L^2}$$

1. When $K = Id$, then $\mathcal{H} = L^2 \Longrightarrow$ ROF model
2. When $K = -\Delta^{-1}$, then $\mathcal{H} = H^{-1} \Longrightarrow$ OSV model

Filtering

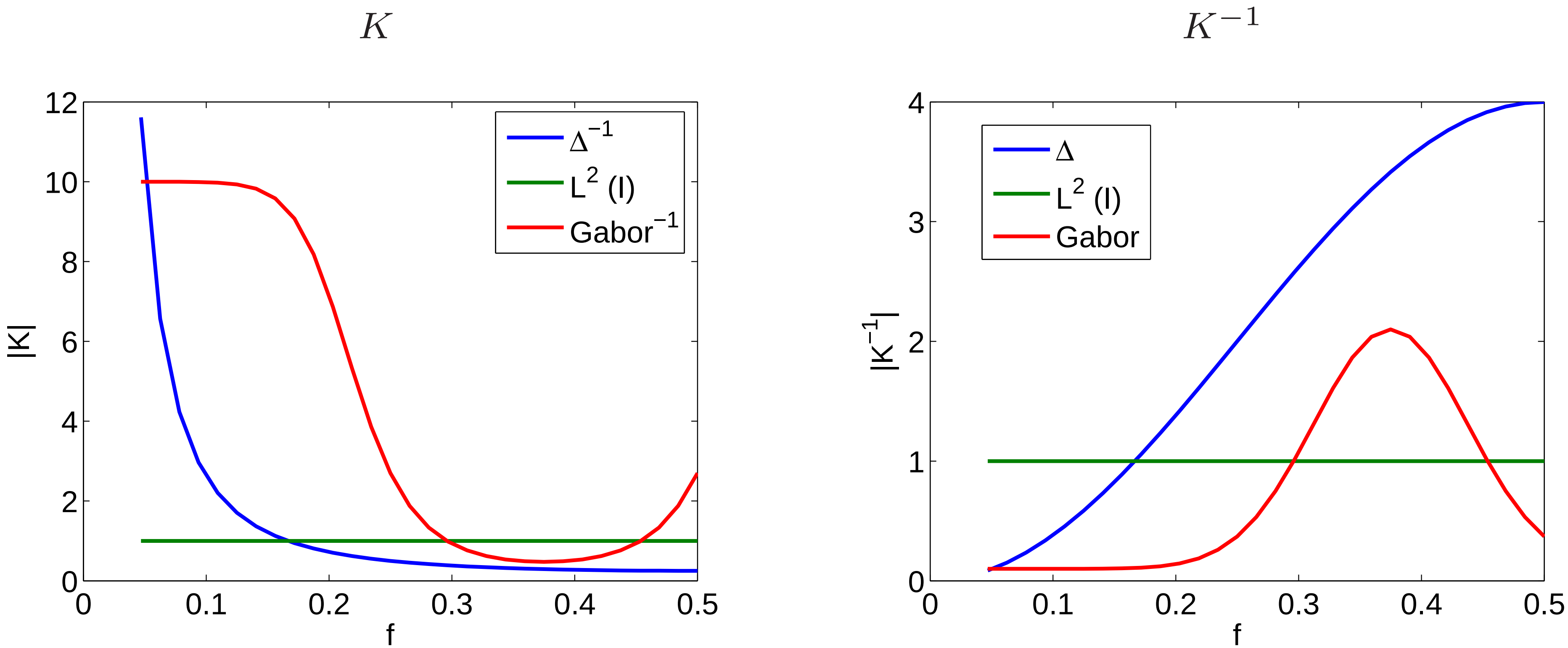


FIG. 1 – The kernel K and its inverse K^{-1} for the OSV, ROF and the proposed TV-Gabor model.

Texture

Textures are periodic elements.

$\Longrightarrow$ A simple way to characterize a texture is by its main frequency and direction.

$\Longrightarrow$ This naturally leads us to consider Gabor functions :

1D Gabor function (frequency ν, bandwidth σ) :

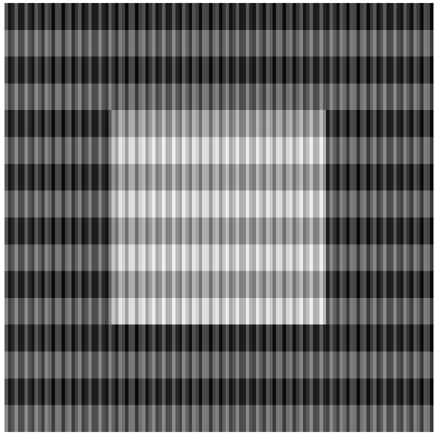
$$g(x) = \cos(2\pi\nu x)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-x^2}{2\sigma^2}\right)$$

2D : consider $g(x)g(y)$ or a rotationally symmetric Gabor function :

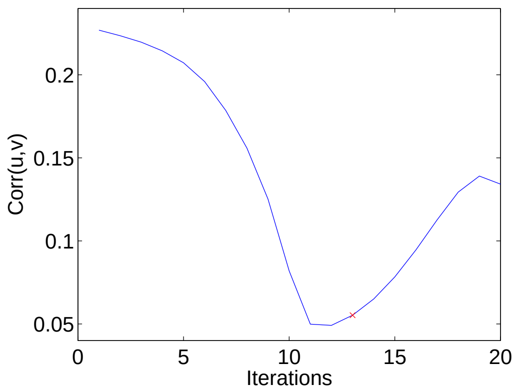
$$g(x,y) = \cos\left(2\pi\nu\sqrt{x^2+y^2}\right)\frac{1}{\sqrt{2\pi}\sigma^2}\exp\left(\frac{-x^2-y^2}{2\sigma^2}\right)$$

A simple example

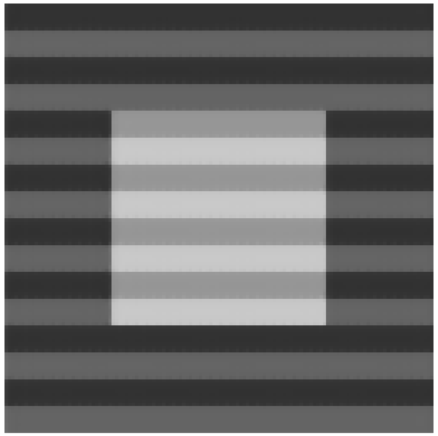
f



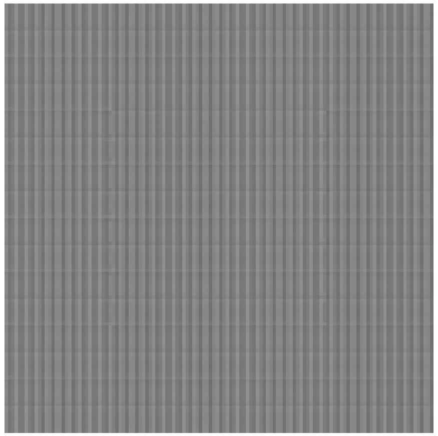
$\mathrm{corr}(u, v)$



u

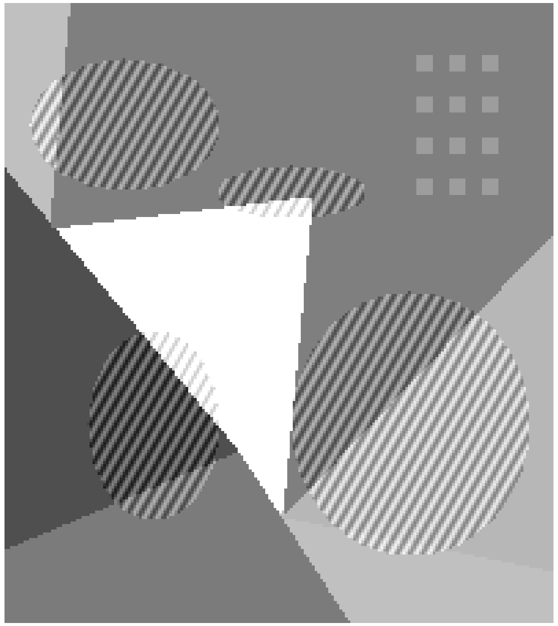


v



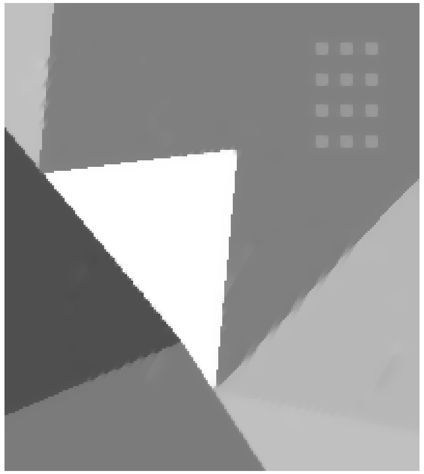
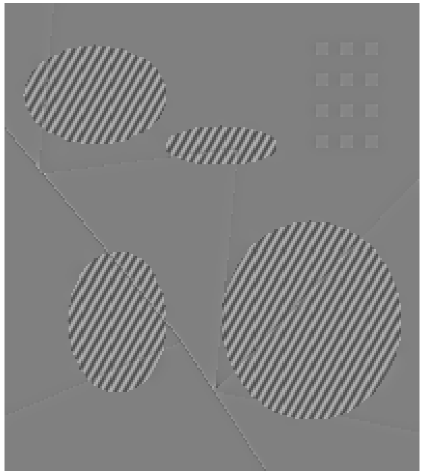
Results (I)

f

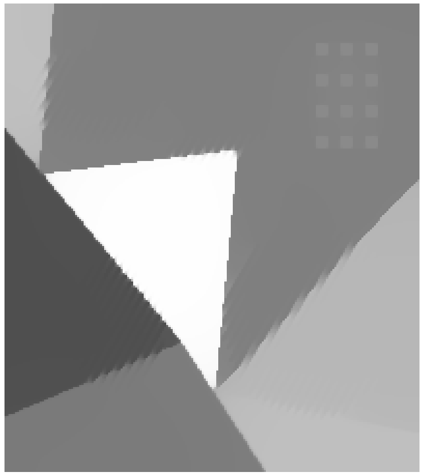
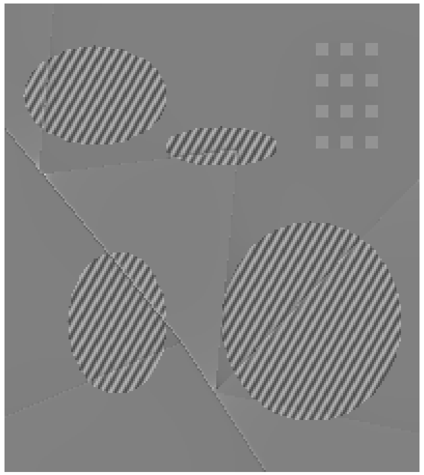


Results (II)

TV-Gabor, u $\quad$ v

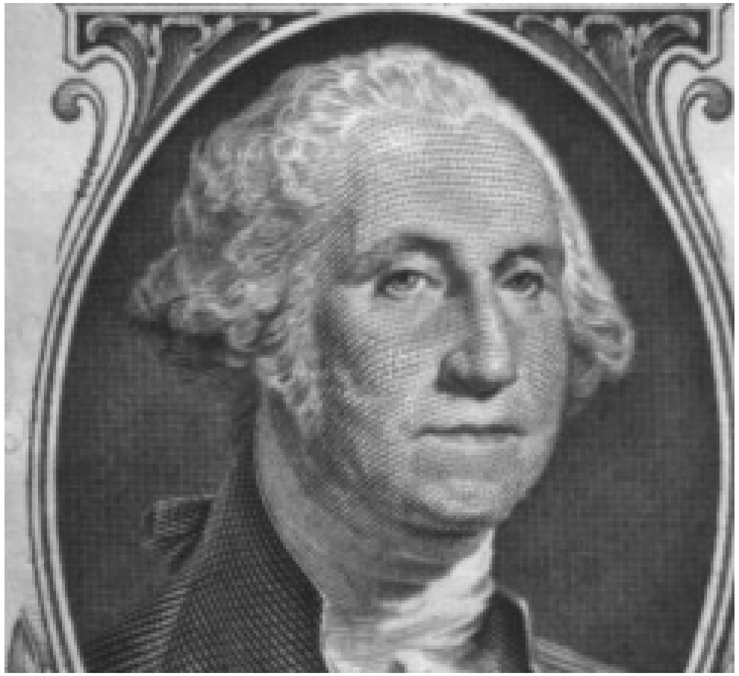
 

TV-L^2, u $\quad$ v

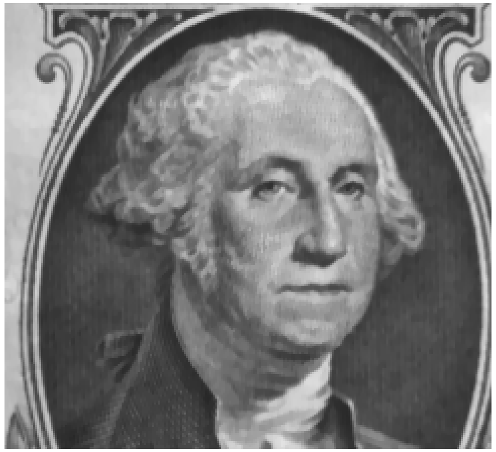
Washington (I)

f

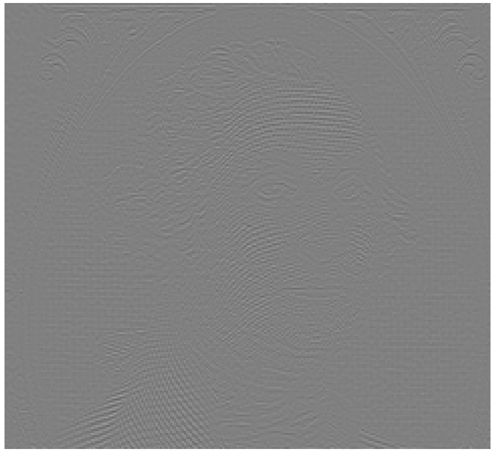


Washington (II)

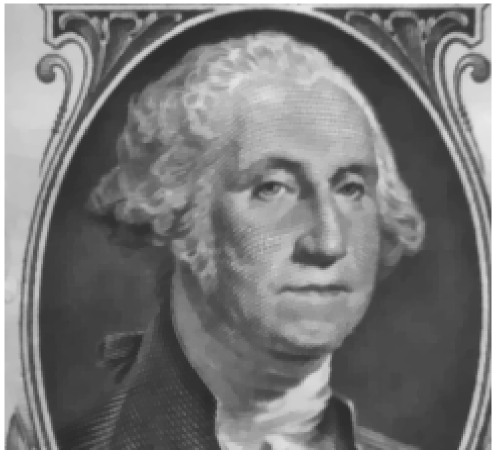
u (TV-Gabor)



v (TV-Gabor)



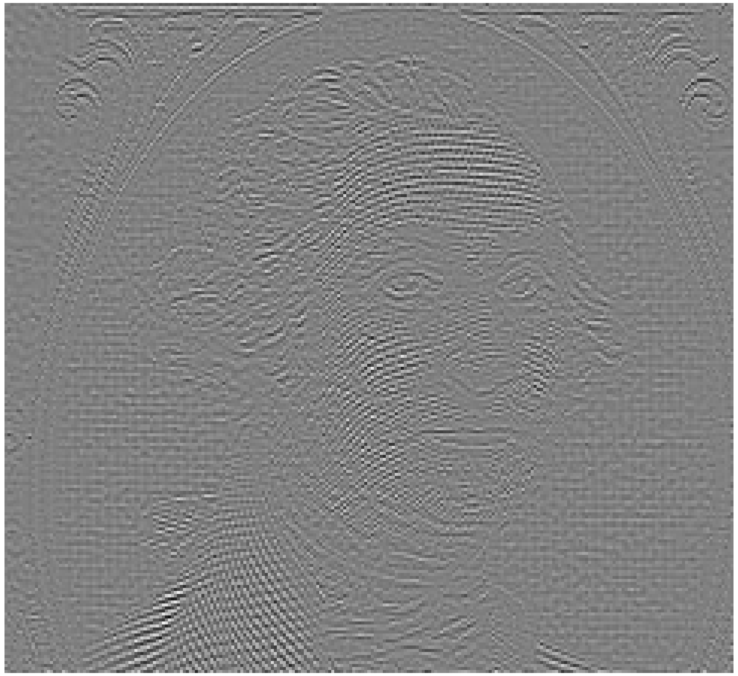
u $(TV - L^2)$



v $(TV - L^2)$

Washington (III)

$4v$ (TV-Gabor)



$4v$ ($TV - L^2$)

Conclusion

Image restoration : We have proposed a general TV-Hilbert framework for image denoising : provided the standard deviation of the noise is known, an optimal solution (in the SNR sens) can be automatically computed.

Image decomposition : We have also shown how to apply this framework to image decomposition. Provided some characteristics of the texture are known, it is possible to incoporate them into the model.

Parameter tuning : In variational approach, the problem of finding the right parameter is of first importance : a good model with a wrong parameter will give a poor result ...
$\Longrightarrow$ open question in the image decomposition problem

References :
Aujol-Gilboa, JMIV 2006
Aujol-Gilboa-Chan-Osher, IJCV 2006