A numerical approach to the MFG

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Outline

Mean field games: introduction

• MFG = model for interaction among a large number of agent / players ... not particles. An agent can decide, based on a set of preferences and by acting on parameters (... control theory).
Note: in standard rumor spreading (or opinion making) modeling agent is supposed to be a mechanical black-box, not the case here. This situation is included as particular case.

• distinctive properties: the existence of a collective behavior (fashion trends, financial crises, real estates valuation, etc.). One agent by itself cannot influence the collective behavior, it only optimizes its own decisions given the environmental situation.
References: Lasry Lions CRAS notes (2006), Lions online course at College de France. Further references latter on.

Mean field games: introduction

• Nash equilibrium: a game of N players is in a Nash equilibrium if, for any player j supposing other $N-1$ remain the same, there is no decision of the player j that can improve its outcome.

• MFG = Nash equilibrium equations for $N \to \infty$. All players are the same.

• Agent follows an evolution equations involving some controlling action. Its decision criterion depend on the others, more precisely on the density of other players.

• Will consider here stochastic diff. equations, but deterministic case is a particular situation and can be treated.

Mathematical framework of MFG

What follows is the most simple model that shows the properties of MFG models. Cf. references for more involved modeling. X_t^x = the characteristics at time t of a player starting in x at time 0. It evolves with SDE:

$$dX_t^x = \alpha(t, X_t^x)dt + \sigma dW_t^x,\ X_0^x = x \tag{1}$$

• $\alpha(t, X_t^x)$ = control can be changed by the agent/ player.
• independent brownians (!)
• $m(t,x)$ = the density of players at time t and position $x \in E$; E is the state space. Optimization problem of the agent:

$$\inf_\alpha \mathbb{E}\left\{ \frac{1}{T}\left[\int_0^T h(X_t^x, \alpha(t, X_t^x)) + V(X_t^x; m(t,\cdot))dt\right] + V_0(X_T^x; m(T,\cdot))\right\} \tag{2}$$

here T can be fixed (fixed horizon) or $T \to \infty$ (static case).

Mathematical framework of MFG: examples

Example: choice of a holiday destination.
Particular case: deterministic, no dependence on the initial condition, no dependence on the control. Each individual minimizes distance to an ideal destination and a term depending on the presence of others:
$V_0(y; m) = F_0(y) + F_1(m)$.
Question: what is the solution ? X_T^x will be chosen as the minimum of $y \mapsto F_0(y) + F_1(m(y))$. Then m is the distribution of such X_T^x.
COUPLING between m and X_T^x !!
Particular case: $F_0(y) = y^2$ on $\mathbb{R}$. Origin is the most preferred point for all individuals, distance increases slowly in neighborhood, fast outside. Take $F_1(m) = cm$.
Modelization: $c > 0$ = crowd aversion, $c < 0$ = propensity to crowd.
Remark: all points y in the the support of m have to be minimums of V_0 !
Solution: $c > 0$: semi-circular distribution $m(y) = \frac{(\lambda - y^2)_+}{c}$, $c < 0$: Dirac masses at minimum of F_0.

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Mathematical objects: SDEs

Brownian motion models a very irregular motion (but continuous). Mathematically it is a set of random variables indexed by time t, denoted W_t, with:

• $W_0 = 0$ with probability 1

• a.e. $t \mapsto W_t(\omega)$ is continuous on $[0, T]$

• for $0 \leq s \leq t \leq T$ the increment $W(t) - W(s)$ is a random normal variable of mean 0 and variance $t - s$: $W(t) - W(s) \approx \sqrt{t-s}\mathcal{N}(0,1)$ ($\mathcal{N}(0,1)$ is the standard normal variable)

• for $0 \leq s < t < u < v \leq T$ the increments $W(t) - W(s)$ $W(v) - W(u)$ are independent.

Recall normal density $\mathcal{N}(0,\lambda)$ is $\frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{x^2}{2\lambda}}$; $W_{t+dt} - W_t$ has as law $\sqrt{dt}\mathcal{N}(0,1)$ (of order $dt^{1/2}$, cf. Ito formula).

Martingales

$(\Omega, \mathcal{A}, P)$ = probability space, $(\mathcal{A}_t)_{t\geq 0}$ filtration.
An adapted family $(M_t)_{t\geq 0}$ of integrable r.v. (i.e. $\mathbb{E}|M_t| < \infty$) is martingale if for all $s \leq t$: $\mathbb{E}(M_t|\mathcal{A}_s) = M_s$.
Thus $\mathbb{E}(M_t) = \mathbb{E}(M_0)$.

Theorem

Let $(W_t)_{t\geq 0}$ be a Brownian motion, then W_t, $W_t^2 - t$, $e^{\sigma W_t - \frac{\sigma^2}{2} t}$ are also martingales.

Ito integral

We want to define $\int_0^T f(t,\omega)dW_t$.
For $\int_0^T h(t)dt$ Riemann sums $\sum_j h(t_j)(t_{j+1}-t_j)$ converge to the Riemann integral when the division $t_0 = 0 < t_1 < t_2 < ... < t_N = T$ of $[0, T]$ becomes finer.
For the Riemann-Stiltjes integral we can replace dt by increments of a bounded variation function $g(t)$ and obtain $\int f(t)dg(t)$
Similarly one can work with Ito sums $\sum_{j=0}^{N-1} h(t_j)(W_{t_{j+1}} - W_{t_j})$ or Stratonovich $\sum_{j=0}^{N-1} h(\frac{t_j+t_{j+1}}{2})(W_{t_{j+1}} - W_{t_j})$ both are the same for deterministic function h.

Ito integral

Example: $h = W$, $t_j = j \cdot dt$.
Ito:

$$\sum_{j=0}^{N-1} h(t_j)(W_{t_{j+1}} - W_{t_j}) = \sum_{j=0}^{N-1} W_{t_j}(W_{t_{j+1}} - W_{t_j}) \tag{3}$$

$$= \frac{1}{2} \sum_{j=0}^{N-1} W_{t_{j+1}}^2 - W_{t_j}^2 - (W_{t_{j+1}} - W_{t_j})^2 \tag{4}$$

$$= \frac{1}{2}\left(W_T^2 - W_0^2\right) - \frac{1}{2} \sum_{j=0}^{N-1} (W_{t_{j+1}} - W_{t_j})^2. \tag{5}$$

The term $\frac{1}{2}\sum_{j=0}^{N-1}(W_{t_{j+1}} - W_{t_j})^2$ has average $Ndt = T$ and variance of order dt so the limit will be $\frac{1}{2}\left(W_T^2 - T\right)$.
Thus $\int_0^T W_t dW_t = \frac{1}{2}\left(W_T^2 - T\right)$; in particular the non-martingale (previsible) part of W_t^2 will be t.

Ito integral

Stratonovich:

$$\sum_{j=0}^{N-1} h\left(\frac{t_j + t_{j+1}}{2}\right)(W_{t_{j+1}} - W_{t_j}) = \sum_{j=0}^{N-1} W_{\frac{t_j+t_{j+1}}{2}}(W_{t_{j+1}} - W_{t_j}) \quad (6)$$

$$\sum_{j=0}^{N-1} \left(\frac{W_{t_j} + W_{t_{j+1}}}{2} + \Delta Z_j\right)(W_{t_{j+1}} - W_{t_j}) \quad (7)$$

Here ΔZ_j is a r.v. independent of W_{t_j}, of null average and variance $dt/4$.
Sum will be $\frac{1}{2}W_T^2$.
Stratonovich is also limit of

$$\sum_{j=0}^{N-1} \frac{h(t_j) + h(t_{j+1})}{2}(W_{t_{j+1}} - W_{t_j}). \quad (8)$$

Ito integral

More generally for H_t adapted to the filtration $(\mathcal{A}_t)_{t\geq 0}$ we can define (as soon as $\int_0^T H_s^2 ds < \infty$) the Ito integral $\int_0^T H_s dW_s$ (martingale if $\mathbb{E}\int_0^T H_s^2 ds < \infty$; sufficient condition). Ito integral is continuous.

Theorem (Ito Isometry)

$$\mathbb{E}\int_0^T H(W_t, t)dW_t = 0 \tag{9}$$

$$\mathbb{E}\Big(\int_0^T H(W_t, t)dW_t\Big)^2 = \int_0^T \mathbb{E}H^2(W_t, t)dt. \tag{10}$$

Proof: first verified on sums...

Ito process

Ito process $(X_t)_{t\geq 0}$: $X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$, with X_0 $\mathcal{A}_0$ measurable, K_t and H_t adapted, $\int_0^T |K_s| ds < \infty$ $\int_0^T H_s^2 ds < \infty$ X_t is the solution of the stochastic differential equation (SDE): $dX_t = Kdt + HdW_t$. When K, H depend on X_t too this is an equality with X_t in both terms.

Theorem (Ito)

For f of C^2 class, if

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t$$

then

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX_t + \frac{1}{2}\beta(t, X_t)^2 \frac{\partial^2 f}{\partial X^2}dt. \tag{11}$$

Rq: similar to development of $f(t, \sqrt{t})$ around $f(0,0) = 0$...
Exercice $\frac{dS_t}{S_t} = \alpha dt + \sigma dW_t$ and $S_t = e^{X_t}$ then $dX_t = (\alpha - \frac{\sigma^2}{2})dt + \sigma dW_t$.

Fokker-Planck

- evolution equation for the density : Fokker-Planck

Theorem (Fokker-Planck)

Let $\rho(t,\cdot)$ be the probability density of X_t that follows

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t \tag{12}$$

then

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}\left(\alpha(t,x)\rho(t,x)\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\beta^2(t,x)\rho(t,x)\right) = 0. \tag{13}$$

Proof: $\mathbb{E}[\varphi(X_t)] = \int_{\mathbb{R}} \varphi(x)\rho(t,x)dx \quad \forall \varphi \in \mathbb{C}_0^\infty$.
By Ito,
$\varphi(X_t) = \varphi(X_0) + \int_0^t \alpha \frac{\partial}{\partial x}\varphi(X_s)\, ds + \int_0^t \beta \frac{\partial}{\partial x}\varphi(X_s)\, dW_s + \frac{1}{2}\int_0^t \beta^2 \frac{\partial^2}{\partial x^2}\varphi(X_s)\, ds$

$\mathbb{E}[\varphi(X_t)] = \int \varphi(x)\rho_0(x)dx + \mathbb{E}[\int_0^t \alpha \frac{\partial}{\partial x}\varphi(X_s)\ ds + \int_0^t \beta \frac{\partial}{\partial x}\varphi(X_s)\ dW_s +$
$\frac{1}{2}\int_0^t \beta^2 \frac{\partial^2}{\partial x^2}\varphi(X_s)\ ds]$

By Ito Isometry $\mathbb{E}[\int_0^t \beta \frac{\partial}{\partial x}\rho\ dW_s] = 0$
$\mathbb{E}[\varphi(X_t)] = \int_{\mathbb{R}} \varphi(x)\rho_0(x)dx + \mathbb{E}[\int_0^t \alpha \frac{\partial}{\partial x}\varphi(X_s)ds] + \frac{1}{2}\mathbb{E}[\int_0^t \beta^2 \frac{\partial^2}{\partial x^2}\varphi(X_s)ds]$
$\mathbb{E}[\varphi(X_t)] = \int_{\mathbb{R}} \varphi(x)\rho_0(x)dx + \int_{\mathbb{R}}\int_0^t \alpha \frac{\partial}{\partial x}\varphi(x)\ \rho(s,x)ds\ dx +$
$\frac{1}{2}\int_{\mathbb{R}}\int_0^t \beta^2 \frac{\partial^2}{\partial x^2}\varphi(x)\rho(s,x)ds\ dx$

By integration by parts
$\mathbb{E}[\varphi(X_t)] = \int_{\mathbb{R}} \varphi(x)\rho_0(x)dx + [\int_0^t \alpha\varphi(x)\rho(s,x)ds]_{-\infty}^{+\infty} -$
$\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x}[\alpha\rho(s,x)]\varphi(x)ds\ dx + \frac{1}{2}\int_{\mathbb{R}}\int_0^t \beta^2 \frac{\partial^2}{\partial x^2}\varphi(x)\rho(s,x)ds\ dx$

$\varphi \in \mathbb{C}_0^\infty$ so $[\int_0^t \alpha\varphi(y)\rho(s,y)ds]_{-\infty}^{+\infty} = 0$

$\int_{\mathbb{R}} \varphi(x)[\rho(t,x) - \rho_0(x) + \int_0^t [\alpha \frac{\partial}{\partial x}\rho(s,x) - \frac{1}{2}\beta^2 \frac{\partial^2}{\partial x^2}\rho(s,x)]ds]dx = 0$

$$\forall\varphi \in \mathbb{C}_0^\infty \Longrightarrow \begin{cases} \frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}\rho(t,x) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\beta^2\rho(t,x) \\ \rho(0,x) = \rho_0(x) \end{cases}$$

Fokker-Planck

• evolution equation for the density : Fokker-Planck for several (independent) noises on same equation.

Theorem (Fokker-Planck)

Let $\xi(x)$ be a probability density on E and for each fixed x consider X_t^x that follows

$$dX_t^x = \alpha(t, X_t^x)dt + \beta(t, X_t^x)dW_t^x, \;\; X_0^x = x. \tag{14}$$

Denote by $\rho_x(t,y)$ the density of X_t^x for fixed x and $\rho(t,y)$ its marginal with respect to x i.e.: $\rho(t,y) = \int \rho_x(t,y)\xi(x)dx$. Then

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}\left(\alpha(t,x)\rho(t,x)\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\beta^2(t,x)\rho(t,x)\right) = 0 \tag{15}$$

$$\rho(0,\cdot) = \xi(\cdot) \tag{16}$$

Proof: by linearity of Fokker-Planck for one noise.

Outline

Optimal control theory

Consider evolution equation (in some Hilbert space):

$$\frac{dx(t)}{dt} = A(t, x(t), u(t)) \tag{17}$$

and optimal control functional to minimize

$$J(u) = \int_0^T f(t, x, u)dt + F(x(T)) \tag{18}$$

Simplest procedure to minimize: gradient descent. Update formula for step $\gamma > 0$:

$$u^{n+1} = u^n - \gamma \nabla_u J(u^n). \tag{19}$$

How to compute the gradient ?
Answer: calculus of variations: variations, Lagrange multiplier, ...

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Nash equlibrium for finite N. Agent k minimizes
$J^k(\alpha^1,...,\alpha^N) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T h(X_t^k,\alpha_t^k) + F^k(X_t^1,...,X_t^N)dt\right]$
The set of decisions $(\underline{\alpha^k})_k$ is a Nash equilibrium if $\forall k$, $\forall \alpha^k$:

$$J^k(\underline{\alpha^1},...\underline{\alpha^{k-1}},\underline{\alpha^k},\underline{\alpha^{k+1}},...,\underline{\alpha^N}) \leq J^k(\underline{\alpha^1},...\underline{\alpha^{k-1}},\alpha^k,\underline{\alpha^{k+1}},...,\underline{\alpha^N}), \tag{20}$$

Theoretical results of Lasry-Lions

Here F^k is symmetric in the other $N-1$ variables and moreover all agents are the same i.e. F^k does not depend on k:
$F^k(X_t^1, ..., X_t^N) = V(X^k; \frac{1}{N-1}\sum_{\ell \neq k} \delta_{X^\ell})$
Define: $H(x,\alpha) = \sup_p \langle p, \alpha \rangle - h(x,p)$; $\nu = \sigma^2/2$.
Limit for $N \to \infty$: static case; the optimality equations converge (up to sub-sequences) to solutions of MFG system

$$+\mathrm{div}(\alpha m) - \nu \Delta m = 0, \int m = 1, \ m \geq 0 \tag{21}$$

$$\alpha = -\frac{\partial}{\partial p} H(x, \nabla u) \tag{22}$$

$$-\nu \Delta u + H(x, \nabla u) + \lambda = V(x,m), \int u = 0. \tag{23}$$

Uniqueness: when V is a strictly monotone operator i.e.
$\int (V(m_1) - V(m_2))(m_1 - m_2) \leq 0$ implies $V(m_1) = V(m_2)$.

Theoretical results of Lasry-Lions

Limit for $N \to \infty$: finite horizon case (i.e. finite T); the optimality equations converge (up to sub-sequences) to solutions of MFG system

$$\partial_t m + \mathrm{div}(\alpha m) - \nu \Delta m = 0, \tag{24}$$

$$m(0,x) = m_0(x), \int m = 1, \ m \geq 0 \tag{25}$$

$$\alpha = -\frac{\partial}{\partial p} H(x, \nabla u) \tag{26}$$

$$\partial_t u + \nu \Delta u - H(x, \nabla u) + V(x,m) = 0, \tag{27}$$

$$u(T,x) = V_0(x, m(T,\cdot)), \int u = 0. \tag{28}$$

Remark: these are not necessarily the critical point equations for an optimization problem ! But will be in some particular cases studied latter.

Outline

Mean field games notations (reminder)

• Mean field games: limits of Nash equilibriums for infinite number of players (P.L.Lions & J.M.Lasry)

• equation for each player $dX_t^x = \alpha dt + \sigma dW_t^x$, $\alpha(t,x) =$ control
• $m(t,x) =$ the density of players at time t and position $x \in Q$
• evolution equation

$$\frac{\partial}{\partial t} m(t,x) - \nu \Delta m(t,x) + div(\alpha(t,x) m(t,x)) = 0,$$
$$m(0,x) = m_0(x).$$

• We consider the optimisation setting: $\min_\alpha J(\alpha)$
$J(\alpha) := \Psi(m(\cdot,T)) + \int_0^T \left\{ \Phi(m(t,\cdot)) + \int_Q L(x,\alpha) m(t,x) dx \right\} dt$
• Φ, Ψ can be linear, concave, ... Typical L : $L(x,\alpha) = \frac{\alpha^2}{2}$.
Rq: MFG equations are critical point equations for the functional J; relationship with individual level: $\nabla_m \Phi = V$, $\nabla_m \Psi = V_0$, $L = h$.

Numerics of MFG : literature overview

• (in)finite horizon: finite-difference discretization: approximation properties, existence and uniqueness, bounds on the solutions. "Mean Field Games: Numerical Methods" Y. Achdou & I. Capuzzo-Dolcetta
• Y. Achdou & I. Capuzzo-Dolcetta: Newton method for the coupled direct-adjoint critical point equations (finite horizon, cx case)
• O. Gueant: study of a prototypical case: solution, stability (09), quadratic Hamiltonian (11)
• solution of the MFG equations from an optimization point of view (A. Lachapelle, J. Salomon, G. Turinici, M3AS 2010)
• Lachapelle & Wolfram (2011) (congestion modelling)

Outline

Optimal control of a Fokker-Plank equation (G. Carlier & J. Salomon)

Evolution equation :

$$\partial_t \rho - \epsilon^2 \Delta \rho + div(v\rho) = 0 \qquad (29)$$

$$\rho(x, t = 0) = \rho_0(x) \qquad (30)$$

• goal: minimize w.r. to v the functional (for some given $V(\cdot)$) :

$$E(v) = \int \int \rho v^2 dxdt + \int \rho(x,1)V(x)$$

$$\begin{cases} i\frac{\partial}{\partial t}\Psi(x,t) = (H_0 - \epsilon(t)^k \mu(x))\Psi(x,t) \\ \Psi(x, t=0) = \Psi_0(x) \end{cases} \tag{31}$$

• vectorial case (rotation control, NMR):
$i\frac{\partial}{\partial t}\Psi(x,t) = [H_0 + (E_1(t)^2 + E_2(t)^2)\mu_1 + E_1(t)^2 \cdot E_2(t)\mu_2]\Psi(x,t)$.
$H_0 = -\Delta + V(x)$, unbounded domain
Evolution on the unit sphere: $\|\Psi(t)\|_{L^2} = 1, \ \forall t \geq 0$.
• evaluation of the quality of a control through a objective functional to minimize
$J(\epsilon) = -2\Re\langle\psi_{target}|\psi(\cdot, T)\rangle + \int_0^T \alpha(t)\epsilon^2(t)dt$
$J(\epsilon) = \|\psi_{target} - \psi(\cdot, T)\|_{L^2}^2 - 2 + \int_0^T \alpha(t)\epsilon^2(t)dt$
$J(\epsilon) = -\langle\Psi(T), O\Psi(T)\rangle + \int_0^T \alpha(t)\epsilon^2(t)dt$

General monotonic algorithms (J. Salomon, G.T.)

state $X \in H$, control $v \in E$, $H, E =$ Hilbert/ Banach spaces.

- $\partial_t X_v + A(t, v(t))X_v = B(t, v(t))$
- $\min_v J(v)$, $\quad J(v) := \int_0^T F\big(t, v(t), X_v(t)\big)\,dt + G\big(X_v(T)\big)$.
- F, G: C^1 + concavity with respect to X (not v!)

$$\forall X, X' \in H,\ \ G(X') - G(X) \leq \langle \nabla_X G(X), X' - X \rangle$$

$\forall t \in \mathbb{R}, \forall v \in E, \forall X, X' \in H$:

$$F(t, v, X') - F(t, v, X) \leq \langle \nabla_X F(t, v, X), X' - X \rangle.$$

Direct-adjoint equations and first lemma

$$\partial_t X_v + A(t, v(t))X_v = B(t, v(t))$$
$$X(0) = X_0$$

$$\partial_t Y_v - A^*\big(t, v(t)\big) Y_v + \nabla_X F\big(t, v(t), X_v(t)\big) = 0$$
$$Y_v(T) = \nabla_X G\big(X_v(T)\big).$$

Lemma

Suppose that A, B, F are differentiable everywhere in $v \in E$, then there exists $\Delta(\cdot, \cdot; t, X, Y) \in C^0(E^2, E)$ such that, for all $v, v' \in E$

$$J(v') - J(v) \leq \int_0^T \Delta(v', v; t, X_{v'}, Y_v) \cdot_E \Big(v' - v\Big) dt \tag{32}$$

Proof: cf. refs.

Well-posedness

$$J(v') - J(v) \leq \int_0^T \Delta(v', v; t, X_{v'}, Y_v) \cdot_E \left(v' - v\right) dt \qquad (33)$$

Remark: useful factorisation because can test at each step if J goes the right way; also can choose $v'(t^*) = v(t^*)$ if pb.
Remark: $\Delta(v', v; t, X, Y)$ has an explicit formula once the problem is given; also note the dependence on Y_v any not $Y_{v'}$.

Lemma

Under hypothesis on $A, B, F, G, \theta > 0$

$$\Delta(v', v; t, X, Y) = -\theta(v' - v) \qquad (34)$$

has an unique solution $v' = \mathcal{V}_\theta(t, v, X, Y) \in E$.

Well-posedness

Theorem (J. Salomon, G.T. Int J Contr, 84(3), 551, 2011)

Under hypothesis ...

- *the following eq. has a solution:*

$$\partial_t X_{v'}(t) + A(t, v')X_{v'}(t) = B(t, v') \tag{35}$$

$$v'(t) = \mathcal{V}_\theta(t, v(t), X_{v'}(t), Y_v(t)) \tag{36}$$

$$X_{v'}(0) = X_0 \tag{37}$$

- $\exists\ (\theta_k)_{k\in\mathcal{N}}$ *such that* $v^{k+1}(t) = \mathcal{V}_{\theta_k}(t, v^k(t), X_{v^{k+1}}(t), Y_{v^k}(t))$
- $J(v^{k+1}) - J(v^k) \leq -\theta_k \|v^{k+1} - v^k\|^2_{L^2([0,T])}$;
- *if* $v^{k+1}(t) = v^k(t)$: $\nabla_v J(v^k) = 0$.

Outline

- large economy: continuum of consumer agents
- time period: $[0, T]$
- any household owns exactly one house and cannot move to another one until T

The Model : the agents

• arbitrage between insulation and heating. A generic player (agent) has an insulation level $x \in [0, 1]$ ($x = 0$: no insulation, $x = 1$: maximal insulation)

• controlled process of the agent: $dX_t^x = \sigma dW_t + v_t dt + dN_t(X_t^x)$, $X_0^x = x$; v is the control parameter (insulation effort), the noise level σ is given.

• note that X_t is a diffusion process with reflexion, in the above equality, $dN_t(X_t)$ has the form $\chi_{\{0,1\}}(X_t)\vec{n}d\xi_t$ ($\xi =$ local time at the boundary $\{0, 1\} = \partial[0, 1]$ cf. Freidlin)

• initial density: $X_0 \sim m_0(dx)$

The Model : the costs

An agent of the economy solves a minimization problem composed of several terms:

• *Insulation acquisition cost:* $h(v) := \frac{v^2}{2}$

• *Insulation maintenance cost:* $g(t, x, m) := \frac{c_0 x}{c_1 + c_2 m(t,x)}$ increasing in x decreasing in m : economy of scale, positive externality. The agents should do the same choice, stay together. The higher is the number of players having chosen an insulation level, the lower are the related costs.

• *Heating cost:* $f(t, x) := p(t)(1 - 0,8x)$ where $p(t)$ is the unit heating cost (unit price of energy, say)

• Define the aggregate state cost:

$$\Phi(m) := \int_0^1 \left(p(t)(1 - 0,8x) + \frac{c_0 x}{c_1 + c_2 m(t,x)} \right) m(t,x) dx$$

and $V = \Phi'$.

• In the model, the agents have rational expectations, i.e they see m as given; we can write the individual agent's problem:

$$\begin{cases} \inf_{v \text{ adm}} \mathbb{E}\left[\int_0^T h(v(t, X_t^x)) + V[m](X_t^x) dt \right] \\ dX_t = v_t dt + \sigma dW_t + dN_t(X_t), X_0 = x \end{cases}$$

The model - The minimization problem and MFG (2)

• We already know that it is linked with the optimal control problem:

$$\begin{cases} \inf\limits_{v \text{ adm}} \int_0^T \int_0^1 h(v(t,x)) + \Phi(m_t)(t)dt \\ \partial_t m - \frac{\sigma^2}{2}\Delta m + \mathrm{div}(vm) = 0 \ , \ m|_{t=0} = m_0(.) \ , \\ m'(.,0) = m'(.,1) = 0 \end{cases}$$

• Finally, if $\nu := \frac{\sigma^2}{2}$, a Mean field equilibrium (Nash equilibrium with an infinite number of players) corresponds to a solution of the following system:

$$\begin{cases} \partial_t m - \nu\Delta m + \mathrm{div}(vm) = 0 \ , \ m|_{t=0} = m_0 \\ \nabla u = v \\ \partial_t u + \nu\Delta u + v \cdot \nabla u - \frac{u^2}{2} = \Phi'(m) \ , \ v|_{t=T} = 0 \end{cases} \tag{38}$$

The model - externality & scale effect

The MFG framework is interesting to describe a situation which lives between two economical ideas: positive externality and economy of scale

• positive externality: positive impact on any agent utility NOT INVOLVED in a choice of an insulation level by a player

• economy of scale: economies of scale are the cost advantages that a firm obtains due to expansion (unit costs decrease)

Criticism of the model:

- stylised from the "industrial" point of view
- not realistic (heating price, maintenance...)
- transition effect (continuous time, continuous space)
- atomised agent (her/his action has no influence on the global density, micro-macro approach)
- non-cooperative equilibrium with rational expectations

Numerical simulations

• Optimization method: Monotonic algorithm

$$\begin{cases} \partial_t m^{k+1} - \nu \Delta m^{k+1} + \mathrm{div}(v^{k+1} m^{k+1}) = 0 \ , \ m^{k+1}(x,0) = m_0 \\ v^{k+1} = \frac{(\theta - 1/2) v^k - \nabla u^k}{(\theta + 1/2)} \\ \partial_t u^{k+1} + \nu \Delta u^{k+1} + v^{k+1} \cdot \nabla u^{k+1} - \frac{(u^{k+1})^2}{2} = \Phi'(m^{k+1}), \ \ v^{k+1}(T) = 0 \end{cases} \tag{39}$$

• Discretization of the PDEs: Godunov scheme (to preserve the positivity of the density m)

• The costs:

heating: $f(t,x) = p(t)(1 - 0,8x)$

insulation: $g(t,x,m) = \frac{x}{0.1 + m(t,x)}$

• *1st example*: $p(t)$ constant / same choices

• *2d example*: $p(t)$ reaching a peak (non constant) / multiplicity of equilibria

Numerical results - First case

- the initial density of the householders is a gaussian centered in $\frac{1}{2}$
- the time period and the noise are respectively $T = 1$ and $\nu = 0.07$
- the energy price is constant $(p(t) \equiv 0,\ 3.2 \text{ and } 10)$

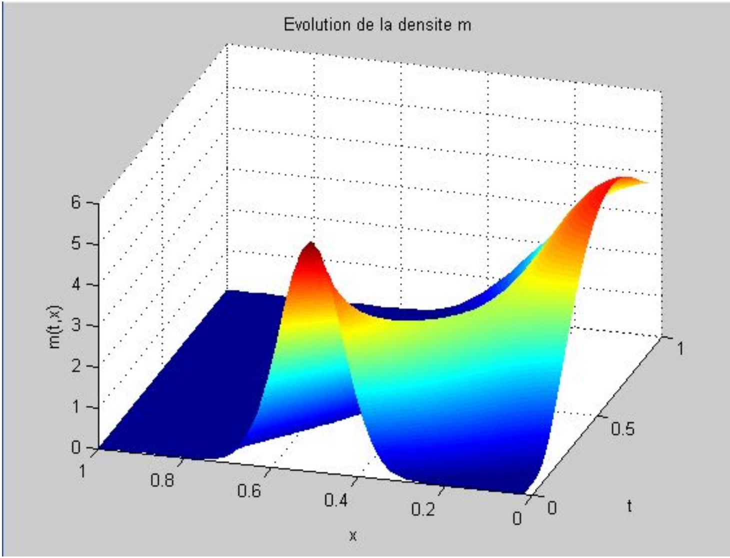

Figure: Numerical results : $p(t) \equiv 0$. Since the cost of energy is null all agents choose to heat their house, move to this choice together.

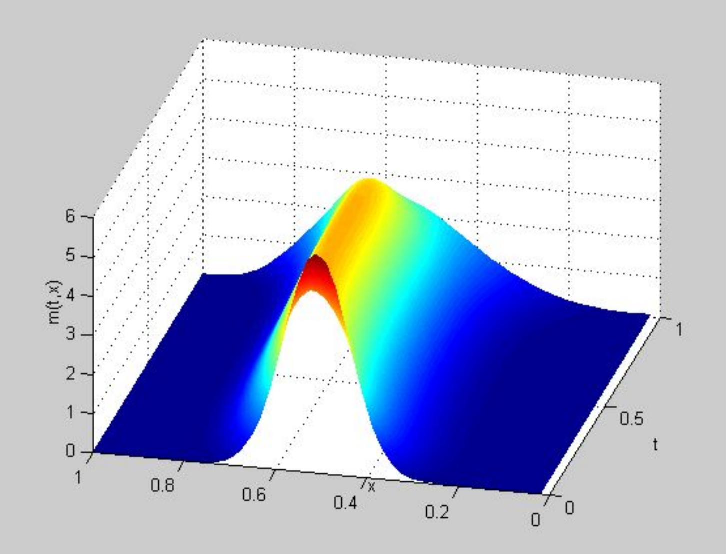

Figure: Numerical results : $p(t) \equiv 3.2$. Cost of energy is intermediary, agents keep their status.

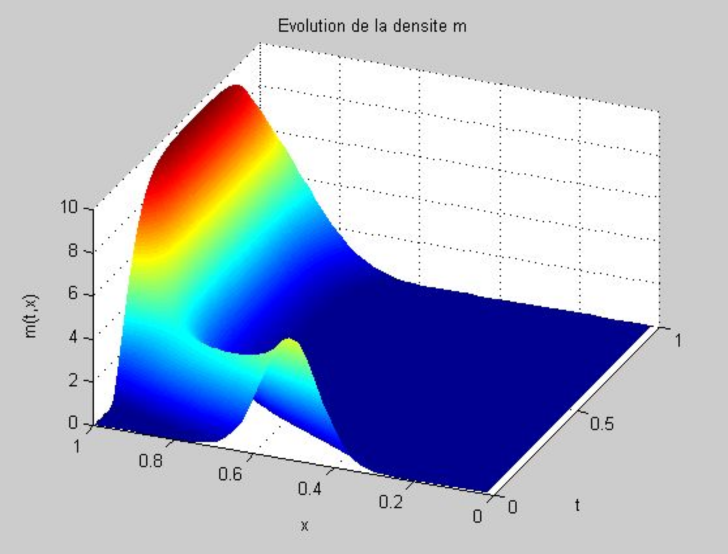

Figure: Numerical results : $p(t) \equiv 10$. Cost of energy is high, agents choose to better insulate, all have the same behavior.

Numerical results - Second case

- the initial density of the agents is an approximation of a Dirac in 0.1 (*i.e* agents are not equipped in insulation material)
- the energy price is not a constant parameter, we look at the following case: the price first reaches a peak and then decreases to its initial level.

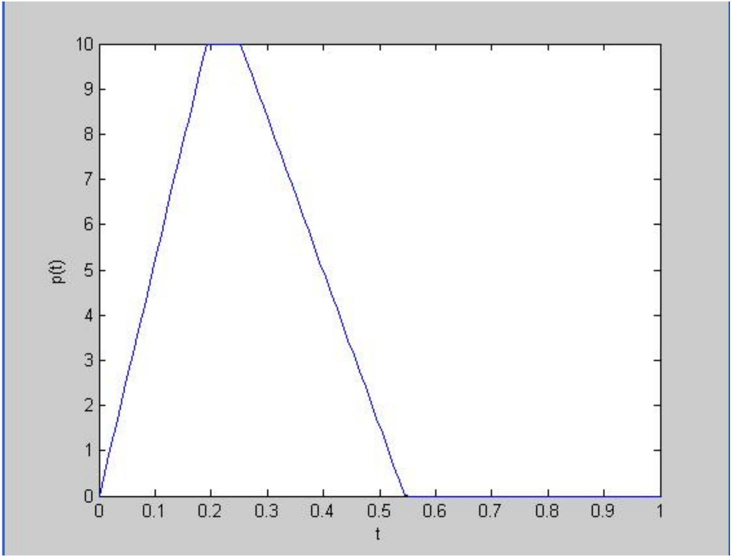

Figure: Numerical results - $p(t)$. Question: In such a case, can we find two Mean Field equilibria, the first related to the expectation of a higher insulation level, the second to the expectation of heating ?

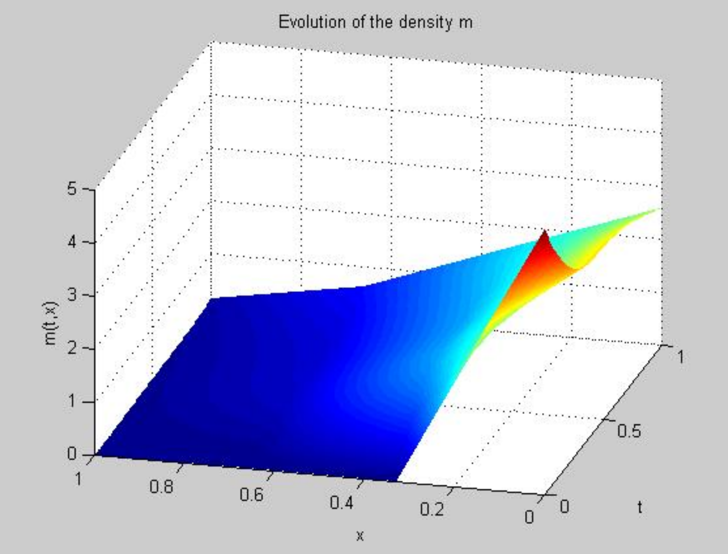

Figure: Numerical results - One of the two equilibria: the energy consumption equilibrium. Agents expect that everybody will keep a low insulation level so there are no gains in insulating.

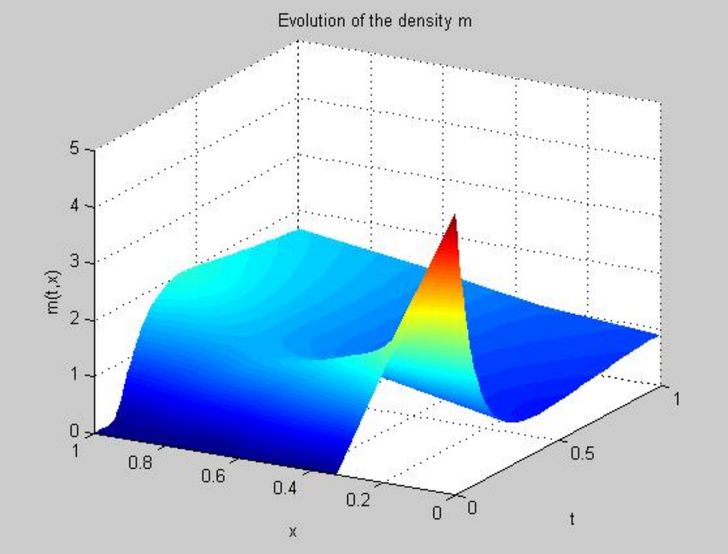

Figure: Numerical results - One of the two equilibria: the insulation equilibrium. Agents expect that everybody will better insulate, which makes insulating attractive.

Multiplicity of equilibria - Incentive policy

- we found an insulation-equilibrium and an energy consumption-equilibrium

- from the ecological point of view: the best is the insulation-equilibrium

- incentive public policies could steer towards the "best" equilibrium (from a certain point of view) when the solution is not unique.

Outline

Liquidity from heterogeneous beliefs and analysis costs (joint work with Min Shen, Université Paris Dauphine)

• Why do agents trade ? Here: heterogenous beliefs and expectations

• Liquidity : many definitions (bid/ask spread, rapidity to recover price after shock, max volume traded at same price etc). Here: trading volume.

• Several approaches: limit order book modeling and optimal order submission (Avellaneda et al. 2008) Heterogenous beliefs: asset pricing (working paper by Emilio Osambela), short sale constraints (Gallmeyer and Hollifield 2008) etc.,

• Specific investigation of this work: question on analysis time/cost

Heterogeneous beliefs and liquidity: the model

• One security of "true" value V.

• each agent has each own estimation (random variable) $V\tilde{A}$ of mean VA and variance $V^2\sigma^2(A)$. The precision on $\tilde{A}$ is $B(A) = 1/\sigma^2(A)$. The agent cannot change its A (which will become its index) but can change $\sigma^2(A)$. Precision can be improved paying $f(B)$ and/or waiting for the estimation to converge or new data to be revealed.

• Agents are distributed with density $\rho(A)$; the mean of this distribution is taken to be 1 (overall neutrality).

• Based on its estimations agent trade $\theta(A)$ units i.e. $V \cdot \theta(A) =$ size of the position of agent at A.

• Each agent has an utility function $U(\mathrm{m}ean(gain), \mathrm{v}ariance(gain))$ (equivalent: expected utility framework for normal variable). Linear situation $U(x, y) = x - \lambda y$. Note gain is function of θ,B (thus also mean and variance).

• Price Vp^A maximizes liquidity and equals offer and demand (this conditions are equivalent if monotonicity ... otherwise not). Note: p^A is not necessarily equal to 1 even if the mean $\mathbb{E}(A) = 1$.

Heterogenous beliefs and liquidity: theoretical results

Technical framework: Mean Field Games by Lasry & Lions; Nash equilibrium

$\mathrm{mean}(\theta, B) = V\theta(A - p^A) - f(B)$; $\mathrm{variance}(\theta, B) = \theta^2 V^2 / B$.

Theorem (M Shen, G.T. 2011)

Under assumptions on functions f and U the equilibrium exists. Offer and demand functions are monotone with respect to p^A.

Theorem (M Shen, G.T. 2011)

Under assumptions on functions f and U if ρ is symetric around p^1 then (liquidity is maximal for $p = p^1$ i.e.) $p^A = p^1$.

Theorem (M Shen, G.T. 2011)

For the linear case the equlibrium relative price is:

$$P^A = \frac{\int_0^\infty AB(A)\rho(A)dA}{\int_0^\infty B(A)\rho(A)dA}. \quad (40)$$

The relative accuracy $B(A)$ cost is

$$B = (f')^{-1}\left(\frac{(A - P^A)^2}{2\lambda}\right). \quad (41)$$

Heterogenous beliefs and liquidity: linear case $U = x - \lambda y$

The relative market price P^A is solution to the equation:

$$\frac{1}{2V\lambda}\int_0^\infty (A - P^A)(f')^{-1}\left(\frac{(A-P^A)^2}{2\lambda}\right)\rho(A)dA = 0 \qquad (42)$$

The trading volume TV_f is

$$TV_f = \frac{P^A}{2\lambda}\int_0^\infty (A - P^A)_+(f')^{-1}\left(\frac{(A-P^A)^2}{2\lambda}\right)\rho(A)dA \qquad (43)$$

Theorem (anti-monotony of trading volume)

Let f, g be two information cost functions such that $g'(b) \geq f'(b)$ for any $b \in \mathbb{R}_+$. Then the trading volume satisfies $TV_f > TV_g$.

Application: for constant total cost $\int f(B)\rho(A)$ which is the greatest volume : is volume brought by best paid analysts ?

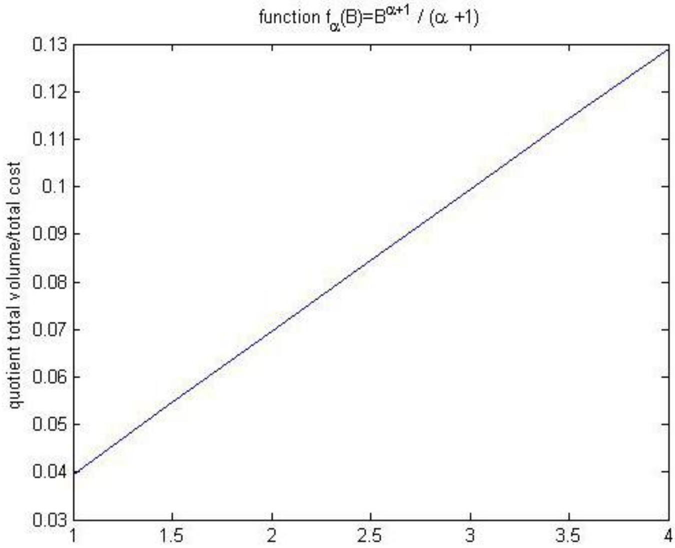

Figure: Quotient of the total volume over total cost for functions $f(B) = \frac{B^{\alpha+1}}{\alpha+1}$