# Fluctuations of the Free Energy in the high temperature Hopfield Model


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**Abstract**

We consider the Hopfield model of size $N$ and with $p \sim tN$ patterns, in the whole high temperature (paramagnetic) region. Our result is that the partition function has log-normal fluctuations. It is obtained by extending to the present model the method of the interpolating Brownian Motions used in [11] for the Sherrington-Kirkpatrick model. We view the load $t$ of the memory as a dynamical parameter, making the partition function a nice stochastic process. Then we write some semi-martingale decomposition for the logarithm of the partition function, and we prove that all the terms in this decomposition converge. In particular, the martingale term converges to a Gaussian martingale.



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## 1 Introduction and result

The Hopfield model took his name and its popularity within the theory of formal neural networks. J.J. Hopfield introduced it in 1982 ([14]) to describe and implement associative memories. In fact, the mathematical model was already defined, and studied in a simple form, by Pastur and Figotin ([24]) in an attempt to describe spin-glasses. In the context of neural networks, the memory is composed of a large number $N$ of neurones (called "spins" in the context of this paper), with only two possible states each. The state of the memory is described by a binary vector

$$\sigma = (\sigma_i)_{i=1}^N \in S_N := \{-1, 1\}^N ,$$


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and the task is to store a large number $p$ of "patterns" (or images), themselves also described by vectors $\xi^1, \ldots, \xi^p$ in $S_N$. Arguing that the natural candidates for minimizing the function $\sigma \mapsto -\sum_{k\leq p}(\sigma \cdot \xi^k)^2$, are the patterns themselves, Hopfield proposed to view the memory dynamics as a gradient algorithm with this function. Performing the descent, the memory, starting from a noisy version of one of the stored patterns, should remove the noise and retrieve the original pattern after a few steps. The picture is roughly correct provided that the load $t = p/N$ of the memory is not too large, and the program can be implemented. The efficiency of the memory can be tested, and traditionally, one would choose the patterns randomly. Besides its impact in neurocomputing and neuromodelling, the model became also very attractive from an information theory point of view, for questions relating to its capacity and the speed of retrieval of the memory ([21], [17], [23], [19], [20], [28], [3]). It became also extremely popular among physicists [22] and probabilists, especially when the minimization is relaxed by introducing a non-zero temperature. With the Sherrington-Kirkpatrick (SK) model, it is one of the two canonical mean-field, spin-glass models. It has the advantage on the other one to bring some information on its ground states for small $t$.

Mathematically, we consider a doubly indexed family $\xi_i^k (i, k \geq 1)$ of independent Bernoulli variables defined on some probability space,

$$\mathbb{P}(\xi_i^k = 1) = \mathbb{P}(\xi_i^k = -1) = 1/2. \tag{1}$$

We will use the symbols $\mathbb{P}, \mathbb{E}$ to denote the probability and the expectation with respect to the variables $\xi_i^k$'s. Let us fix $N$ for some time. With a slight abuse of notations, we will denote by $\xi^k = (\xi_1^k, \ldots, \xi_N^k)$ the $k$-th pattern, and for $\sigma \in S_N$,

$$\sigma \cdot \xi^k = \sum_{i=1}^N \sigma_i \xi_i^k \; . \tag{2}$$

The Hamiltonian of the Hopfield model is defined as

$$H_N(t, \sigma) = \frac{N}{2} \sum_{k=1}^{[tN]} (\frac{\sigma \cdot \xi^k}{N})^2 \tag{3}$$

with $H_N(t, \sigma) = 0$ when $0 \leq t < 1/N$. The partition function at the inverse temperature $\beta > 0$ is given by

$$Z_N^\beta(t) = E_\sigma \exp\{\beta H_N(t, \sigma)\}. \tag{4}$$

In contrast with $\mathbb{P}, \mathbb{E}$ introduced above, we will use the symbols $P_\sigma$ and $E_\sigma$ to denote the uniform probability measure and expectation on $S_N$. For instance, the above right-hand side is an average over $\sigma \in S_N$. The Gibbs measure is the probability measure $\sigma \in S_N \mapsto (2^N Z_N^\beta(t))^{-1} \exp\{\beta H_N(t, \sigma)\}$, which concentrates on the minima of $-H_N$ as $\beta$ is increased.

At a rigorous level, the behavior of the model starts now to be well understood in various domains of the parameter space [25], [8], [27], [4], [5], [6], [28], [7], [29], [10]. The model has three different phases: the high temperature phase, the retrieval (ferromagnetic) phase, and the spin-glass phase –in the order of increasing complexity–. The high temperature region is, by definition, the set of $(\beta, t) \in \mathbb{R}_+^2$ such that the mean of the quenched free energy and the annealed free energy are of the same order in the thermodynamic limit, i.e. $\mathbb{E} \ln Z_N^\beta(t) \sim \ln \mathbb{E} Z_N^\beta(t)$ as $N \to \infty$. (By Jensen inequality, we see that the inequality "$\leq$" holds between the two terms.) This region is equal ([2]) to the closure of the set

$$\beta(1+\sqrt{t}) < 1. \tag{5}$$

The boundary of this region already has some intricate features. Physicists predict that at the critical temperature, there is a spin-glass behavior in the case of non-zero $t$. For this reason, the critical case with $p/N \to 0$ has been recently studied in [12], [13], [30].

In this paper, we will stay in the interior of the high temperature region. For such $\beta, t$ as in (5), it is shown in [28] that the random variable $Z_N^\beta(t)/\mathbb{E} Z_N^\beta(t)$ remains bounded away from 0 and $+\infty$ in probability, and its concentration properties are investigated. We will prove that this variable converges in law to a log-normal limit. The result is the same as for the high temperature SK model ([1], [11]), and, like in the latter reference, it is part of a functional convergence result: We will view the normalized partition function

$$\widetilde{Z}_N^\beta(t) = \frac{Z_N^\beta(t)}{\mathbb{E} Z_N^\beta(t)} \tag{6}$$

as a (sequence in $N$ of) random process(es), depending on the load $t = p/N$.

With these notations we are ready to formulate our result:

**Theorem 1** *Let $\beta_0 \in [0,1)$ and $t_0 > 0$ be such that (5) holds. Then the process $\widetilde{Z}_N^{\beta_0}(t)$ converges in distribution as $N \to \infty$ on the Skorohod space $D([0, t_0[, \mathbf{R}^+)$ to the process*

$$\widetilde{Z}_\infty^{\beta_0}(t) = \exp\{M_\infty^{\beta_0}(t) - \frac{1}{2}\Gamma(\beta_0, t)\} \tag{7}$$

*where $M_\infty^{\beta_0}(t)$ is an independent increments Gaussian process on $[0, t_0[$ with continuous paths, mean zero and variance $\Gamma(\beta_0, t)$. The function $\Gamma(\beta, t)$ is defined as*

$$\Gamma(\beta, t) = -\frac{1}{2} \ln\Big(1 - t\Big(\frac{\beta}{1-\beta}\Big)^2\Big) - \frac{t}{2}\Big(\frac{\beta}{1-\beta}\Big)^2. \tag{8}$$

**Remarks** (i) We note that the parameter $t = p/N$, not only has a natural interpretation as the load of the memory –or, equivalently, the rank of the Hamiltonian–, but also has a natural role in coupling in a fine way the various Hopfield models for different values of $p$, in such a way that convergence holds at the process level.

(ii) We can view the high temperature region as the disordered region, and its complement as the ordered one (grouping spin-glass phase and ferromagnetic phase). We note that the limiting variance $\Gamma$ becomes infinite as $(\beta, t)$ approaches the boundary of the high temperature region from inside: the order is brought by the fluctuations.

(iii) The result of convergence at the level of variables, was published in the paper [26], but we must admit that we were unable to complete a number of details and steps in the proof. The strategy in [26] is to use the moment method together with a cluster expansion, following a method used by Aizenman, Lebowitz and Ruelle for the SK model; this leads here to rather intricate computations. In contrast, our method here *essentially* uses the *second moment* –due to the martingale structure and the simplicity of the limit–, and stochastic calculus –instead of expansion–. The Hopfield model, which Hamiltonian is quadratic in the disorder variables $\xi$, is notoriously more complicated to treat than the SK model, which is linear, and it is essential here to have a handy method requiring few computations. In particular, we have to face the explosion of moments deep inside the high temperature region, and use truncation. A striking illustration is that we cannot work out the proof using the bracket of the martingale, which corresponds to perform a minimal amount of integration, but this is already too much to cover the whole high temperature region. Instead, we use the sum of squares of our (pure jump) martingale, without performing any integration. ■

**Outline of the proof.** For each fixed $N$ and each $\sigma \in S_N$, the random process $t \mapsto H_N(t,\sigma)$ is a random walk with jumps at times $t = k/N, k \geq 1$. The corresponding exponential martingale is the normalized Boltzmann factor

$$e_N^\beta(t,\sigma) = \exp\{\beta H_N(t,\sigma) - [tN]\phi_N(\beta,\sigma)\}\ , \tag{9}$$

where

$$\phi_N(\beta,\sigma) = \log \mathbb{E} \exp\{\beta \frac{N}{2}(\frac{\sigma\cdot\xi}{N})^2\}. \tag{10}$$

The filtration we use is the one generated by the i.i.d. sequence of patterns,

$$\mathcal{F}_t = \sigma(\xi_i^k; i \geq 1, 1 \leq k \leq [tN])\ . \tag{11}$$

We will see in the sequel that the function $\phi_N(\beta,\sigma)$ depends only on $\beta$ but not on $\sigma$ (therefore we will denote it by $\phi_N(\beta,\sigma) = \phi_N(\beta)$), and that

$$\phi_N(\beta) = -\frac{1}{2}\ln(1-\beta) - \frac{1}{4N}\Big(\frac{\beta}{1-\beta}\Big)^2 + o(1/N), \qquad N \to \infty. \tag{12}$$

Recall (6), and note that

$$\widetilde{Z}_N^\beta(t) = E_\sigma e_N^\beta(t,\sigma) \tag{13}$$

is a $(\mathcal{F}_t)_{t\in[0,\infty)}$-martingale, for every fixed $N$. Let us denote by $\Delta$ the difference operator

$$\Delta f(t) = f(t+1/N) - f(t) \tag{14}$$

To the martingale $\widetilde{Z}_N^{\beta_0}(t)$ we associate its logarithm martingale $M_N^{\beta_0}(t)$ defined as

$$M_N^{\beta_0}(t) = \sum_{k=0}^{[Nt]-1} \Delta M_N^{\beta_0}\Big(\frac{k}{N}\Big), \qquad \Delta M_N^{\beta_0}(t) = \frac{\Delta \widetilde{Z}_N^{\beta_0}(t)}{\widetilde{Z}_N^{\beta_0}(t)}, \tag{15}$$

in particular $M_N^{\beta_0}(0) = 0$, and $\Delta M_N^{\beta_0}(\frac{k}{N}) > -1$. Since we keep $\beta_0 < 1$ fixed all through the paper, we will omit this index writing $M_N(t) = M_N^{\beta_0}(t)$.

The reason for the name "logarithm martingale" is the identity

$$\begin{aligned}\ln \widetilde{Z}_N(t) &= \ln \Pi_{k=0}^{[Nt]-1}\Big(1+\Delta M_N(\frac{k}{N})\Big)\\ &= \sum_{k=0}^{[Nt]-1} \ln\Big(1+\Delta M_N(\frac{k}{N})\Big)\\ &= M_N(t) - \frac{1}{2}\sum_{k=0}^{[Nt]-1}\Big[\Delta M_N(\frac{k}{N})\Big]^2 + \sum_{k=0}^{[Nt]-1} r\Big(\Delta M_N(\frac{k}{N})\Big) \qquad (16)\end{aligned}$$

where $r(u) = \ln(1+u) - u + u^2/2$ is continuous on $(-1,\infty)$ and $r(u) = O(u^3)$ as $u \to 0$.

In order to show that the sequence of martingales $M_N(\cdot)$ converges in law to the independent increment process $M_\infty(\cdot)$, it is enough to verify the following three conditions:

(i) For every $0 \le t < t_0$

$$\mathrm{I}^N = \sum_{k=0}^{[Nt]-1} \mathbb{E}\Big(\Big[\Delta M_N(\frac{k}{N})\Big]^2 \mathbf{1}_{|\Delta M_N(\frac{k}{N})|<1}\,\Big|\,\mathcal{F}_{\frac{k}{N}}\Big) \xrightarrow{\mathbb{P}} \Gamma(\beta_0, t).$$

(ii) For every $\alpha > 0$

$$\mathrm{II}^N = \sum_{k=0}^{[Nt_0]-1} \mathbb{P}\Big(|\Delta M_N(\frac{k}{N})| > \alpha\,\Big|\,\mathcal{F}_{\frac{k}{N}}\Big) \xrightarrow{\mathbb{P}} 0.$$

(iii) For every $0 \le t < t_0$

$$\mathrm{III}^N = \sum_{k=0}^{[Nt]-1} \mathbb{E}\Big(\Delta M_N(\frac{k}{N})\mathbf{1}_{|\Delta M_N(\frac{k}{N})|>1}\,\Big|\,\mathcal{F}_{\frac{k}{N}}\Big) \xrightarrow{\mathbb{P}} 0.$$

Indeed, the desired convergence is then a consequence of Theorem 2.21 in [15], page 365. From our condition (ii) we first see that we can transform our truncations with indicator functions here, into ones with continuous truncation functions as in [15]. The condition [sup-$\beta$] in [15] follows from (iii) together with the martingale property of $M_N$ and from (ii), while the condition [$\gamma$] follows from (i) and (ii), and the condition [$\delta$] follows from (ii).

But in addition, the second term in (16) converges in probability to the deterministic function $\Gamma(\beta_0, t)$. This follows from Theorem VIII-3.12b in [16], which states that, under our condition (iii), the convergence in law of $M_N$ to $M_\infty$ implies that of the quadratic variations, in probability. Note that, due to $\Delta M_N > -1$, we have

$$\mathrm{III}^N = \sum_{k=0}^{[Nt]-1} \mathbb{E}\Big(|\Delta M_N(\frac{k}{N})|\mathbf{1}_{|\Delta M_N(\frac{k}{N})|>1}|\mathcal{F}_{\frac{k}{N}}\Big)\,,$$

i.e., the property VIII-3.13 holds true.

Finally, our conditions clearly imply that the third term converges to 0 in probability, uniformly on compact time intervals in $[0, t_0)$.

Combining all this, we see that the full theorem simply follows from checking these three conditions. ■

**Remark** We stress that, as mentioned above and in contrast to [11], we do not use the bracket $<M_N>$ of the martingale in our computations. By definition, the bracket is defined by $<M_N>(0) = 0$ and

$$\Delta <M_N> (\frac{k}{N}) = \mathbb{E}\Big([\Delta M_N(\frac{k}{N})]^2 \Bigm| \mathcal{F}_{\frac{k}{N}}\Big) .$$

It turns out that, in a non-empty part of the region (5), this conditional expectation is already ruined by the explosion of moments: as $N \to \infty$, $< M_N > (t)$ does not converge to the limit $\Gamma(\beta_0, t)$ in the whole high temperature region. We need to work with the quadratic variation of the martingale $M_N$, instead of its bracket. ■

The whole paper is dedicated to checking these conditions. It is organized as follows. In Section 2, we formulate several lemmas as well as Proposition 1 which contains all the necessary asymptotic expansions. They are used in the following sections (3, 4, 5), where the conditions (i)–(iii) are verified. The proofs of the lemmas and Proposition 1 are postponed to an Appendix in Section 6.

**Warm-up calculation**: In the rest of this section we would like to sketch in a heuristic manner, the proof of the Theorem in the easier case of $\beta_0$ and $t_0$ small enough. In this particular case the proof is technically much lighter – no truncation is then needed, and then the argument are natural extensions to the discrete setting of those in [11]–, we will grab some intuition which will (hopefully) make the full proof more transparent. Let us first introduce some notations to simplify the formulas. We denote by $P_\sigma^{N,t}$ the Gibbs (probability) measure and $E_\sigma^{N,t}$ the corresponding expectation,

$$P_\sigma^{N,t}(\sigma) = \frac{e_N(t,\sigma)}{2^N \widetilde{Z}_N(t)}, \quad E_\sigma^{N,t}(f) = \sum_{\sigma \in S_N} f(\sigma) P_\sigma^{N,t}(\sigma) ,$$

and by $Re_N(t,\sigma)$ the relative increment

$$Re_N(t,\sigma) = \frac{\Delta e_N(t,\sigma)}{e_N(t,\sigma)} = e^{\frac{\beta_0}{2}(\frac{\sigma.\xi^{[tN]+1}}{\sqrt{N}})^2 - \phi_N(\beta_0)} - 1 .$$

With these notations,

$$\Delta M_N(\frac{k}{N}) = E_\sigma^{N,k/N} Re_N(\frac{k}{N}, \sigma) . \tag{17}$$

Clearly, among the above three conditions, the most "important" is Condition (i) providing the variance $\Gamma(\beta_0, t)$ of the limiting Gaussian process. Assume that the truncation $\mathbf{1}_{|\Delta M_N(\frac{k}{N})|<1}$ is omitted in this condition. Then we could compute the conditional expectations explicitly. Condition (i) reduces to the convergence to zero of

$$\sum_{k=0}^{[tN]-1} E^{N,k/N}_{\sigma^1,\sigma^2}\mathbb{E}\left[Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2)-\Delta\Gamma(\frac{k}{N})\right] \ , \tag{18}$$

for every $0\le t<t_0$. Here $\Delta\Gamma(t)=\Gamma(\beta_0,t+1/N)-\Gamma(\beta_0,t)$ and $E^{N,k/N}_{\sigma^1,\sigma^2}$ stands for the integration (in $\sigma^1,\sigma^2$) with respect the product of the Gibbs measure on $S_N^2$.

• We start with a much simpler question. We study in the previous sum, the expected value of the numerator of, say, the $k=[sN]$-th summand $(0<s<t<t_0)$,

$$\begin{aligned} S^N(s) &= \mathbb{E}\left[\widetilde{Z}_N(s)^2 E^{N,k/N}_{\sigma^1,\sigma^2}\mathbb{E}\left[Re_N(s,\sigma^1)Re_N(s,\sigma^2)-\Delta\Gamma(\frac{k}{N})\right]\right] \\ &= \sum_{\theta=0,\pm1/N,\dots,\pm1}\mathbb{E}\left[Re_N(s,\sigma^1)Re_N(s,\sigma^2)-\Delta\Gamma(\frac{[Ns]}{N})\right] \\ &\qquad\qquad \times\mathbb{E}[e_N(s,\sigma^1)e_N(s,\sigma^2)]\ P_{\sigma^1,\sigma^2}(\sigma^1\cdot\sigma^2=\theta N), \end{aligned}\tag{19}$$

where we denote by $P_{\sigma^1,\sigma^2}(\cdot)$ the uniform probability measure on the product space $S_N^2$. For any two spin configurations $\sigma^i,\sigma^j$ we will often use the short notation

$$\theta^{i,j}=\frac{\sigma^i\cdot\sigma^j}{N}. \tag{20}$$

and $\theta=\theta^{1,2}$ when no confusion is possible. Note that

$$P_{\sigma^1,\sigma^2}(\sigma^1\cdot\sigma^2=\theta N)=\binom{N}{(1+\theta)N/2}2^{-N}\sim\frac{1}{\sqrt{2\pi N(1+\theta)(1-\theta)}}e^{-NI(\theta)} \tag{21}$$

as $N\to\infty$ with

$$I(\theta)=\left[(1+\theta)\ln(1+\theta)+(1-\theta)\ln(1-\theta)\right]/2\sim-\theta^2/2 \tag{22}$$

as $\theta\to0$. We now loosely argue that for $\beta_0$ and $s$ small enough

$$NS^N(s)\to0. \tag{23}$$

For this purpose we only need the following asymptotic expansion

$$\ln\mathbb{E}e^{\beta\left(\frac{\sigma^1\cdot\xi}{\sqrt{N}}\right)^2+\beta\left(\frac{\sigma^2\cdot\xi}{\sqrt{N}}\right)^2-2\phi_N(\beta)}=-\frac{1}{2}\ln\left(1-\frac{\beta^2\theta^2}{(1-\beta)^2}\right)-\frac{1}{2N}\frac{\beta^2}{(1-\beta)^2}+\cdots \tag{24}$$

which holds true for all $\beta$ and $\theta=(\sigma^1\cdot\sigma^2)/N$ as soon as both $\beta(1+\theta)<1$ and $\beta(1-\theta)<1$ hold, see Proposition 1. (We denote by $\cdots$ negligible terms to be specified in this proposition.) Consequently (24) is valid for all $\beta<1/2$ whatever the value of $\theta$ is. It follows that, for $\beta_0<1/2$,

$$\begin{aligned} &\ln\mathbb{E}[Re_N(s,\sigma^1)Re_N(s,\sigma^2)-\Delta\Gamma(\frac{[Ns]}{N})] \\ &= -\frac{1}{2}\ln\left(1-\frac{\beta_0^2\theta^2}{(1-\beta_0)^2}\right)-\frac{\beta_0^2}{2N(1-\beta_0)^2(1-s\beta_0^2/(1-\beta_0)^2)}+\cdots, \end{aligned}\tag{25}$$

and

$$\mathbb{E}e_N(s,\sigma^1)e_N(s,\sigma^2) = \exp\Big\{[Ns]\Big( -\frac{1}{2}\ln\Big(1-\frac{\beta_0^2\theta^2}{(1-\beta_0)^2}\Big) - \frac{1}{2N}\frac{\beta_0^2}{(1-\beta_0)^2} + \cdots\Big)\Big\}. \tag{26}$$

Now let us split the sum (19) into two terms $S_1^N(s)$ and $S_2^N(s)$ where in the first term the sum is taken over $|\theta| < \varepsilon$ with $\varepsilon$ fixed small enough and the second – over $|\theta| > \varepsilon$. Then $S_2^N(s)$ converges to zero exponentially fast as $N \to \infty$ for all $s$ small enough. In fact, one can find $h > 0$ such that

$$\sup_{|\theta|>\varepsilon}\Big\{ s\Big( -\frac{1}{2}\ln\Big(1-\frac{\beta_0^2\theta^2}{(1-\beta_0)^2}\Big)\Big) - I(\theta)\Big\} \le -h.$$

Since the terms in (25) are bounded and the sum $S^N(t)$ contains at most $2N$ terms then by the expansions (21) and (26) we get

$$\limsup_{N\to\infty}\frac{1}{N}\ln|NS_2^N(s)| \le -h. \tag{27}$$

The analysis of $S_1^N(s)$ is more subtle. Since $\theta$ is small, let us replace the logarithms in (25) and (26) and $I(\theta)$ by their asymptotic expansions of order $\theta^2$ as $\theta \to 0$. This gives

$$NS_1^N(s) = \tag{28}$$
$$\frac{e^{-\frac{s\beta_0^2}{2(1-\beta_0)^2}}}{\sqrt{2\pi N}}\sum_{\substack{\theta=0,\pm 1/N,\ldots,\\ |\theta|<\varepsilon}}\frac{\beta_0^2}{2(1-\beta_0)^2}\Big(N\theta^2-\Big[1-\frac{s\beta_0^2}{(1-\beta_0)^2}\Big]^{-1}\Big)e^{-\Big(1-\frac{s\beta_0^2}{(1-\beta_0)^2}\Big)N\frac{\theta^2}{2}}+\cdots$$

Let us make the change of variables $\theta\sqrt{N} = u$. Then the sum (28) converges to the Gaussian integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\beta_0^2}{2(1-\beta_0)^2}\Big(u^2-\Big[1-\frac{s\beta_0^2}{(1-\beta_0)^2}\Big]^{-1}\Big)e^{-\Big(1-\frac{s\beta_0^2}{(1-\beta_0)^2}\Big)\frac{u^2}{2}}du = 0. \tag{29}$$

This "shows" (23).

• In fact, the proof of (23) gives us a serious hint for completing the proof of Condition (i). One could proceed in the following way. The absolute value of the summand in the sum (18) can be estimated using the Cauchy-Schwarz inequality as in Comets and Neveu [11]: writing $s = \frac{k}{N}$ for shorter notations,

$$\begin{aligned} &E_{\sigma^1}\Big(e_N(s,\sigma^1)^{1/2}\widetilde{Z}_N(s)^{-1/2}\times\Big|E_{\sigma^2}\mathbb{E}[Re_N(s,\sigma^1)Re_N(s,\sigma^2)-\Delta\Gamma(s)] \\ &\qquad\times e_N(s,\sigma^1)^{1/2}e_N(s,\sigma^2)\widetilde{Z}_N(s)^{-3/2}\Big|\Big) \\ &\le\Big[E_{\sigma_1,\sigma_2,\sigma_3}\Big(\mathbb{E}[Re_N(s,\sigma^1)Re_N(s,\sigma^2)-\Delta\Gamma(s)]\mathbb{E}[Re_N(s,\sigma^1)Re_N(s,\sigma^3)-\Delta\Gamma(s)] \\ &\qquad\times e_N(s,\sigma^1)e_N(s,\sigma^2)e_N(s,\sigma^3)\Big)\widetilde{Z}_N(s)^{-3}\Big]^{1/2}. \end{aligned}\tag{30}$$

First of all, let us get rid of the term $\widetilde{Z}_N(\frac{k}{N})^{-3}$ in (30). In fact, it is not difficult to prove using Talagrand's results that $\mathbb{P}(\inf_{0\le t<t_0} \widetilde{Z}_N(t) \le 1/\eta) \le C/\eta$ for some $C > 0$, all $N$ and all $\eta > 0$, see Lemma 5 for detail. Thus $\widetilde{Z}_N(\frac{k}{N})^{-3}$ can be bounded by $\eta^3$ with probability arbitrarily close to 1 by the choice of $\eta$. As we need the convergence in probability only, we can concentrate on the convergence of the positive term $E_{\sigma^1,\sigma^2,\sigma^3}(\cdot)$ of (30). One can verify proceeding in the same way as for $S^N(s)$ that, for $\beta$ and $s$ small enough, the expectation of this term converges to zero. Just substitute the expansion (25) and the analogue of (26) with three terms and finally get the analogue of (29) in $\mathbf{R}^3$.

We sketched the proof of the result for $\beta_0$ and $t_0$ small. Let us outline difficulties when these parameters are larger. First of all, the expansion (24) is true only when $\beta(1+\theta) < 1$ and $\beta(1-\theta) < 1$ hold, otherwise it would explode. (It behaves like $e^{\beta(g_1^2+g_2^2)}$ where $g_1, g_2$ are standard Gaussian with covariance $\theta$.) Thus $\mathbb{E}Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2)$ would be exponentially large for some $\theta$ when $\beta > 1/2$. The other difficulty to overcome, even if (24) is bounded, is that, for $t$ large, $I(\theta)$ from (22) would not necessarily dominate $\frac{1}{N}\ln \mathbb{E}e_N(s,\sigma^1)e_N(s,\sigma^2)$ as it was for $t$ small. Then the expectation of the Boltzmann factors would dominate $P_{\sigma^1,\sigma^2}(\theta)$ and the terms like $S_2^N(s)$ would be exponentially large.

To overcome all this obstacles we make truncations, one of them we borrow from Talagrand's paper [28]. We truncate separately the increments independent from $\mathcal{F}_{\frac{k}{N}}$ (this is done in Lemma 4) and the Boltzmann factors (this is the subject of Lemmas 2 and 3). The main idea is to keep the order of the first moment but to reduce essentially the second. The price to pay is that, it becomes more difficult to use martingale techniques in the presence of truncations, as experienced in [32] in the case of continuous martingale for the SK model with continuous or vector spins.

The combination of the martingale method together with truncation technics has been also applied in [9] in order to compute the fluctuations of the free energy for the $p$-spin SK model. Truncations allowed to extend the result for all $\beta$ smaller than a certain bound $\beta_p$, that has the same asymptotic behaviour as the inverse critical temperature i.e. $\beta_p \sim \sqrt{2\ln 2}$ when $p \to \infty$. Later in [18], by some more ingenious truncation procedure, this result has been extended to a bigger bound $\tilde{\beta}_p$ which is precisely Talagrand's bound for the critical temperature, see [31],(6.6). Finally, let us note that truncations are also useful in the extension of the results on fluctuations of overlaps to the whole of the high-temperature phase. In [18] fluctuations of the overlaps for the $p$-spin SK model have been found for all $\beta$ up to Talagrand's bound and those of the Hopfield model have been established in the whole high-temperature phase (5).

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# 2 Technical lemmas

In this section, we state a few necessary technical results, and we also choose the various parameters which enter the truncation events.

The following asymptotic expansions are essential tools to derive our lemmas.

They are generalizations of (12), which is valid for $\beta < 1$.

**Proposition 1** **a)** *The function $\phi_N(\beta,\sigma)$ does not depend on $\sigma$ and the expansion (12) holds. For every $\beta_i \in (0,1)$ and $\sigma^1, \sigma^2, \sigma^3 \in S_N$ let us denote by*

$$\psi_N(\beta_1,\beta_2,\sigma^1,\sigma^2) = \log \mathbb{E} \exp\left\{\frac{N}{2}\left[\beta_1(\frac{\sigma^1\cdot\xi}{N})^2 + \beta_2(\frac{\sigma^2\cdot\xi}{N})^2\right]\right\} \tag{31}$$

*and*

$$\Lambda_N(\beta_1,\beta_2,\beta_3,\sigma^1,\sigma^2,\sigma^3) = \log \mathbb{E} \exp\left\{\frac{N}{2}\left[\beta_1(\frac{\sigma^1\cdot\xi}{N})^2 + \beta_2(\frac{\sigma^2\cdot\xi}{N})^2 + \beta_3(\frac{\sigma^3\cdot\xi}{N})^2\right]\right\}. \tag{32}$$

*These functions depend on the $\sigma^i$'s only through the $\theta^{i,j}$'s.*

**b)** *Furthermore, if $\beta_1, \beta_2$ and $\theta^{1,2}$ satisfy*

$$(1-\beta_1)(1-\beta_2) - (\theta^{1,2})^2\beta_1\beta_2 > 0 \tag{33}$$

*then*

$$\begin{aligned}\psi_N(\beta_1,\beta_2,\theta^{1,2}) &= \psi(\beta_1,\beta_2,\theta^{1,2}) - \frac{1}{4N}\left[\frac{\beta_1^2}{(1-\beta_1)^2} + \frac{\beta_2^2}{(1-\beta_2)^2}\right] \\ &\quad -\frac{1}{2N}\frac{\beta_1\beta_2}{(1-\beta_1)(1-\beta_2)} + O\Big(\frac{(\theta^{1,2})^2}{N}\Big) + o\Big(\frac{1}{N}\Big)\end{aligned} \tag{34}$$

*where*

$$\psi(\beta_1,\beta_2,\theta^{1,2}) = -\frac{1}{2}\log\Big((1-\beta_1)(1-\beta_2) - (\theta^{1,2})^2\beta_1\beta_2\Big) \tag{35}$$

*and $o(1/N)$ is uniform in $\theta^{1,2}$ belonging to compact subsets of (33). If the $\beta_i$'s and the $\theta^{i,j}$'s, $i,j = 1,2,3$ satisfy both (33) and*

$$\begin{aligned}&(1-\beta_1)(1-\beta_2)(1-\beta_3) - (1-\beta_1)\beta_2\beta_3(\theta^{2,3})^2 - (1-\beta_2)\beta_1\beta_3(\theta^{1,3})^2 \\ &\qquad -(1-\beta_3)\beta_1\beta_2(\theta^{1,2})^2 - 2\beta_1\beta_2\beta_3\theta^{1,2}\theta^{1,3}\theta^{2,3} > 0\,,\end{aligned} \tag{36}$$

*then the expansion*

$$\begin{aligned}\Lambda_N(\beta_1,\beta_2,\beta_3,\theta^{1,2},\theta^{1,3},\theta^{2,3}) &= \Lambda(\beta_1,\beta_2,\beta_3,\theta^{1,2},\theta^{1,3},\theta^{2,3}) \\ &\quad -\frac{1}{4N}\left[\frac{\beta_1^2}{(1-\beta_1)^2} + \frac{\beta_2^2}{(1-\beta_2)^2} + \frac{\beta_3^2}{(1-\beta_3)^2}\right] \\ &\quad -\frac{1}{2N}\left[\frac{\beta_1\beta_2}{(1-\beta_1)(1-\beta_2)} + \frac{\beta_1\beta_3}{(1-\beta_1)(1-\beta_3)} + \frac{\beta_2\beta_3}{(1-\beta_2)(1-\beta_3)}\right] \\ &\quad + \sum_{i,j=1,2,3} O\Big(\frac{(\theta^{i,j})^2}{N}\Big) + o\Big(\frac{1}{N}\Big)\end{aligned} \tag{37}$$

*holds uniformly on compacts, where*

$$\Lambda(\beta_1,\beta_2,\beta_3,\theta^{1,2},\theta^{1,3},\theta^{2,3}) = -\frac{1}{2}\log\Big((1-\beta_1)(1-\beta_2)(1-\beta_3) - (1-\beta_1)\beta_2\beta_3(\theta^{2,3})^2$$
$$-(1-\beta_2)\beta_1\beta_3(\theta^{1,3})^2 - (1-\beta_3)\beta_1\beta_2(\theta^{1,2})^2 - 2\beta_1\beta_2\beta_3\theta^{1,2}\theta^{1,3}\theta^{2,3}\Big)\ . \quad (38)$$

**c)** *Finally, for every $\beta \in (0,1)$ and every $\sigma^i \in S_N, i=1,\dots,4$, let*

$$\kappa_N(\beta,\sigma^1,\sigma^2,\sigma^3,\sigma^4) = \log\mathbb{E}\exp\left\{\frac{N\beta}{2}\sum_{i=1}^{4}(\frac{\sigma^i\cdot\xi}{N})^2\right\}. \quad (39)$$

*This function depends on the $\sigma^i$'s only through the $\theta^{i,j}$'s, $i,j=1,\dots,4$. Furthermore there exists an $\tau=\tau(\beta)>0$ small enough such that if $\max_{i,j=1,\dots,4}|\theta^{i,j}|<\tau$ then (33) and (36) with $\beta_i=\beta$ are satisfied and*

$$\kappa_N(\beta,\sigma^1,\sigma^2,\sigma^3,\sigma^4) = \kappa(\beta,\sigma^1,\sigma^2,\sigma^3,\sigma^4) - \frac{4}{N}\frac{\beta^2}{(1-\beta)^2} + \sum_{i,j=1}^{4} O\Big(\frac{(\theta^{i,j})^2}{N}\Big) + o\Big(\frac{1}{N}\Big) \quad (40)$$

*with $o(1/N)$ uniform in the $\theta^{i,j}$'s and*

$$\begin{aligned}\kappa(\beta,\Theta) \ &= -\tfrac{1}{2}\log\ \Big[(1-\beta)^4 - \beta^2(1-\beta)^2||\Theta||_2^2 \\ &-2\beta^3(1-\beta)\Big(\theta^{1,3}\theta^{1,2}\theta^{2,3}+\theta^{1,4}\theta^{1,2}\theta^{2,4}+\theta^{1,4}\theta^{1,3}\theta^{3,4}+\theta^{2,4}\theta^{2,3}\theta^{3,4}\Big) \\ &+\beta^4\Big((\theta^{1,2})^2(\theta^{3,4})^2+(\theta^{1,3})^2(\theta^{2,4})^2+(\theta^{1,4})^2(\theta^{2,3})^2\Big) \\ &-2\beta^4(\theta^{1,2}\theta^{1,4}\theta^{2,3}\theta^{3,4}+\theta^{1,2}\theta^{1,3}\theta^{2,4}\theta^{3,4}+\theta^{1,3}\theta^{2,4}\theta^{2,3}\theta^{1,4})\Big]\end{aligned} \quad (41)$$

*where $\Theta=(\theta^{i,j})_{i,j=1,\dots,4}$. We can (and we will) choose $\tau(\beta)$ small enough and some $p=p(\beta)>1$ such that $\max_{i,j=1,\dots,4}|\theta^{i,j}|<\tau$ implies*

$$\sup_N\mathbb{E}\exp\left\{\frac{Np\beta}{2}\sum_{i=1}^{4}(\frac{\sigma^i\cdot\xi}{N})^2\right\}<\infty. \quad (42)$$

Remark that if $\beta_1=\beta_2=\beta$ then (33) is equivalent to $\beta(1+\theta^{1,2})<1$ and $\beta(1-\theta^{1,2})<1$ simultaneously hold. With these expansions in hand we can prove

**Lemma 1** *There exists an $\varepsilon_0$, with $0<\varepsilon_0<\tau(\beta_0)$, such that for any three $\varepsilon_{i,j}$ $i,j=1,2,3$ satisfying $0<\varepsilon_{i,j}<\varepsilon_0<\tau(\beta_0)$ we have*

$$\lim_{N\to\infty}\sup_{0\le t\le t_0} N^2E_{\sigma^1,\sigma^2,\sigma^3}\Big[(\prod_{i,j=1,2,3}\mathbf{1}_{|\theta^{i,j}|<\varepsilon_{i,j}})\mathbb{E}\Big(Re_N(t,\sigma^1)Re_N(t,\sigma^2)-\Delta\Gamma(t)\Big)$$
$$\mathbb{E}\Big(Re_N(t,\sigma^1)Re_N(t,\sigma^3)-\Delta\Gamma(t)\Big)\mathbb{E}\Big(\textstyle\prod_{i=1}^3 e_N(t,\sigma^i)\Big)\Big]=0, \quad (43)$$

*and for every* $0 < \varepsilon < \varepsilon_0$ *there exists a* $C < \infty$ *such that*

$$\lim_{N\to\infty} \sup_{0\le t\le t_0} N^2 E_{\sigma^1,\sigma^2,\sigma^3,\sigma^4} \mathbf{1}_{\max|\theta^{i,j}|<\varepsilon} \Big|\mathbb{E}\prod_{i=1}^{4} Re_N(t,\sigma^i)\Big| \mathbb{E}\Big[\prod_{i=1}^{4} e_N(t,\sigma^i)\Big] < C. \quad (44)$$

The convergence (43) is the key estimate to the proof of our result. It falls short of completing it if $t_0$ were small enough. As we already pointed out, in order to extend our result to the domain (5), we first of all need to truncate the Boltzmann factors $e_N(k/N,\sigma) \in \mathcal{F}_{\frac{k}{N}}$. For that purpose let us make
*Step 1.* Let us fix $E$ small enough to ensure that

$$\frac{\beta_0^2 t_0}{(1-\beta_0)(1-\beta_0-E)} < 1, \qquad 0 < \frac{E}{1-\beta_0-E} < 1. \quad (45)$$

Note that this choice of $E > 0$ is possible only under condition (5) of Theorem 1. Let us now introduce the events truncating the Hamiltonian:

$$B_{N,t,\sigma,E} = \left\{ H_N(t,\sigma) < \frac{[t_0 N]}{2(1-\beta_0-E)} \right\} \quad (46)$$

**Lemma 2** *Let $E$ be fixed according to (45). For every $\varepsilon > 0$ there exists a constant $h_1 = h_1(\varepsilon, E) > 0$ such that for all $N \ge N_1(\varepsilon, E)$*

$$\sup_{0\le t\le t_0} \mathbb{E} E_\sigma e_N(t,\sigma) \mathbf{1}_{\overline{B_{N,t,\sigma,E}}} \le e^{-h_1 N} \quad (47)$$

*and*

$$\sup_{0\le t\le t_0} \mathbb{E} E_{\sigma^1,\sigma^2} e_N(t,\sigma^1) e_N(t,\sigma^2) \mathbf{1}_{B_{N,t,\sigma^1,E}\cap B_{N,t,\sigma^2,E}} \mathbf{1}_{|\theta^{1,2}|>\varepsilon} \le e^{-h_1 N}. \quad (48)$$

Define now the event

$$\begin{aligned} A_{N,t,h,\varepsilon,E} &= \left\{ E_\sigma e_N(t,\sigma) \mathbf{1}_{\overline{B_{N,t,\sigma,E}}} < e^{-\frac{hN}{2}} \right\} \\ &\bigcap \left\{ E_{\sigma^1,\sigma^2} e_N(t,\sigma^1) e_N(t,\sigma^2) \mathbf{1}_{B_{N,t,\sigma^1,E}\cap B_{N,t,\sigma^2,E}} \mathbf{1}_{|\theta^{1,2}|>\varepsilon} < e^{-\frac{hN}{2}} \right\}, \end{aligned}$$

the lemma implies immediately, via Chebyshev's inequality, the following corollary

**Corollary 1** *Let $E$ be fixed by (45). For all $\varepsilon > 0$ and all $N \ge N_1(\varepsilon, E)$*

$$\mathbb{P}\Big( \bigcup_{k=0}^{[Nt_0]-1} \overline{A_{N,k/N,h_1,\varepsilon,E}} \Big) \le 2[Nt_0]\exp(-h_1 N/2). \quad (49)$$

*where $h_1 = h_1(\varepsilon, E)$ and $N \ge N_1(\varepsilon, E)$ are from Lemma 2.*

Moreover, we will need the following Lemma 3

**Lemma 3** *Let $E$ be fixed again according to (45). For every $\gamma > 0$ small enough there exists a constant $h_2 = h_2(\gamma, E) > 0$ and $\varepsilon_1 = \varepsilon_1(\gamma) > 0$ such that for every $\varepsilon < \varepsilon_1$ and all $N \geq N_2(\varepsilon, \gamma, E)$*

$$\mathbb{E}E_{\sigma^1,\sigma^2,\sigma^3}\left[\left(\prod_{i=1}^{3} e_N(t,\sigma^i)\right)\mathbf{1}_{|\theta^{1,2}|<\varepsilon,|\theta^{1,3}|<\varepsilon,|\theta^{2,3}|<\gamma}\mathbf{1}_{\overline{B_{N,t,\sigma^1,E}}}\right] \leq e^{-h_2 N} \tag{50}$$

*and*

$$\mathbb{E}E_{\sigma^1,\sigma^2,\sigma^3}\left[\left(\prod_{i=1}^{3} e_N(t,\sigma^i)\right)\mathbf{1}_{|\theta^{1,2}|<\varepsilon,|\theta^{1,3}|<\varepsilon,|\theta^{2,3}|>\gamma}\prod_{i=1}^{3}\mathbf{1}_{B_{N,t,\sigma^i,E}}\right] \leq e^{-h_2 N} \tag{51}$$

Now we are ready to make
*Step 2.* Let us fix $\gamma < \varepsilon_0$, where $\varepsilon_0$ is borrowed from Lemma 1, and find $\varepsilon_1(\gamma)$ according to Lemma 3.
*Step 3.* Let us fix $\varepsilon < \min(\varepsilon_1(\gamma), \varepsilon_0)$. In particular, $\varepsilon < \tau(\beta_0)$ with $\tau$ introduced above (39).

All exponential estimates of Lemmas 1–3 and Corollary 1 hold true with parameters $E$, $\varepsilon$ and $\gamma$ fixed in steps 1–3 and appropriate $h_1(\varepsilon, E)$, $h_2(\gamma, E)$.

To extend our result to the whole domain (5), we also need to truncate separately the relative increment of the Boltzmann weight $\exp\left\{\beta_0\left(\frac{\sigma\cdot\xi^{k+1}}{\sqrt{N}}\right)^2 - \phi_N(\beta_0)\right\} - 1$, that we will call the "independent from $\mathcal{F}_{\frac{k}{N}}$ increment". To do so, we consider the events

$$C_{N,t,\sigma,\delta} = \left\{\frac{\beta_0}{2}\left(\frac{\sigma\cdot\xi^{[tN]+1}}{\sqrt{N}}\right)^2 < N\delta\right\}. \tag{52}$$

We have the following lemma.

**Lemma 4** *Let $C_{N,\sigma,\delta} = \left\{\frac{\beta_0}{2}\left(\frac{\sigma\cdot\xi}{\sqrt{N}}\right)^2 < N\delta\right\}$. For every $\varepsilon < \varepsilon_0$ and every $\delta > 0$ there exists a constant $h_3 = h_3(\delta, \varepsilon) > 0$ such that for every $\sigma^1, \sigma^2, \sigma^3, \sigma^4 \in S_N$ with $\max_{i,j=1,\ldots,4}|\theta^{i,j}| < \varepsilon$ we have*

$$\mathbb{E}\left[\left(\prod_{i=1}^{n} e^{\frac{\beta_0}{2}(\frac{\sigma^i\cdot\xi}{\sqrt{N}})^2-\phi_N(\beta_0)}\right)\mathbf{1}_{\overline{C_{N,\sigma^1,\delta}}}\right] \leq e^{-h_3 N}\ , \tag{53}$$

*for $n = 0, 1, \ldots 4$. (Convention: the product is equal to 1 for $n = 0$.)*

*Step 4.* We choose $\delta > 0$ such that

$$0 < \delta < \min(h_1(\varepsilon, E), h_2(\gamma, E))/8. \tag{54}$$

Then we also have the estimates of Lemma 4 for these fixed $\delta$ and $\varepsilon$ with $h_3 = h_3(\delta, \varepsilon)$.

Finally, in the proof of convergences (i)–(iii) we will often need to bound from below the denominator of $M_N(t)$ –that is $\widetilde{Z}_N(t)$– to get rid of the so-called "small denominator problem". Introduce

$$D_{N,\eta} = \left\{\inf_{0\leq t\leq t_0}\widetilde{Z}_N(t) > \frac{1}{\eta}\right\}. \tag{55}$$

**Lemma 5** *There exists a constant $C$ such that for every $N$ and every $\eta > 0$*

$$\mathbb{P}\Big(\overline{D_{N,\eta}}\Big) \le \frac{C}{\eta}.$$

# 3 Proof of Condition (ii)

The key point in the proof of this condition is the fact that the scale of fluctuations of the fourth moment of $\Delta Z_N$ is of order $o(1/N)$ uniformly for $0 \le t \le t_0$ by Lemma 1, see (44). Thus one may guess, that $\mathbb{II}^N$ should be of order $o(1)$. But (44) is valid only for small enough overlaps. Thus, to take advantage of it, we should make several truncations to cut exponentially growing terms for $\theta$ large.

To start with, let us bound $\mathbb{II}^N$ by the sum of two terms $\mathbb{II}^N \le \mathbb{II}_1^N + \mathbb{II}_2^N$ where the "independent from $\mathcal{F}_{\frac{k}{N}}$ increments" are truncated: recalling the notations for the Gibbs measure, its expectation, (17) and (52),

$$\begin{aligned}
\mathbb{II}_1^N &= \sum_{k=0}^{[tN]-1} \mathbb{P}\Big(\Big|E_\sigma^{N,k/N} Re_N(\frac{k}{N},\sigma)\mathbf{1}_{\overline{C_{N,\frac{k}{N},\sigma,\delta}}}\Big| > \frac{\alpha}{2} \;\Big|\; \mathcal{F}_{\frac{k}{N}}\Big),\\
\mathbb{II}_2^N &= \sum_{k=0}^{[tN]-1} \mathbb{P}\Big(\Big|E_\sigma^{N,k/N} Re_N(\frac{k}{N},\sigma)\mathbf{1}_{C_{N,\frac{k}{N},\sigma,\delta}}\Big| > \frac{\alpha}{2} \;\Big|\; \mathcal{F}_{\frac{k}{N}}\Big).
\end{aligned}$$

Then by Chebyshev's inequality

$$0 \le \mathbb{II}_1^N \le \frac{2}{\alpha}\sum_{k=0}^{[tN]-1} \mathbb{E}\Big(E_\sigma^{N,k/N}|Re_N(\frac{k}{N},\sigma)|\mathbf{1}_{\overline{C_{N,\frac{k}{N},\sigma,\delta}}} \;\Big|\; \mathcal{F}_{\frac{k}{N}}\Big). \tag{56}$$

A fact to keep in mind all along the different proofs is that the disorder variables involved in the definition of the $Re_N(\frac{k}{N},\sigma)$ and $\overline{C_{N,\frac{k}{N},\sigma,\delta}}$ are independent from those defining $\mathcal{F}_{\frac{k}{N}}$. Since

$$\Big|Re_N(\frac{k}{N},\sigma)\Big| \le e^{\frac{\beta_0}{2}\left(\frac{\sigma\cdot\xi^{k+1}}{\sqrt{N}}\right)^2 - \phi_N(\beta_0)} + 1, \tag{57}$$

according to Lemma 4 we obtain an exponentially small bound for $\mathbb{II}_1^N$:

$$\begin{aligned}
|\mathbb{II}_1^N| &\le \frac{2}{\alpha}\sum_{k=0}^{[tN]-1} E_\sigma^{N,k/N}\Big[\mathbb{E}\Big(e^{\frac{\beta_0}{2}\left(\frac{\sigma\cdot\xi^{k+1}}{\sqrt{N}}\right)^2 - \phi_N(\beta_0)}\mathbf{1}_{\overline{C_{N,\frac{k}{N},\sigma,\delta}}}\Big) + \mathbb{E}\mathbf{1}_{\overline{C_{N,\frac{k}{N},\sigma,\delta}}}\Big]\\
&\le \frac{4}{\alpha}[tN]e^{-h_2 N} \to 0.
\end{aligned} \tag{58}$$

Next, we bound $\mathbb{II}_2^N$ by the sum of two terms $\mathbb{II}_2^N \le \mathbb{II}_{2,1}^N + \mathbb{II}_{2,2}^N$ where the factors $e_N(\frac{k}{N},\sigma) \in \mathcal{F}_{\frac{k}{N}}$ are truncated:

$$\mathbb{I}_{2,1}^N = \sum_{k=0}^{[tN]-1} \mathbb{P}\Big(\Big|E_\sigma^{N,k/N} Re_N(\frac{k}{N},\sigma)\mathbf{1}_{C_{N,\frac{k}{N},\sigma,\delta}}\mathbf{1}_{\overline{B_{N,\frac{k}{N},\sigma,E}}}\Big| > \frac{\alpha}{4}\ \Big|\ \mathcal{F}_{\frac{k}{N}}\Big), \tag{59}$$

and

$$\mathbb{I}_{2,2}^N = \sum_{k=0}^{[tN]-1} \mathbb{P}\Big(\Big|E_\sigma^{N,k/N} Re_N(\frac{k}{N},\sigma)\mathbf{1}_{C_{N,\frac{k}{N},\sigma,\delta}}\mathbf{1}_{B_{N,\frac{k}{N},\sigma,E}}\Big| > \frac{\alpha}{4}\ \Big|\ \mathcal{F}_{\frac{k}{N}}\Big). \tag{60}$$

Then

$$\begin{aligned} |\mathbb{I}_{2,1}^N| &\le \frac{4}{\alpha}\sum_{k=0}^{[tN]-1} \mathbb{E}\Big(E_\sigma^{N,k/N}|Re_N(\frac{k}{N},\sigma)|\mathbf{1}_{C_{N,\frac{k}{N},\sigma,\delta}}\mathbf{1}_{\overline{B_{N,\frac{k}{N},\sigma,E}}}\ \Big|\ \mathcal{F}_{\frac{k}{N}}\Big) \\ &\le \frac{4}{\alpha}\sum_{k=0}^{[tN]-1} [e^{N\delta}+1]E_\sigma^{N,k/N}(\mathbf{1}_{\overline{B_{N,\frac{k}{N},\sigma,E}}}) \end{aligned} \tag{61}$$

where in the second line we used the definition of $C_{N,\frac{k}{N},\sigma,\delta}$, together with (57). Let us remind that we only need to prove the convergence in probability. Then it suffices to show the convergence to zero in $L^1(\mathbb{P})$ of the right-hand side of (61) multiplied by $\mathbf{1}_{\mathrm{D}_{N,\eta}} \in \mathcal{F}_{\frac{k}{N}}$ for *all* fixed $\eta > 0$. Indeed, $\mathbb{P}(\overline{\mathrm{D}_{N,\eta}})$ can be made arbitrarily close to zero for all $N$ by the choice of $\eta$ according to Lemma 5. But by Lemma 2

$$\begin{aligned} &\mathbb{E}\Big[\frac{4}{\alpha}\sum_{k=0}^{[tN]-1}[e^{N\delta}+1]E_\sigma^{N,k/N}(\mathbf{1}_{\overline{B_{N,\frac{k}{N},\sigma,E}}})\mathbf{1}_{\mathrm{D}_{N,\eta}}\Big] \\ &\quad\le \frac{4\eta}{\alpha}\sum_{k=0}^{[tN]-1}\Big[e^{N\delta}+1\Big]\,\mathbb{E}\Big[E_\sigma e_N(\frac{k}{N},\sigma)\mathbf{1}_{\overline{B_{N,\frac{k}{N},\sigma,E}}}\Big] \le \frac{4\eta}{\alpha}\sum_{k=0}^{[tN]-1}\Big[e^{N\delta}+1\Big]\,e^{-h_1 N} \end{aligned}$$

and the latter quantity converges to 0 exponentially fast because of the choice of $\delta$ (54) we made. Hence, $\mathbb{I}_{2,1}^N \to 0$ in probability.

Let us turn to $\mathbb{I}_{2,2}^N$. Once again, by Chebyshev's inequality with the fourth moment, the problem is reduced to the convergence in probability of

$$\sum_{k=0}^{[tN]-1}\mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\prod_{i=1}^4[Re_N(\frac{k}{N},\sigma^i)\mathbf{1}_{C_{N,\frac{k}{N},\sigma^i,E}}\mathbf{1}_{B_{N,\frac{k}{N},\sigma^i,E}}]\ \Big|\ \mathcal{F}_{\frac{k}{N}}\Big) = \mathbb{I}_{2,2,>\varepsilon}^N + \mathbb{I}_{2,2,<\varepsilon}^N \tag{62}$$

where $\mathbb{I}_{2,2,<\varepsilon}^N$ is the preceding term when the summation $E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}$ runs over the set

$$L_\varepsilon = \big\{(\sigma^1,\sigma^2,\sigma^3,\sigma^4)\in S_N^4 : \forall i,j\ \ |\theta^{i,j}| < \varepsilon\big\}$$

and $\mathbb{I}_{2,2,>\varepsilon}^N$ when the summation runs over $\overline{L_\varepsilon}$. Note that, on the contrary to their sum, neither $\mathbb{I}_{2,2,>\varepsilon}^N$ nor $\mathbb{I}_{2,2,<\varepsilon}^N$ are necessarily positive. Now we can benefit from the truncations in estimating $\mathbb{I}_{2,2,>\varepsilon}^N$. Its absolute value $|\mathbb{I}_{2,2,>\varepsilon}^N|$ is not greater than the sum of 6 terms like the left-hand side of (62) with $|Re_N(\frac{k}{N},\sigma^i)|$ and where the expectation $E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}$ is taken over $\sigma^i$, $i=1,\dots,4$, with $|\theta^{i,j}| > \varepsilon$ *for one among* *the six possible pairs of* $i, j$. Because of the definition of the $C_{N,\frac{k}{N},\sigma,\delta}$'s, the term with e.g. $|\theta^{1,2}| > \varepsilon$ does not exceed

$$\sum_{k=0}^{[tN]-1} \left[e^{N\delta}+1\right]^4 E^{N,k/N}_{\sigma^3,\sigma^4}\Big[\prod_{i=3}^{4}\mathbf{1}_{B_{N,\frac{k}{N},\sigma^i,E}}\Big] E^{N,k/N}_{\sigma^1,\sigma^2}\Big(\mathbf{1}_{|\theta^{1,2}|\geq\varepsilon}\prod_{i=1}^{2}\mathbf{1}_{B_{N,\frac{k}{N},\sigma^i,E}}\Big) \tag{63}$$

Again, in view of Lemma 5 and Corollary 1, in order to establish the convergence in probability, it suffices to show the convergence to zero of (63) on the set $\mathrm{D}_{N,\eta}\bigcap \mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}$. This will follow from the convergence to zero in $L^1(\mathbb{P})$ of the product $\pi_N$ of the variable in (63) multiplied by $\mathbf{1}_{\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}}\mathbf{1}_{\mathrm{D}_{N,\eta}}$ for all $\eta > 0$. But by the definition of $\mathrm{D}_{N,\eta}$ and $\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}$ the term in round brackets $E_{\sigma^1,\sigma^2}(\cdot)$ is not greater than $\eta^2 e^{-h_1 N/2}$ on these events. Hence $\pi_N$ is bounded in $L^1(\mathbb{P})$ by $\eta^2\sum_{k=0}^{[tN]-1}\left[e^{N\delta}+1\right]^4 e^{-h_1N/2}$, which vanishes due to the choice of $\delta < h_1/8$ we made in (54). We conclude that $\mathrm{I\!I}^N_{2,2,>\varepsilon}\to 0$ in probability.

Next, let us proceed with $\mathrm{I\!I}^N_{2,2,<\varepsilon}$. Contrary to $\mathrm{I\!I}^N_{2,2,>\varepsilon}$, here truncations by $C_{N,\frac{k}{N},\sigma^i,E}$ and $B_{N,\frac{k}{N},\sigma^i,E}$ are *obstacles to overcome*: Otherwise we could bound the denominator $\widetilde{Z}_N(\frac{k}{N})$ by $1/\eta$ on $\mathrm{D}_{N,\eta}$ and the result would immediately follow from (44) of Lemma 1. Let us first remove the truncation by $C_{N,\frac{k}{N},\sigma,\delta}$. By Lemma 4

$$\begin{aligned}
&\sum_{k=0}^{[tN]-1}\Big|\mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\mathbf{1}_{L_\varepsilon}\prod_{i=1}^{4}\Big(Re_N(\frac{k}{N},\sigma^i)\mathbf{1}_{B_{N,\frac{k}{N},\sigma^i,E}}\Big)\mathbf{1}_{\cup_{i=1}^4\overline{C_{N,\frac{k}{N},\sigma^i,\delta}}}\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big)\Big|\\
&\qquad\leq \sum_{j=1}^{4}\sum_{k=0}^{[tN]-1}\mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\mathbf{1}_{L_\varepsilon}\prod_{i=1}^{4}|Re_N(\frac{k}{N},\sigma^i)|1_{\overline{C_{N,\frac{k}{N},\sigma^j,\delta}}}\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big)\\
&\qquad= \sum_{j=1}^{4}\sum_{k=0}^{[tN]-1}E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\mathbf{1}_{L_\varepsilon}\mathbb{E}[\prod_{i=1}^{4}|Re_N(\frac{k}{N},\sigma^i)|1_{\overline{C_{N,\frac{k}{N},\sigma^j,\delta}}}]\,.
\end{aligned}\tag{64}$$

For every fixed $j = 1,\ldots,4$, the term $\mathbb{E}[\,\cdot\,]$ is not greater than

$$\begin{aligned}
\mathbb{E}\Big[\Big(&e^{\sum_{i=1}^4\frac{\beta_0}{2}(\frac{\sigma^i.\xi}{\sqrt{N}})^2-4\phi_N(\beta_0)}+\sum_{(i,k,l)\in A}e^{\frac{\beta_0}{2}(\frac{\sigma^i.\xi}{\sqrt{N}})^2+\frac{\beta_0}{2}(\frac{\sigma^k.\xi}{\sqrt{N}})^2+\frac{\beta_0}{2}(\frac{\sigma^l.\xi}{\sqrt{N}})^2-3\phi_N(\beta_0)}\\
&+\sum_{(i,k)\in B}e^{\frac{\beta_0}{2}(\frac{\sigma^i.\xi}{\sqrt{N}})^2+\frac{\beta_0}{2}(\frac{\sigma^k.\xi}{\sqrt{N}})^2-2\phi_N(\beta_0)}+\sum_{i=1,\ldots,4}e^{\frac{\beta_0}{2}(\frac{\sigma^i.\xi}{\sqrt{N}})^2-\phi_N(\beta_0)}+1\Big)1_{\overline{C_{N,\frac{k}{N},\sigma^j,\delta}}}\Big]
\end{aligned}$$

where $A$ contains the four terms obtained by deleting one of the four configurations and $B$ contains the 6 terms obtained by deleting two configurations. Distributing the indicator function over the elements of this sum results in either terms controlled at Lemma 4, or in expressions like e.g.

$$\mathbb{E}\Big(e^{\frac{\beta_0}{2}(\frac{\sigma^1.\xi}{\sqrt{N}})^2+\frac{\beta_0}{2}(\frac{\sigma^2.\xi}{\sqrt{N}})^2+\frac{\beta_0}{2}(\frac{\sigma^3.\xi}{\sqrt{N}})^2-3\phi_N(\beta_0)}\Big)1_{\overline{C_{N,\frac{k}{N},\sigma^4,\delta}}}.$$

According to Proposition 1, there exists $p(\beta_0)$ such that $\max_{i,j=1,\ldots,4}|\theta^{i,j}| < \varepsilon$

$$\mathbb{E}\exp\left\{\frac{N}{2}[p\beta_0(\frac{\sigma^1\cdot\xi}{N})^2+p\beta_0(\frac{\sigma^2\cdot\xi}{N})^2+p\beta_0(\frac{\sigma^3\cdot\xi}{N})^2]\right\}<\infty.$$

This combined with Lemma 4 and the Holder inequality show that (64) converges to 0 exponentially fast. Therefore we may consider $\mathbb{I}^N_{2,2<\varepsilon}$ without $\mathbf{1}_{C_{N,\frac{k}{N},\sigma^i,\delta}}$. Next, to get rid of $\mathbf{1}_{B_{N,\frac{k}{N},\sigma^1,E}}$ as well, we have to show the convergence in probability to zero of

$$\sum_{k=0}^{[tN]-1}\left|\mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\mathbf{1}_{L_\varepsilon}\prod_{i=1}^4 Re_N(\frac{k}{N},\sigma^i)\times\mathbf{1}_{\cup_{i=1}^4\overline{B_{N,\frac{k}{N},\sigma^i,E}}}\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big)\right| \tag{65}$$

$$\leq \sum_{j=1}^4\sum_{k=0}^{[tN]-1}E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\mathbf{1}_{L_\varepsilon}\mathbb{E}[\prod_{i=1}^4|Re_N(\frac{k}{N},\sigma^i)|]\mathbf{1}_{\overline{B_{N,\frac{k}{N},\sigma^j,E}}}.$$

By Proposition 1 and (42), the expectation in the numerator $\mathbb{E}[\prod_{i=1}^4|Re_N(\frac{k}{N},\sigma^i)|]$ with $|\theta^{i,j}|<\varepsilon$ is bounded. Then the right-hand side of (65) is not greater than

$$C\sum_{k=0}^{[tN]-1}\sum_{i=1}^4 E^{N,k/N}_{\sigma^i}\mathbf{1}_{\overline{B_{N,\frac{k}{N},\sigma^i,E}}}$$

for some constant $C$ and all $N$ large enough. We split this term into two parts with $\mathbf{1}_{\mathrm{D}_{N,\eta}}$ and with $\mathbf{1}_{\overline{\mathrm{D}_{N,\eta}}}$. The last one is equal to zero with probability arbitrarily close to 1 with the appropriate choice of $\eta$ due to Lemma 5, while the expectation of the first one is bounded by $4C[tN]\eta e^{-h_1N}$ by Lemma 2. Then (65) converges to zero in probability.

Finally, it remains to show the convergence in probability of $\mathbb{I}^N_{2,2,<\varepsilon}$ without truncations by $\mathbf{1}_{C_{N,\frac{k}{N},\sigma^i,E}}\mathbf{1}_{B_{N,\frac{k}{N},\sigma^i,E}}$. Once again, by virtue of Lemma 5 we can multiply each term by $\mathbf{1}_{\mathrm{D}_{N,\eta}}$, and we are left to proving convergence in $L^1$ again. The $L^1$-norm under consideration is equal to

$$\begin{aligned}
\mathbb{E}\mathbf{1}_{\mathrm{D}_{N,\eta}}\sum_{k=0}^{[tN]-1}&\left|\mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\mathbf{1}_{L_\varepsilon}\prod_{i=1}^4 Re_N(\frac{k}{N},\sigma^i)\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big)\right|\\
&= \mathbb{E}\sum_{k=0}^{[tN]-1}\left|E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\mathbf{1}_{L_\varepsilon}\mathbb{E}(\prod_{i=1}^4 Re_N(\frac{k}{N},\sigma^i))\mathbf{1}_{\mathrm{D}_{N,\eta}}\right|\\
&\leq \mathbb{E}\sum_{k=0}^{[tN]-1}E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\mathbf{1}_{L_\varepsilon}|\mathbb{E}(\prod_{i=1}^4 Re_N(\frac{k}{N},\sigma^i))|\mathbf{1}_{\mathrm{D}_{N,\eta}}\\
&\leq \eta^4\sum_{k=0}^{[tN]-1}E_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\mathbf{1}_{L_\varepsilon}|\mathbb{E}(\prod_{i=1}^4 Re_N(\frac{k}{N},\sigma^i))|\mathbb{E}\prod_{i=1}^4 e_N(\frac{k}{N},\sigma^i)
\end{aligned}\tag{66}$$

By (44) of Lemma 1 each term of the sum (66) is bounded by $CN^{-2}$ with some constant $C>0$. Then the sum (66) is of the order $O(N^{-1})$. This completes the proof of Condition (ii).

# 4 Proof of Condition (i)

The key point in the proof of this condition is the result (43) of Lemma 1. Since it is valid only for small overlaps, we need to make again a series of truncations as in the previous proof. By Condition (ii) with $\alpha = 1$ it suffices to show that

$$\sum_{k=0}^{[tN]-1} \mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2}\Big[Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2) - \Delta\Gamma(\frac{k}{N})\Big]\mathbf{1}_{|\Delta M_N(\frac{k}{N})|<1}\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big) \tag{67}$$

converges to zero in probability. First, notice that we can restrict ourselves to the study of

$$\mathrm{I}^N_1 = \sum_{k=0}^{[tN]-1} \mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2}\Big[\prod_{i=1}^{2} Re_N(\frac{k}{N},\sigma^i)\mathbf{1}_{C_{N,\frac{k}{N},\sigma^i,\delta}} - \Delta\Gamma(\frac{k}{N})\Big]\mathbf{1}_{|\Delta M_N(\frac{k}{N})|<1}\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big). \tag{68}$$

Indeed, due to truncation by $\mathbf{1}_{|\Delta M_N(\frac{k}{N})|<1}$ we have

$$\begin{aligned}
&\sum_{k=0}^{[tN]-1} \mathbb{E}\Big(|E^{N,k/N}_{\sigma^1,\sigma^2}[Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2)\mathbf{1}_{\overline{C_{N,\frac{k}{N},\sigma^1,\delta}}\cup\overline{C_{N,\frac{k}{N},\sigma^2,\delta}}}]|\mathbf{1}_{|\Delta M_N(\frac{k}{N})|<1}\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big)\\
&\quad\le 2\sum_{k=0}^{[tN]-1} \mathbb{E}\Big(\Big|E^{N,k/N}_{\sigma}[Re_N(\frac{k}{N},\sigma)\mathbf{1}_{\overline{C_{N,\frac{k}{N},\sigma,\delta}}}]\Big|\mathbf{1}_{|\Delta M_N(\frac{k}{N})|<1}\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big)\\
&\quad\le 2\sum_{k=0}^{[tN]-1} \mathbb{E}\Big(E^{N,k/N}_{\sigma}|Re_N(\frac{k}{N},\sigma)|\mathbf{1}_{\overline{C_{N,\frac{k}{N},\sigma,\delta}}}\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big)\\
&\quad= 2\sum_{k=0}^{[tN]-1} E^{N,k/N}_{\sigma}\mathbb{E}(|Re_N(\frac{k}{N},\sigma)|\mathbf{1}_{\overline{C_{N,\frac{k}{N},\sigma,\delta}}})\\
&\quad\le 4[Nt]e^{-h_3N}
\end{aligned} \tag{69}$$

where the last exponential estimate holds by Lemma 4. Next, let us split $\mathrm{I}^N_1 = \mathrm{I}^N_{1,1} + \mathrm{I}^N_{1,2}$ where in the first term the Boltzmann factors $e_N(\frac{k}{N},\sigma^1)e_N(\frac{k}{N},\sigma^2)$ are truncated by $1_{B_{N,\frac{k}{N},\sigma^1,E}\cap B_{N,\frac{k}{N},\sigma^2,E}}$ and in the second by $1_{\overline{B_{N,\frac{k}{N},\sigma^1,E}}\cup\overline{B_{N,\frac{k}{N},\sigma^2,E}}}$. First, we show the convergence to zero of the second term which is

$$\begin{aligned}
\mathrm{I}^N_{1,2} \;=\; &\sum_{k=0}^{[tN]-1} \mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2}[Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2)\mathbf{1}_{C_{N,\frac{k}{N},\sigma^1,\delta}\cap C_{N,\frac{k}{N},\sigma^2,\delta}} - \Delta\Gamma(\frac{k}{N})]\\
&\times \mathbf{1}_{\overline{B_{N,\frac{k}{N},\sigma^1,E}}\cup\overline{B_{N,\frac{k}{N},\sigma^2,E}}}\mathbf{1}_{|\Delta M_N(\frac{k}{N})|<1}\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big).
\end{aligned} \tag{70}$$

Since we need the convergence in probability only, by Lemma 5 and Corollary 1 it suffice to analyze $\widetilde{\mathrm{I}}^N_{1,2}$ which is the right-hand side of (70) with each term in the sum over $k$ multiplied by $\mathbf{1}_{\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}}\mathbf{1}_{\mathrm{D}_{N,\eta}}$. By the definition of the events $\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}$, $\mathrm{D}_{N,\eta}$ and $C_{N,\frac{k}{N},\sigma,\delta}$ we get

$$|\widetilde{\mathrm{I}}^N_{1,2}| \;\le\; \sum_{k=0}^{[tN]-1} [(e^{N\delta}+1)^2 + 2\sup_{0\le t\le t_0}\Gamma(t)]E^{N,k/N}_{\sigma^1,\sigma^2}\mathbf{1}_{\overline{B_{N,\frac{k}{N},\sigma^1,E}}\cup\overline{B_{N,\frac{k}{N},\sigma^2,E}}}\mathbf{1}_{\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}}\mathbf{1}_{\mathrm{D}_{N,\eta}}$$

$$\leq \quad 2\eta \sum_{k=0}^{[tN]-1} [(e^{N\delta}+1)^2 + \sup_{0\leq t\leq t_0} \Gamma(t)]e^{-h_1 N/2} \tag{71}$$

Since $2\delta < h_1/2$, due to the choice of $\delta$ (54) we made, the bound (71) converges to zero exponentially fast. Thus $\mathrm{I}^N_{1,2} \to 0$ in probability. So, we are lead to study $\mathrm{I}^N_{1,1}$ that we split it its turn: $\mathrm{I}^N_{1,1} = \mathrm{I}^N_{1,1,1} - \mathrm{I}^N_{1,1,2}$ where

$$\begin{aligned}\mathrm{I}^N_{1,1,1} &= \sum_{k=0}^{[tN]-1} \mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2}[Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2)\mathbf{1}_{C_{N,\frac{k}{N},\sigma^1,\delta}\cap C_{N,\frac{k}{N},\sigma^2,\delta}} - \Delta\Gamma(\frac{k}{N})] \\ &\qquad \times \mathbf{1}_{B_{N,\frac{k}{N},\sigma^1,E}\cap B_{N,\frac{k}{N},\sigma^2,E}} \Bigm| \mathcal{F}_{\frac{k}{N}}\Big),\end{aligned}$$

$$\begin{aligned}\mathrm{I}^N_{1,1,2} &= \sum_{k=0}^{[tN]-1} \mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2}[Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2)\mathbf{1}_{C_{N,\frac{k}{N},\sigma^1,\delta}\cap C_{N,\frac{k}{N},\sigma^2,\delta}} - \Delta\Gamma(\frac{k}{N})] \\ &\qquad \times \mathbf{1}_{B_{N,\frac{k}{N},\sigma^1,E}\cap B_{N,\frac{k}{N},\sigma^2,E}}\mathbf{1}_{|\Delta M_N(\frac{k}{N})|>1} \Bigm| \mathcal{F}_{\frac{k}{N}}\Big).\end{aligned}$$

Let us show that $\mathrm{I}^N_{1,1,2}$ converges to zero in probability. Let us estimate it as $|\mathrm{I}^N_{1,1,2}| \leq \bar{I}^N_{1,1,2} + 2\sup_{0\leq t\leq t_0}\Gamma(t)\sum_{k=0}^{[tN]-1}\mathbb{P}(|\Delta M_N(\frac{k}{N})| > 1 \mid \mathcal{F}_{\frac{k}{N}})$. Here the second term being $\mathbb{I}^N$ with $\alpha = 1$, vanishes in probability by Condition $(ii)$. The first term equals

$$\begin{aligned}\bar{\mathrm{I}}^N_{1,1,2} &= \sum_{k=0}^{[tN]-1} \mathbb{E}\Big(\Big|E^{N,k/N}_{\sigma^1,\sigma^2}[Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2)]\mathbf{1}_{C_{N,\frac{k}{N},\sigma^1,\delta}\cap C_{N,\frac{k}{N},\sigma^2,\delta}} \\ &\qquad \times \mathbf{1}_{B_{N,\frac{k}{N},\sigma^1,E}\cap B_{N,\frac{k}{N},\sigma^2,E}}\Big|\mathbf{1}_{|\Delta M_N(\frac{k}{N})|>1} \Bigm| \mathcal{F}_{\frac{k}{N}}\Big).\end{aligned}$$

Applying Cauchy-Schwarz inequality twice, we bound it by the following product:

$$\begin{aligned}|\widetilde{\mathrm{I}}^N_{1,1,2}|^2 &\leq \sum_{k=0}^{[tN]-1} \mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\prod_{i=1}^{4}[Re_N(\frac{k}{N},\sigma^i)\mathbf{1}_{C_{N,\frac{k}{N},\sigma^i,\delta}}\mathbf{1}_{B_{N,\frac{k}{N},\sigma^i,E}}] \Bigm| \mathcal{F}_{\frac{k}{N}}\Big) \\ &\qquad \times \sum_{k=0}^{[tN]-1}\mathbb{P}\Big(|\Delta M_N(\frac{k}{N})| > 1 \Bigm| \mathcal{F}_{\frac{k}{N}}\Big).\end{aligned}$$

The first factor is nothing but $\mathbb{I}^N_{2,2,>\varepsilon} + \mathbb{I}^N_{2,2,<\varepsilon}$ from the proof of Condition (ii), see (62). We have already established that it converges to 0. The second factor is $\mathbb{I}^N$ the quantity controlled by Condition (ii) with $\alpha = 1$. Whence, $\mathrm{I}^N_{1,1,2}$ converges to zero in probability. Now let us write $\mathrm{I}^N_{1,1,1} = \mathrm{I}^N_{1,1,1,>\varepsilon} + \mathrm{I}^N_{1,1,1,<\varepsilon}$ where in the first term the expectation $E_{\sigma^1,\sigma^2}$ is taken over $\sigma^1,\sigma^2$ with $|\theta^{1,2}| > \varepsilon$ and in the second – over $\sigma^1,\sigma^2$ with $|\theta^{1,2}| < \varepsilon$. Let us benefit from truncations in the analysis of the first term. As usual, by Lemma 5 and Corollary 1 instead of $\mathrm{I}^N_{1,1,1,>\varepsilon}$ we show the convergence in probability only of $\widetilde{\mathrm{I}}^N_{1,1,1,>\varepsilon}$ where each term is multiplied by $\mathbf{1}_{\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}}$ and $\mathbf{1}_{\mathrm{D}_{N,\eta}}$. But by definition of $\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}$, $C_{N,\frac{k}{N},\sigma,\delta}$ and $\mathrm{D}_{N,\eta}$

$$|\widetilde{\mathrm{I}}^N_{1,1,1,>\varepsilon}| \leq [tN]\eta^2[(e^{\delta N}+1)^2 + 2\sup_{0\leq t\leq t_0}\Gamma(t)]e^{-h_1 N/2}.$$

Hence, $I^N_{1,1,1>\varepsilon} \to 0$ in probability. Thus, we have to investigate the term $\mathrm{I}^N_{1,1,1,<\varepsilon}$ only, and, as in the previous section, we now need to remove the truncations which prevent us to perform the expectations. First of all, we get rid of the truncation of the increments $Re_N(\frac{k}{N},\sigma)$ by $C_{N,\frac{k}{N},\sigma^i,\delta}$ exactly in the same way as in (64). (Only, take two replicas instead of four and use Lemma 4 in the last step provided that $|\theta^{1,2}|<\varepsilon$.) Hence, we need to study the following conditional expectation which we compute explicitly:

$$\begin{aligned}&\sum_{k=0}^{[tN]-1}\mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2}\mathbf{1}_{|\theta^{1,2}|<\varepsilon}[Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2)-\Delta\Gamma(\frac{k}{N})]\\&\qquad\times\mathbf{1}_{B_{N,\frac{k}{N},\sigma^1,E}\cap B_{N,\frac{k}{N},\sigma^2,E}}\ \Big|\ \mathcal{F}_{\frac{k}{N}}\Big)\\&=\sum_{k=0}^{[tN]-1}E^{N,k/N}_{\sigma^1,\sigma^2}\mathbf{1}_{|\theta^{1,2}|<\varepsilon}\chi_N(\theta^{1,2})\mathbf{1}_{B_{N,\frac{k}{N},\sigma^1,E}\cap B_{N,\frac{k}{N},\sigma^2,E}}\end{aligned}\tag{72}$$

with the notation

$$\chi_N(\theta^{1,2})=\mathbb{E}\Big(Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2)-\Delta\Gamma(\frac{k}{N})\Big).$$

Again by Lemma 5 we have to show the convergence to zero of the right hand-side of (72) with each term multiplied by $\mathbf{1}_{\mathrm{D}_{N,\eta}}$. This means that we may restrict ourselves to the analysis of

$$J_N=\eta^2\sum_{k=0}^{[tN]-1}\Big|E_{\sigma^1,\sigma^2}\mathbf{1}_{|\theta^{1,2}|<\varepsilon}\chi_N(\theta^{1,2})e_N(\frac{k}{N},\sigma^1)e_N(\frac{k}{N},\sigma^2)\mathbf{1}_{B_{N,\frac{k}{N},\sigma^1,E}\cap B_{N,\frac{k}{N},\sigma^2,E}}\Big|.$$

Let us apply the Cauchy-Schwarz inequality:

$$\begin{aligned}\mathbb{E}J_N&\le\eta^2\sum_{k=0}^{[tN]-1}\mathbb{E}E_{\sigma^1}(e_N(\frac{k}{N},\sigma^1))^{1/2}\\&\quad\times\Big|E_{\sigma^2}\mathbf{1}_{|\theta^{1,2}|<\varepsilon}\chi_N(\theta^{1,2})[e_N(\frac{k}{N},\sigma^1)]^{1/2}e_N(\frac{k}{N},\sigma^2)\mathbf{1}_{B_{N,\frac{k}{N},\sigma^1,E}\cap B_{N,\frac{k}{N},\sigma^2,E}}\Big|\\&\le\eta^2\sum_{k=0}^{[tN]-1}\Big(\mathbb{E}E_{\sigma^1,\sigma^2,\sigma^3}\Big[\mathbf{1}_{|\theta^{1,2}|<\varepsilon,|\theta|^{1,3}<\varepsilon}\chi_N(\theta^{1,2})\chi_N(\theta^{1,3})\\&\qquad\times\prod_{i=1}^{3}[e_N(\frac{k}{N},\sigma^i)\mathbf{1}_{B_{N,\frac{k}{N},\sigma^i,E}}]\Big]\Big)^{1/2}\end{aligned}\tag{73}$$

where we used the fact that $\mathbb{E}E_{\sigma^1}e^\beta_N(\frac{k}{N},\sigma^1)=1$. The expectation in (73) equals $\mathbb{E}E_{\sigma^1,\sigma^2,\sigma^3}[\cdot]=L^N_1(\frac{k}{N})+L^N_2(\frac{k}{N})-L^N_3(\frac{k}{N})$ with

$$\begin{aligned}L^N_1(\frac{k}{N})&=\mathbb{E}E_{\sigma^1,\sigma^2,\sigma^3}\mathbf{1}_{|\theta^{1,2}|<\varepsilon,|\theta^{1,3}|<\varepsilon,|\theta^{2,3}|>\gamma}\chi_N(\theta^{1,2})\chi_N(\theta^{1,3})\\&\qquad\times\prod_{i=1}^{3}[e_N(\frac{k}{N},\sigma^i)\mathbf{1}_{B_{N,\frac{k}{N},\sigma^i,E}}]\end{aligned}$$

$$\begin{aligned} L_2^N(\frac{k}{N}) &= \mathbb{E}E_{\sigma^1,\sigma^2,\sigma^3}\mathbf{1}_{|\theta^{1,2}|<\varepsilon,|\theta^{1,3}|<\varepsilon,|\theta^{2,3}|<\gamma}\chi_N(\theta^{1,2})\chi_N(\theta^{1,3}) \\ &\qquad \times \prod_{i=1}^{3} e_N(\frac{k}{N},\sigma^i) \\ L_3^N(\frac{k}{N}) &= \mathbb{E}E_{\sigma^1,\sigma^2,\sigma^3}\mathbf{1}_{|\theta^{1,2}|<\varepsilon,|\theta^{1,3}|<\varepsilon,|\theta^{2,3}|<\gamma}\chi_N(\theta^{1,2})\chi_N(\theta^{1,3}) \\ &\qquad \times \prod_{i=1}^{3}[e_N(\frac{k}{N},\sigma^i)]\mathbf{1}_{\cup_{i=1}^{3}\overline{B_{N,\frac{k}{N},\sigma^i,E}}} \end{aligned}$$

Since $|\theta^{1,2}|<\varepsilon$, $|\theta^{1,3}|<\varepsilon$, the expansions of Proposition 1 are valid for $\chi_N(\theta^{1,2})$ and $\chi_N(\theta^{1,3})$. In particular $|\chi_N(\theta^{1,2})\chi_N(\theta^{1,3})|<C$ for some constant $C$ all $N\geq 1$ and all $\theta^{1,2}$ $\theta^{1,3}$ with $|\theta^{1,2}|<\varepsilon$, $|\theta^{1,3}|<\varepsilon$. Then by Lemma 3

$$\begin{aligned} \sup_{0\leq k\leq [Nt_0]-1}|L_1^N(\frac{k}{N})| &\leq Ce^{-h_2 N} \\ \sup_{0\leq k\leq [Nt_0]-1}|L_3^N(\frac{k}{N})| &\leq 3Ce^{-h_2 N} \end{aligned}$$

Finally, recall that $\gamma$ and $\varepsilon$ were chosen smaller than $\varepsilon_0$ in steps 2 and 3 of section 2. From the estimate (43) of Lemma 1 it follows that

$$\lim_{N\to\infty}\sup_{0\leq k\leq [Nt_0]-1}N^2L_2^N(\frac{k}{N})=0,$$

and Condition (i) is proved.

# 5 Proof of Condition (iii)

As in the previous cases, the starting point of the proof is the splitting of $\text{Ш}^N$ into two terms, where in one term $\text{Ш}_1^N$ the increments $Re_N(\frac{k}{N},\sigma)$ are truncated by $\mathbf{1}_{C_{N,\frac{k}{N},\sigma,\delta}}$ and in the second $\text{Ш}_2^N$ by $\mathbf{1}_{\overline{C_{N,\frac{k}{N},\sigma,\delta}}}$. Then proceeding along the lines of (58) and applying Lemma 4 we get $\mathbb{E}|\text{Ш}_2^N|\leq [tN]e^{-h_3 N}$.

Second, we split $\text{Ш}_1^N=\text{Ш}_{1,1}^N+\text{Ш}_{1,2}^N$ where in the first term $\text{Ш}_{1,1}^N$ the Boltzmann factors $e_N(\frac{k}{N},\sigma)$ are truncated by $\mathbf{1}_{B_{N,\frac{k}{N},\sigma,E}}$ and in the second $\text{Ш}_{1,2}^N$ by $\mathbf{1}_{\overline{B_{N,\frac{k}{N},\sigma,E}}}$. Proceeding along the line of (61) we get $|\text{Ш}_{1,2}^N|\to 0$.

Now, to complete the proof, we need to show the convergence in probability of $\text{Ш}_{1,1}^N$. We apply Holder inequality twice to obtain

$$\begin{aligned} |\text{Ш}_{1,2}^N| \leq \sum_{k=0}^{[tN]-1}&\Big[\mathbb{E}\Big(E_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}^{N,k/N}\prod_{i=1}^{4}Re_N(\frac{k}{N},\sigma^i)\mathbf{1}_{C_{N,\frac{k}{N},\sigma^i,\delta}}\mathbf{1}_{B_{N,\frac{k}{N},\sigma^i,E}}\,\Big|\,\mathcal{F}_{\frac{k}{N}}\Big)^{1/4} \\ &\times\mathbb{P}\Big(|\Delta M_N(\frac{k}{N})|>1\,\Big|\,\mathcal{F}_{\frac{k}{N}}\Big)^{3/4}\Big] \end{aligned}$$

$$
\begin{aligned}
&\leq \Big[\sum_{k=0}^{[tN]-1}\mathbb{E}\Big(E^{N,k/N}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\prod_{i=1}^{4}Re_N(\frac{k}{N},\sigma^i)\mathbf{1}_{C_{N,\frac{k}{N},\sigma^i,\delta}}\mathbf{1}_{B_{N,\frac{k}{N},\sigma^i,E}}|\mathcal{F}_{\frac{k}{N}}\Big)\Big]^{1/4}\\
&\quad\times\Big[\sum_{k=0}^{[tN]-1}\mathbb{P}\Big(|\Delta M_N(\frac{k}{N})|>1\;\Big|\;\mathcal{F}_{\frac{k}{N}}\Big)\Big]^{3/4}\\
&= \Big[\mathrm{II}^N_{2,2,>\varepsilon}+\mathrm{II}^N_{2,2,<\varepsilon}\Big]^{1/4}\times\Big[\mathrm{II}^N\Big]^{3/4}
\end{aligned}
$$

Condition (iii) is proved.

# 6 Appendix: Proofs for Section 2

To complete the paper, we are left to check the technical results.

## 6.1 Proof of Proposition 1

First notice that the expansion (12) is a particular case of (34), which is itself a particular case of (37). For the sake of shortness we will only prove (34). The proofs of (37) and (40) rely on the same ideas and methods. The bound (42) is a consequence of (40). Let $\gamma(\cdot)$ be the logarithmic moment generating function of $\xi_1^1$,

$$\gamma(x)=\log\mathbb{E}e^{x\xi_1^1}=\log\operatorname{ch}(x).$$

Let $g_1$ be the density function of the $\mathcal{N}(0,1)$ probability measure on $\mathbb{R}$ with respect to the Lebesgue measure. For every $u\in\mathbb{R}$

$$e^{u^2/2}=\int_{\mathbb{R}}e^{ux}g_1(x)dx. \tag{74}$$

Hence,

$$
\begin{aligned}
\mathbb{E}e^{\frac{\beta_1}{2}(\frac{\sigma^1\cdot\xi}{\sqrt{N}})^2+\frac{\beta_2}{2}(\frac{\sigma^2\cdot\xi}{\sqrt{N}})^2} &= \mathbb{E}\int_{\mathbb{R}^2}e^{(\sqrt{\frac{\beta_1}{N}}\sum_{i=1}^N\sigma_i^1\xi_i)u+(\sqrt{\frac{\beta_2}{N}}\sum_{i=1}^N\sigma_i^2\xi_i)v}g_1(u)g_1(v)dudv\\
&= \mathbb{E}\int_{\mathbb{R}^2}e^{\sum_{i=1}^N(\sqrt{\frac{\beta_1}{N}}\sigma_i^1u+\sqrt{\frac{\beta_2}{N}}\sigma_i^2v)\xi_i}g_1(u)g_1(v)dudv\\
&= \int_{\mathbb{R}^2}e^{\sum_{i=1}^N\gamma(\sqrt{\frac{\beta_1}{N}}\sigma_i^1u+\sqrt{\frac{\beta_2}{N}}\sigma_i^2v)}g_1(u)g_1(v)dudv\\
&= \int_{\mathbb{R}^2}e^{\frac{N}{2}\sum_{\pm}(1\pm\theta)\gamma(\sqrt{\frac{\beta_1}{N}}u\pm\sqrt{\frac{\beta_2}{N}}v)}g_1(u)g_1(v)dudv
\end{aligned}
$$

where we wrote $\theta$ for $\theta^{1,2}$ to simplify the notations. Let $g_\Gamma$ be the density function of the centered Gaussian probability measure on $\mathbb{R}^2$ with Variance-Covariance matrix

$$\Gamma=\frac{1}{(1-\beta_1)(1-\beta_2)-\theta^2\beta_1\beta_2}\begin{pmatrix}1-\beta_2 & \sqrt{\beta_1\beta_2}\theta\\ \sqrt{\beta_1\beta_2}\theta & 1-\beta_1\end{pmatrix}.$$

Notice that this Gaussian law is defined if and only if (33) holds. We multiply and divide by $g_\Gamma(u,v)$ in the last integral, and we obtain

$$\mathbb{E}e^{\frac{\beta_1}{2}(\frac{\sigma^1\cdot\xi}{\sqrt{N}})^2+\frac{\beta_2}{2}(\frac{\sigma^2\cdot\xi}{\sqrt{N}})^2} = e^{\psi(\beta_1,\beta_2,\theta)}\Big[1+K^N\Big]$$

with

$$K^N = \int_{\mathbb{R}^2}\Big(e^{-\frac{u^2\beta_1+v^2\beta_2+2\theta\sqrt{\beta_1\beta_2}uv}{2}}e^{\frac{N}{2}\sum_\pm(1\pm\theta)\gamma(\sqrt{\frac{\beta_1}{N}}u\pm\sqrt{\frac{\beta_2}{N}}v)}-1\Big)g_\Gamma(u,v)dudv\ .$$

Let us write $K^N$ as the sum of two terms $K_1^N+K_2^N$ where the first one stands for the integral over $||(u,v)||_2 < N^{1/8}$ and the second over $||(u,v)||_2 \geq N^{1/8}$. Since $\gamma(x)=\log\mathrm{ch}(x)\leq x^2/2$ we have for every $\theta$

$$-\frac{u^2\beta_1+v^2\beta_2+2\theta\sqrt{\beta_1\beta_2}uv}{2}+\frac{N}{2}\sum_\pm(1\pm\theta)\gamma\Big(\sqrt{\frac{\beta_1}{N}}u\pm\sqrt{\frac{\beta_2}{N}}v\Big)\leq 0.$$

Thus, since Gaussian tails decay exponentially fast, $K_2^N=o(1/N)$ uniformly in $\theta$. On the other hand we know that

$$\gamma(x)=\frac{x^2}{2}-\frac{x^4}{12}+o(x^4)$$

and

$$e^x-1=x+o(x^2)$$

when $x\to 0$. Thus, for $||(u,v)||_2 < N^{1/8}$ we have

$$e^{-\frac{u^2\beta_1+v^2\beta_2+2\theta\sqrt{\beta_1\beta_2}uv}{2}}e^{\frac{N}{2}\sum_\pm(1\pm\theta)\gamma(\sqrt{\frac{\beta_1}{N}}u\pm\sqrt{\frac{\beta_2}{N}}v)}-1=$$
$$-\frac{1}{12N}(\beta_1^2u^4+\beta_2^2v^4)-\frac{\theta}{3N}(\beta_1\beta_2)^{1/2}uv(\beta_1u^2+\beta_2v^2)-\frac{1}{2N}\beta_1\beta_2u^2v^2+o(\frac{1}{N})$$

where $o(\frac{1}{N})$ is uniform in $\theta$. Thus

$$K_1^N=-\frac{1}{4N}\left(\frac{\beta_1^2}{(1-\beta_1)^2}+\frac{\beta_2^2}{(1-\beta_2)^2}\right)-\frac{1}{2N}\frac{\beta_1\beta_2}{(1-\beta_1)(1-\beta_2)}+O(\frac{(\theta)^2}{N})+o(\frac{1}{N}),$$

and (34) is then established. ■

## 6.2 Proof of the lemmas

*Proof of Lemma 1*

First of all, remember that there exists an $\tau=\tau(\beta_0)$ small enough such that

$$\max_{i,j=1,2,3}|\theta^{i,j}|<\tau \tag{75}$$

implies (33) and (36) with $\beta_1 = \beta_2 = \beta_3 = \beta_0$. Now, according to Proposition 1, for all $\sigma^1, \sigma^2, \sigma^3 \in S_N$ satisfying (75) we have

$$\begin{aligned} \chi_N(\theta^{1,2}) &= \mathbb{E}(Re_N(t,\sigma^1)Re_N(t,\sigma^2) - \Delta\Gamma(t)) \\ &= -\frac{1}{2}\ln\Big(1 - \frac{\beta_0^2(\theta^{1,2})^2}{(1-\beta_0)^2}\Big) - \frac{\beta_0^2}{2N(1-\beta_0)^2} + O(\frac{(\theta^{1,2})^2}{N}) + o(\frac{1}{N}) \\ &\quad - \Big(\frac{\partial\Gamma(t)}{\partial t}\frac{1}{N} + o(\frac{1}{N})\Big) \\ &= \frac{\beta_0^2}{2(1-\beta_0)^2}\Big((\theta^{1,2})^2 - \frac{1}{N(1-t\beta_0^2/(1-\beta_0)^2)}\Big) + O(\frac{(\theta^{1,2})^2}{N}) + o(\frac{1}{N}), \end{aligned} \tag{76}$$

$$\begin{aligned} &\Lambda_N(\beta_0,\theta^{1,2},\theta^{1,3},\theta^{2,3}) - 3\phi_N(\beta_0) \\ &= -\frac{1}{2}\ln\Big(1 - \frac{\beta_0^3}{(1-\beta_0)^3}\|\Theta\|^2 - 2\frac{\beta_0^2}{(1-\beta_0)^2}\theta^{1,2}\theta^{1,3}\theta^{2,3}\Big) - \frac{3}{2N}\frac{\beta_0^2}{(1-\beta_0)^2} \\ &\quad + O(\frac{\|\Theta\|^2}{N}) + o(\frac{1}{N}) \end{aligned} \tag{77}$$

where we denote by

$$\Theta = (\theta^{1,2},\theta^{1,3},\theta^{2,3})\ .$$

(We use the same notation as in the case of four spins, see below (41), but no confusion can result.) Furthermore, $O(1)$ and $o(1)$ involved are uniform for $\Theta$ in this domain and also for $t \in [0,t_0]$. By Stirling's formula

$$\begin{aligned} P_{\sigma^1\sigma^2\sigma^3}(\Theta) &:= P_{\sigma^1\sigma^2\sigma^3}(\sigma^1\cdot\sigma^2 = \theta^{1,2}N, \sigma^1\cdot\sigma^3 = \theta^{1,3}N, \sigma^2\cdot\sigma^3 = \theta^{2,3}N) \\ &= 2^{-2N}\binom{N}{N(1+\theta^{1,2})/2}\binom{N(1+\theta^{1,2})/2}{N(1+\theta^{1,2}+\theta^{1,3}+\theta^{2,3})/4} \\ &\quad \times \binom{N(1-\theta^{1,2})/2}{N(1-\theta^{1,2}+\theta^{1,3}-\theta^{2,3})/4} \\ &= \frac{16\exp\{-NI(\theta^{1,2},\theta^{1,3},\theta^{1,3})\}}{\sqrt{(2\pi)^3N^3}} \\ &\quad \times [(1+\theta^{1,2}+\theta^{1,3}+\theta^{1,3})(1-\theta^{1,2}-\theta^{1,3}+\theta^{1,3})]^{-1/2} \\ &\quad \times [(1+\theta^{1,2}-\theta^{1,3}-\theta^{1,3})(1-\theta^{1,2}+\theta^{1,3}-\theta^{1,3})]^{-1/2} \\ &\quad \times (1+O(1/N)) \end{aligned}$$

with $O(1)$ uniform for $\theta^{i,j}$ with $\max_{i,j}|\theta^{i,j}| < \tau$. Now, we have

$$\begin{aligned} I(\Theta) &= \frac{1}{4}\Big((1+\theta^{1,2}+\theta^{1,3}+\theta^{1,3})\ln(1+\theta^{1,2}+\theta^{1,3}+\theta^{1,3}) \\ &\quad + (1-\theta^{1,2}-\theta^{1,3}+\theta^{1,3})\ln(1-\theta^{1,2}-\theta^{1,3}+\theta^{1,3}) \\ &\quad + (1+\theta^{1,2}-\theta^{1,3}-\theta^{1,3})\ln(1+\theta^{1,2}-\theta^{1,3}-\theta^{1,3}) \\ &\quad + (1-\theta^{1,2}+\theta^{1,3}-\theta^{1,3})\ln(1-\theta^{1,2}+\theta^{1,3}-\theta^{1,3})\Big) \\ &= -\frac{1}{2}\|\Theta\|^2 + O((|\theta^{1,2}|+|\theta^{1,3}|+|\theta^{2,3}|)^3)\ , \end{aligned}$$

as $\max_{i,j}|\theta^{i,j}| \to 0$. It follows that

$$\begin{aligned} & t\Lambda_N(\beta_0,\Theta) - 3t\phi_N(\beta_0) + \frac{1}{N}\log P_{\sigma^1\sigma^2\sigma^3}(\Theta) \\ &= \quad -\frac{1}{2}\Big(1-\frac{t\beta_0^2}{(1-\beta_0)^2}\Big)\|\Theta\|^2 + t\alpha(\Theta)(|\theta^{1,2}|+|\theta^{1,3}|+|\theta^{2,3}|)^3 \\ &\qquad -\frac{3t}{2}\frac{\beta_0^2}{(1-\beta_0)^2N} + O(\frac{\|\Theta\|^2}{N}) + o(\frac{1}{N}) \end{aligned} \tag{78}$$

where $\alpha(\Theta)$ is bounded as $\max_{i,j}|\theta^{i,j}| \to 0$, $O(1)$ and $o(1)$ are uniform in $t \in [0,t_0]$ and $\Theta$ with $\max_{i,j}|\theta^{i,j}| < \tau$ as $N \to \infty$.

Thus, for any $0 < h < \frac{1}{2}\Big(1-\frac{t_0\beta^2}{(1-\beta)^2}\Big)$ one can find $\varepsilon_0$ with $\varepsilon_0 < \tau(\beta_0)$ such that

$$e^{[Nt]\Lambda_N(\Theta)}P_{\sigma^1\sigma^2\sigma^3}(\Theta) \le Ce^{-hN\|\Theta\|^2} \tag{79}$$

for some constant $C$, all $\Theta$ with $\max_{i,j}|\theta^{i,j}| < \varepsilon_0$, all $N \ge 1$ and all $t \in [0,t_0]$. We fix e.g. $h = \frac{1}{4}\Big(1-\frac{t_0\beta^2}{(1-\beta)^2}\Big)$ and choose $\varepsilon_0$ accordingly.

Now we are ready to proceed with the proof of (43). First, let us rewrite it as

$$\lim_{N\to\infty}\sup_{0\le t\le t_0}\sum_{\substack{\theta^{1,2},\theta^{1,3},\theta^{2,3}\\ |\theta^{i,j}|<\varepsilon_{i,j}}} N^2\chi_N(\theta^{1,2})\chi_N(\theta^{1,3})e^{[Nt]\Lambda_N(\Theta)}P_{\sigma^1\sigma^2\sigma^3}(\Theta) = 0. \tag{80}$$

Let us split the left-hand side into two sums: the first $J_1^N$ will be over $\Theta$ with $\max_{i,j}|\theta^{i,j}| < N^{1/3-\eta}$ and the second $J_1^N$ over the remaining terms. Note that by (76) there exists a constant $K$ such that $|\chi_N(\theta^{1,2})\chi_N(\theta^{1,3})| \le K$ for all $\Theta$ and $t$ satisfying the fixed conditions and all $N \ge 1$. Hence, $|J_2^N| \le (\varepsilon_0 N)^3KN^2Ce^{-hN^{1/3-2\eta}}$, whence it vanishes uniformly in $t \in [0,t_0]$.

Next, let us concentrate on $J_1^N = J_{1,1}^N + J_{1,1}^N$ that we divide into two terms as well

$$J_{1,1}^N = \frac{16e^{-\frac{3t\beta_0^2}{2(1-\beta_0)^2}}}{(2\pi N)^{3/2}}\sum_{\substack{\theta^{1,2},\theta^{1,3},\theta^{2,3}\\ \max_{i,j}|\theta^{i,j}|<N^{-1/3-\eta}}} N^2\chi_N(\theta^{1,2})\chi_N(\theta^{1,3})e^{-\frac{1}{2}\Big(1-\frac{t\beta_0^2}{(1-\beta_0)^2}\Big)\|\Theta\|^2}$$

$$\begin{aligned} J_{1,2}^N &= \frac{16e^{-\frac{3t\beta_0^2}{2(1-\beta_0)^2}}}{(2\pi N)^{3/2}}\sum_{\substack{\theta^{1,2},\theta^{1,3},\theta^{2,3}\\ \max_{i,j}|\theta^{i,j}|<N^{-1/3-\eta}}} N^2\chi_N(\theta^{1,2})\chi_N(\theta^{1,3})e^{-\frac{1}{2}\Big(1-\frac{t\beta_0^2}{(1-\beta_0)^2}\Big)\|\Theta\|^2} \\ &\quad \times\Big[\frac{(2\pi N)^{3/2}}{16e^{-\frac{3t\beta_0^2}{2(1-\beta_0)^2}}}e^{[Nt]\Lambda_N(\Theta)}P_{\sigma^1\sigma^2\sigma^3}(\Theta)e^{\frac{1}{2}\Big(1-\frac{t\beta_0^2}{(1-\beta_0)^2}\Big)\|\Theta\|^2} - 1\Big]. \end{aligned}$$

After the change of variables $s^{i,j} = \theta^{i,j}\sqrt{N}$ the first term $J_{1,1}^N$ becomes

$$\begin{aligned} J_{1,1}^N &= \frac{16e^{-\frac{3t\beta_0^2}{2(1-\beta_0)^2}}}{\sqrt{(2\pi)^3}}\frac{\beta_0^4}{4(1-\beta_0)^4}\frac{1}{N^{3/2}}\sum_{\substack{S\in N^{-1/2}\mathbb{Z}^3\\ \max_{i,j}|s^{i,j}|<N^{1/6-\eta}}}\Big((s^{1,2})^2 - \frac{1}{1-t\beta_0^2/(1-\beta_0)^2} + o(1)\Big) \\ &\quad \times\Big((s^{1,3})^2 - \frac{1}{1-t\beta_0^2/(1-\beta_0)^2} + o(1)\Big)e^{-\frac{1}{2}\Big(1-\frac{t\beta_0^2}{(1-\beta_0)^2}\Big)\|S\|^2} \end{aligned} \tag{81}$$

with $o(1)$ uniform for $S = (s^{1,2}, s^{1,3}, s^{2,3})$ and $t$ in their corresponding domains. These are integral's sums converging to the integral

$$J = \frac{16e^{-\frac{3t\beta_0^2}{2(1-\beta_0)^2}}}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} \frac{\beta_0^4}{4(1-\beta_0)^4}\Big((s^{1,2})^2 - \frac{1}{1-t\beta_0^2/(1-\beta_0)^2}\Big) \tag{82}$$
$$\times \Big((s^{1,3})^2 - \frac{1}{1-t\beta_0^2/(1-\beta_0)^2}\Big) e^{-\frac{1}{2}\Big(1-\frac{t\beta_0^2}{(1-\beta_0)^2}\Big)\|S\|^2} ds^{1,2}ds^{1,3}ds^{2,3} = 0.$$

We need just a slightly more accurate argument to specify the uniform convergence in $t \in [0, t_0]$. Denote by $h(t) = 1 - \frac{t\beta_0^2}{(1-\beta_0)^2}$. The summand in (81) and the integrand (82) are bounded by $(K_1(s^{1,2})^2 + K_2)(K_3(s^{1,3})^2 + K_4)e^{-h(t_0)\|S\|^2/2}$ with some constants $K_1, \dots, K_4$ for all $N \geq 1$ all $s$ and all $t \in [0, t_0]$. Then one can find $A$ large enough to make the sum of (81) over $S$ with $\max_{i,j}|s^{i,j}| > A$ and the rest of the integral (82) over $S$ with $\max_{i,j}|s^{i,j}| > A$ arbitrarily small. Thus the problem is reduced to show the uniform convergence of the integral's sum over $S$ with $\max_{i,j}|s^{i,j}| < A$ to the corresponding integral in a finite volume. By the change of variables $r^{i,j} = \sqrt{h(t)}s^{i,j}$ this sum equals

$$\frac{16\beta_0^4 e^{-\frac{3t\beta_0^2}{2(1-\beta_0)^2}}}{\sqrt{(2\pi)^3}4(1-\beta_0)^4 h^2(t)} \times \frac{1}{N^{3/2}} \sum_{\substack{r^{1,2}, r^{1,3}, r^{2,3} \\ =0, \pm\sqrt{h(t)/N}, \dots, \pm\sqrt{h(t)}A}} f(R) \tag{83}$$

with $f(R) = ((r^{1,2})^2 - 1)((r^{1,3})^2 - 1)e^{-\|R\|^2/2}$. Then the difference between (83) and the integral by absolute value is not greater than

$$\frac{16\beta_0^4 e^{-\frac{3t\beta_0^2}{2(1-\beta_0)^2}}}{\sqrt{(2\pi)^3}4(1-\beta_0)^4 h^2(t)} (\sqrt{h(t)}A)^3 \sup_{\substack{|r^{1,2}|, |r^{1,3}|, |r^{2,3}|, |\tilde{r}^{1,2}|, |\tilde{r}^{1,3}|, |\tilde{r}^{2,3}| < \sqrt{h(t)}A \\ |r^{1,2} - \tilde{r}^{1,2}| < \sqrt{h(t)/N}, |r^{1,3} - \tilde{r}^{1,3}| < \sqrt{h(t)/N}, |r^{2,3} - \tilde{r}^{2,3}| < \sqrt{h(t)/N}}} |f(R) - f(\widetilde{R})|$$
$$\leq \frac{16\beta_0^4}{\sqrt{(2\pi)^3}4(1-\beta_0)^4 h^2(t_0)} A^3 \sup_{\substack{|r^{1,2}|, |r^{1,3}|, |r^{2,3}|, |\tilde{r}^{1,2}|, |\tilde{r}^{1,3}|, |\tilde{r}^{2,3}| < A \\ |r^{1,2} - \tilde{r}^{1,2}| < 1/\sqrt{N}, |r^{1,3} - \tilde{r}^{1,3}| < 1/\sqrt{N}, |r^{2,3} - \tilde{r}^{2,3}| < 1/\sqrt{N}}} |f(R) - f(\widetilde{R})| \to 0$$

due to the uniform continuity of $f(R)$ on the compact $[-A, A]^3$. Hence, $J_{1,1}^N$ converges to $J$ uniformly.

Finally, consider $J_{1,2}^N$. Again by the expansions (78) and (78)

$$\sup_{\substack{|\theta^{i,j}| < N^{-1/3-\eta} \\ t \in [0, t_0]}} \Bigg| \frac{\sqrt{(2\pi)^3 N^3}}{16e^{-\frac{3t\beta_0^2}{2(1-\beta_0)^2}}} e^{[Nt]\Lambda_N(\Theta)} P_{\sigma^1\sigma^2\sigma^3}(\Theta) e^{\frac{1}{2}\Big(1-\frac{t\beta_0^2}{(1-\beta_0)^2}\Big)\|\Theta\|^2}$$
$$- 1\Bigg| \to 0, \qquad N \to \infty.$$

By the same change of variables $r^{i,j} = \sqrt{h(t)}\theta^{i,j}\sqrt{N}$ and the same arguments as for $J_{1,1}^N$ the sum of the remaining terms taken with absolute values

$$\frac{16e^{-\frac{3t\beta_0^2}{2(1-\beta_0)^2}}}{\sqrt{(2\pi)^3 N^3}} \sum_{\substack{\theta^{1,2}, \theta^{1,3}, \theta^{2,3} \\ \max_{i,j}|\theta^{i,j}| < N^{-1/3-\eta}}} |N^2 \chi_N(\theta^{1,2})\chi_N(\theta^{1,3})| e^{-\frac{1}{2}\Big(1-\frac{t\beta_0^2}{(1-\beta_0)^2}\Big)\|\Theta\|^2} \tag{84}$$

converges uniformly in $[0, t_0]$ to the integral

$$\frac{16\beta_0^4}{\sqrt{2\pi^3}4(1-\beta_0)^4 h^5(t)} \int_{\mathbb{R}^3} |f(R)| dr^{1,2} dr^{1,3} dr^{2,3} = \frac{C}{h^5(t)} \leq \frac{C}{h^5(t_0)}$$

where $C$ does not depend on $t$. Then (84) is bounded uniformly for $t \in [0, t_0]$ and $N \geq 1$. It follows that $J_{1,2}^N$ converges to zero uniformly in $[0, t_0]$ finishing the proof of (43).

The proof of (44) is similar. Let us rewrite its left-hand side as

$$\sup_{0 \leq t \leq t_0} \sum_{\substack{\theta^{i,j}, \\ \max_{i,j=1,\ldots,4} |\theta^{i,j}| < \varepsilon}} N^2 |\zeta_N(\Theta)| e^{[Nt]\kappa_N(\Theta)} P_{\sigma^1\sigma^2\sigma^3\sigma^4}(\Theta). \tag{85}$$

Here by Proposition (1)

$$\begin{aligned} \zeta_N(\Theta) &= \mathbb{E} \prod_{i=1}^4 \left( e^{\beta_0 \left( \frac{\sigma^i \cdot \xi}{\sqrt{N}} \right)^2 - \phi_N(\beta_0)} - 1 \right) \\ &= e^{\kappa_N(\beta_0, \Theta) - 4\phi_N(\beta_0)} - \sum_{\Theta \in A} e^{\Lambda_N(\beta_0, \Theta) - 3\phi_N(\beta_0)} + \sum_{\Theta \in B} e^{\psi_N(\beta_0, \Theta) - 2\phi_N(\beta_0)} - 3 \\ &= O(\frac{\|\Theta\|^2}{N}) + o(\frac{1}{N}), \end{aligned} \tag{86}$$

$$\begin{aligned} & t\kappa_N(\Theta) + \frac{1}{N} \log P_{\sigma^1\sigma^2\sigma^3\sigma^4}(\Theta) \\ &\quad = -\frac{1}{2}\Big(1 - \frac{t\beta^2}{(1-\beta)^2}\Big)\|\Theta\|^2 + t\alpha(\Theta)\|\Theta\|^3 + O(\frac{1}{N}) \end{aligned} \tag{87}$$

as $N \to \infty$ uniformly for $\Theta$ with $\max_{i,j} |\theta^{i,j}| < \varepsilon_0$ and $t \in [0, t_0]$. The function $\alpha(\Theta)$ is bounded as $\max_{i,j} |\theta^{i,j}| \to 0$. The set $A$ contains 4 terms obtained by deleting one of the four configurations and computing three scalar products entering $\Lambda_N$ and $B$ contains 6 terms involving $\psi_N$.

Using the same change of variables $s^{i,j} = \theta^{i,j}\sqrt{N}$ as in the proof of (43) and completely analogous arguments one proves that the sum in (85) converges uniformly in $[0, t_0]$ to a four-dimensional integral over the Gaussian density of a polynomial taken by absolute value. This integral is, indeed, finite and its value is bounded uniformly in $[0, t_0]$ by some constant. This proves (44). ■

*Proof of Lemma 2*

Using Talagrand's idea we write

$$\begin{aligned} \mathbb{E} e^{\beta_0 H_N(t,\sigma) - [tN]\phi_N(\beta_0)} \mathbf{1}_{\overline{B_{N,t,\sigma,E}}} &= \mathbb{E} e^{-EH_N(t,\sigma)} e^{(\beta_0+E)H_N(t,\sigma) - [tN]\phi_N(\beta_0)} \mathbf{1}_{\overline{B_{N,t,\sigma,E}}} \\ &\leq e^{-\frac{E[t_0N]}{2(1-\beta_0-E)} - [tN]\phi_N(\beta_0)} \mathbb{E} e^{(\beta_0+E)H_N(t,\sigma)} \mathbf{1}_{\overline{B_{N,t,\sigma,E}}}. \end{aligned}$$

But, according to Lemma 2.1 in [28], we have

$$\begin{aligned}\mathbb{E}e^{(\beta_0+E)H_N(t,\sigma)}\mathbf{1}_{\overline{B_{N,t,\sigma,E}}} &\leq \mathbb{E}e^{(\beta_0+E)H_N(t,\sigma)} \\ &\leq e^{-\frac{[tN]}{2}\log(1-\beta_0-E)}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}e_N(t,\sigma)\mathbf{1}_{\overline{B_{N,t,\sigma,E}}} &\leq e^{-\frac{E[t_0N]}{2(1-\beta_0-E)}}e^{-\frac{[tN]}{2}\log(1-\beta_0-E)-[tN]\phi_N(\beta_0)} \\ &\leq e^{-\frac{[tN]}{2}\left(\frac{E}{1-\beta_0-E}-\log(1+\frac{E}{1-\beta_0-E})\right)}.\end{aligned}$$

It is a well known fact that for every $0 < x < 1$ we have $x - \log(1+x) > 0$. Because of (5) $0 < \frac{E}{1-\beta_0-E} < 1$. Thus, stating

$$h_i(E) = \frac{E}{1-\beta_0-E} - \log(1 + \frac{E}{1-\beta_0-E})$$

we obtain the desired result. In order to prove (48) we first need to prove that for every $\sigma^1, \sigma^2 \in S_N$

$$\mathbb{E}e_N(t,\sigma_1)e_N(t,\sigma_2)\mathbf{1}_{B_{N,t,\sigma^1,E}\cap B_{N,t,\sigma^2,E}} \leq e^{\frac{[t_0N]}{2}\frac{\beta_0^2\theta^2}{(1-\beta_0)(1-\beta_0-E)}} \tag{88}$$

Let us denote by $\lambda(\theta)$ the lowest solution of the equation

$$(1-x)^2 - \theta^2x^2 = (1-\beta_0)^2, \tag{89}$$

i.e. the lowest root of the polynomial function

$$Q_\theta(x) = x^2(1-\theta^2) - 2x + \beta_0(2-\beta_0).$$

Since for every $\theta \in [-1,1]$ we have $Q_\theta(\beta_0) \leq 0$ we necessarily have $\lambda(\theta) \leq \beta_0$ as well. Let us also notice that due to (89) the condition (33) with $\beta_1 = \beta_2 = \lambda(\theta)$ that here reduces to

$$(1-\lambda(\theta))^2 - \theta^2\lambda(\theta)^2 > 0$$

is necessarily fulfilled. Thus, applying Talagrand's idea one more time we write

$$\begin{aligned}&\mathbb{E}\prod_{k=1}^{2}e_N(t,\sigma^k)\mathbf{1}_{B_{N,t,\sigma^k,E}} \\ &= \mathbb{E}e^{\lambda(\theta)[\sum_{k=1}^2 H_N(t,\sigma^k)]+(\beta_0-\lambda(\theta))[\sum_{k=1}^2 H_N(t,\sigma^k)]-2[tN]\phi_N(\beta_0)}\prod_{k=1}^{2}\mathbf{1}_{B_{N,t,\sigma^k,E}} \\ &\leq e^{(\beta_0-\lambda(\theta))\frac{[t_0N]}{1-\beta_0-E}}e^{[tN]\psi_N(\lambda(\theta),\lambda(\theta),\theta)-2[tN]\phi_N(\beta_0)}.\end{aligned} \tag{90}$$

Hence, according to Proposition 1 we get

$$
\begin{aligned}
[t_0N]\frac{(\beta_0-\lambda(\theta))}{1-\beta_0-E} &+ [tN]\psi_N(\lambda(\theta),\lambda(\theta),\theta) \\
&= [t_0N]\frac{(\beta_0-\lambda(\theta))}{1-\beta_0-E} - \frac{[tN]}{2}\left[\log((1-\lambda(\theta))^2-\lambda(\theta)^2\theta^2)+O(1/N)\right] \\
&= [t_0N]\frac{(\beta_0-\lambda(\theta))}{1-\beta_0-E} - \frac{[tN]}{2}\left[\log((1-\beta_0)^2)+O(1/N)\right]
\end{aligned}
$$

where $O(1/N)$ is uniform in $\theta$. Meanwhile, due to the fact that $\lambda(\theta)$ solves (89) we have

$$
\beta_0-\lambda(\theta)\le\frac{\lambda(\theta)^2\theta^2}{2-\lambda-\beta_0}\le\frac{\beta_0^2\theta^2}{2(1-\beta_0)}.
$$

This, together with (12) and (90) yield (88). Now we can prove (48). Because of (5) we have

$$
\frac{\beta_0^2t}{(1-\beta_0-E)(1-\beta_0)}\le\frac{\beta_0^2t_0}{(1-\beta_0)^2}<1
$$

Thus, according to (88) there exists a c, $0<c<1$ such that

$$
\mathbb{E}\prod_{k=1}^2 e_N(t,\sigma^k)\mathbf{1}_{B_{N,t,\sigma^k,E}}\le e^{N\frac{c}{2}\theta^2}.
$$

Let us denote by $I$ the rate function governing the large deviations of $\theta=\frac{\sigma^1.\sigma^2}{N}$ under the uniform measure on $S_N^2$. We know that $I(u)=\frac{1}{2}u^2+o(u^2)$. Thus according to Varadhan's Lemma

$$
\begin{aligned}
\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}E_{\sigma^1,\sigma^2}\prod_{k=1}^2 e_N(t,\sigma^k)\mathbf{1}_{B_{N,t,\sigma^k,E}} &\le \sup_{|u|>\varepsilon}\left\{\frac{c}{2}u^2-\frac{1}{2}u^2\right\} \\
&= -h_{ii}(\varepsilon)
\end{aligned}
$$

where $h_{ii}$ is obviously a non-negative function. Defining $h_1(\varepsilon,E)=\min(h_i(E),h_{ii}(\varepsilon))$ ends the proof, since the bounds are uniform in t. ■

*Proof of Lemma 3*

The proof of Lemma 3 relies on the ideas already in use in the proof of Lemma 2. That is the reason why we will not give full details. Let $g$ be any continuous function defined on $[-1,1]^3$, with negative values and such that $g(0,0,0)=0$. Let us define $u(x,y,z)=E+g(x,y,z)$. For every $\sigma^1,\sigma^2,\sigma^3\in S_N$ we write, with $\Theta=(\theta^{1,2},\theta^{1,3},\theta^{2,3})$,

$$
\begin{aligned}
&\mathbb{E}e^{\beta_0\sum_{k=1}^3 H_N(t,\sigma^k)-3[tN]\phi_N(\beta_0)}\mathbf{1}_{\overline{B_{N,t,\sigma^1,E}}} \\
\le\ & e^{-\frac{u(\Theta)[t_0N]}{2(1-\beta_0-E)}}\mathbb{E}e^{(\beta_0+u(\Theta))H_N(t,\sigma^1)+\beta_0\sum_{k=2}^3 H_N(t,\sigma^k)-3[tN]\phi_N(\beta_0)}\mathbf{1}_{\overline{B_{N,t,\sigma^1,E}}}
\end{aligned}
$$

Notice that if $(\theta^{1,2},\theta^{1,3},\theta^{2,3}) = (0,0,0)$ then conditions (33) and (36) with $\beta_1 = \beta_0 + u(0,0,0)$ and $\beta_2 = \beta_3 = \beta_0$ are satisfied. Since $u$ is continuous there exists a $\rho_1 > 0$ small enough to ensure that if $\max_{i,j=1,2,3}|\theta^{i,j}| < \rho_1$ then (33) and (36) with $\beta_1 = \beta_0 + u(\Theta)$ and $\beta_2 = \beta_3 = \beta_0$. are satisfied. According to Proposition 1 we obtain

$$\begin{aligned}
& \mathbb{E}e^{(\beta_0+u(\Theta))H_N(t,\sigma^1)+\beta_0\sum_{k=2}^3 H_N(t,\sigma^k)-3[tN]\phi_N(\beta_0)}\mathbf{1}_{\overline{B_{N,t,\sigma^1,E}}} \\
\leq\ & \mathbb{E}e^{(\beta_0+u(\Theta))H_N(t,\sigma^1)+\beta_0\sum_{k=2}^3 H_N(t,\sigma^k)-3[tN]\phi_N(\beta_0)} \\
=\ & \exp\Big(-\frac{[Nt]}{2}\log\Big((1-\beta_0)^2(1-\beta_0-u(\Theta))-(1-\beta_0-u(\Theta))\beta_0^2(\theta^{2,3})^2 \\
& -(1-\beta_0)\beta_0(\beta_0+u(\Theta))((\theta^{1,3})^2+(\theta^{1,2})^2)-\beta_0^2(\beta_0+u(\Theta))\theta^{1,2}\theta^{1,3}\theta^{2,3}\Big) \\
& +\frac{3[Nt]}{2}\log(1-\beta_0)+O(1)\Big) \\
=\ & \exp-\Big(\frac{[Nt]}{2}\log\Big(\frac{1-\beta_0-u(\Theta)}{1-\beta_0}\Big)-\frac{[Nt]}{2}\log\Big(1+v(\Theta)\Big)+O(1)\Big)
\end{aligned}$$

where $v$ is a continuous function defined on $[-1,1]^3$ and vanishing at $(0,0,0)$. Thus

$$\begin{aligned}
& \mathbb{E}e^{\beta_0 H_N(t,\sigma^1)+\beta_0\sum_{k=2}^3 H_N(t,\sigma^k)-3[tN]\phi_N(\beta_0)}\mathbf{1}_{\overline{B_{N,t,\sigma^1,E}}} \\
\leq\ & \exp\Big(-\frac{[tN]}{2}\Big(-\frac{u(\Theta)}{1-\beta_0-E}-\log\Big(1+\frac{u(\Theta)}{1-\beta_0-u(\Theta)}\Big) \\
& +\log\Big(1+v(\Theta)\Big)+O(1)\Big)
\end{aligned}$$

Since

$$\frac{u(0,0,0)}{1-\beta_0-E}-\log\Big(1+\frac{u(0,0,0)}{1-\beta_0-u(0,0,0)}\Big)\Big)=\frac{E}{1-\beta_0-E}-\log(1+\frac{E}{1-\beta_0-E})>0$$

one can find $\rho_2 < \rho_1$ small enough to ensure that

$$\sup_{|\theta^{i,j}|<\rho_2}\frac{u(\Theta)}{(1-\beta_0-E)}-\log\Big(1+\frac{u(\Theta)}{1-\beta_0-u(\Theta)}\Big)+\log\Big(1+v(\Theta)\Big)>0$$

and this ends the proof of (50). Now we prove (51). As for the proof of (48), we first need to prove that there exists $\rho_3$ small enough to ensure that if $|\theta^{1,2}| < \rho_3$ and $|\theta^{1,3}| < \rho_3$ then

$$\mathbb{E}\mathbf{1}_{|\theta^{1,2}|<\rho_3,|\theta^{1,3}|<\rho_3}\prod_{k=1}^3 e_N(t,\sigma^k)\mathbf{1}_{B_{N,t,\sigma^k,E}} \leq e^{Nc/2(\theta^{2,3})^2} \tag{91}$$

where $0 < c < 1$. Let us denote by $\lambda(\Theta)$ the lowest root of the equation

$$\begin{aligned}
(1-\beta_0)^3 \ =\ & (1-\beta_0)(1-\lambda)^2-\lambda^2(1-\beta_0)(\theta^{2,3})^2 \\
& -\lambda(1-\lambda)\beta_0((\theta^{1,3})^2+(\theta^{1,2})^2)-2\lambda^2\beta_0\theta^{1,3}\theta^{1,2}\theta^{2,3}.
\end{aligned}$$

Notice that in view of (89), it suffices to make only $\theta^{1,2}$ and $\theta^{1,3}$ (not $\theta^{2,3}$) small in order to ensure that (92) admits solutions. For every $\sigma^1, \sigma^2, \sigma^3 \in S_N$ such that $|\theta^{1,2}| < \rho_3$ and $|\theta^{1,3}| < \rho_3$ we write

$$\begin{aligned}
& \mathbb{E} e^{\beta_0 \sum_{k=1}^3 H_N(t,\sigma^k) - 3[tN]\phi_N(\beta_0)} \prod_{k=1}^{3} \mathbf{1}_{B_{N,t,\sigma^k,E}} \\
\leq \;& e^{[tN]\frac{\beta_0-\lambda(\Theta)}{(1-\beta_0-E)}} \mathbb{E} e^{\beta_0 H_N(t,\sigma^1)+\lambda(\Theta)H_N(t,\sigma^2))+\lambda(\Theta)H_N(t,\sigma^3))-3[tN]\phi_N(\beta_0)} \\
\leq \;& e^{[tN]\frac{\beta_0-\lambda(\Theta)}{(1-\beta_0-E)}}
\end{aligned}$$

It is long but not difficult to prove that for any $\gamma > 0$ and $|\theta^{2,3}| > \gamma$ there exists $\rho_4(\gamma)$ such that for every $|\theta^{1,2}| < \rho_4$ and $|\theta^{1,3}| < \rho_4$ we get

$$[tN]\frac{\beta_0 - \lambda(\Theta)}{(1-\beta_0-E)} \leq N\frac{c}{2}(\theta^{2,3})^2.$$

The final expression (51) follows by the large deviation argument for Laplace integrals we already used to prove (48). ■

*Proof of Lemma 4*

The statement with $n = 0$ is nothing but an immediate consequence of the exponential decay of probabilities associated to the law of large numbers. The proof of the other statements work the same way. For the sake of shortness we will only prove (53) for $n = 2$. Let $p = p(\beta_0)$ be the quantity given by Proposition 1, and $q$ its conjugate. Due to Holder's inequality we have

$$\begin{aligned}
& \mathbb{E} e^{\frac{\beta_0}{2}(\frac{\sigma^1.\xi}{\sqrt{N}})^2 + \frac{\beta_0}{2}(\frac{\sigma^2.\xi}{\sqrt{N}})^2 - 2\phi_N(\beta_0)} \mathbf{1}_{\overline{C_{N,\sigma^1,\delta}}} \qquad \leq \\
& \qquad e^{-2\phi_N(\beta_0)} \Big( \mathbb{E} \exp\Big\{ \frac{N}{2}[p\beta_0(\frac{\sigma^1 \cdot \xi}{N})^2 + p\beta_0(\frac{\sigma^2 \cdot \xi}{N})^2] \Big\} \Big)^{1/p} \Big( \mathbb{P}(\overline{C_{N,\sigma^1,\delta}}) \Big)^{1/q}
\end{aligned}$$

The first two factors are bounded, and the last one decays exponentially fast to 0. ■

*Proof of Lemma 5*

It follows from its definition that $\tilde{Z}_N(t)$ is a strictly positive $\mathcal{F}_t$-martingale. Thus, $\tilde{Z}_N(t)^{-1}$ is a strictly positive sub-martingale and, according to Doob's inequality, for all $\eta > 0$ we get

$$\eta \mathbb{P}\Big( \sup_{0 \leq t \leq t_0} \tilde{Z}_N(t)^{-1} \geq \eta \Big) \leq \mathbb{E}\left[ \tilde{Z}_N(t_0)^{-1} \right]. \qquad (92)$$

We know from Theorem 1.1 in [28] that for every $N$

$$\mathbb{P}(\tilde{Z}_N(t_0)^{-1} > m) \leq K e^{-(\frac{\log m}{K})^2}.$$

Hence we have $\mathbb{E}[\tilde{Z}_N(t_0)^{-1}] < C < \infty$ and this together with (92) yield the announced result. ■

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