Exact Optimization for Markov Random Fields: Total Variation, Levelable and Convex cases

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Context

- Many computer vision/image proccessing problem can be expressed as an energy minimization problem
  - restoration
  - segmentation

$$E(u|v) = \underbrace{\int_\Omega D(u,v)}_{\text{Data fidelity}} + \beta \underbrace{\int_\Omega R(u)}_{\text{Régularisation}}$$

- High dimensional problem
- Generally non-convex
- Fast algorithms
- Exact solution

Context

- Optimization: continuous approaches
  - Gradient descent, Euler-Lagrange, [Rudin, Osher, Fatemi Phisica D. 199
  - Duality [Chambolle 2004 JMIV]
  - ”Graduated Non Convexity” (GNC) [Blake et Zisserman 1987]
- Optimization: Discrete approaches, Markov Random Fields (MRFs)
  - Dynamic programming (global minimizer) [Amini *et al* PAMI 1990]
  - Simulated Annealing (global optimizer) [Geman et Geman PAMI 1984]
  - Iterated Conditional Mode (local minimizer) [Besag JRSC 1986]
  - Graph cuts:
    - global optimum for some binary MRFs binaires [Greig *et al.* JRSC 1989]
    - Approximate solution [Boykov *et al.* PAMI 2001]
    - Exact solution [Ishikiwa PAMI 2003]
    - Exact solution for Convex MRFs[Kolmogorov TR 2005]
- Our approach:
  - Reformulate energies as one (or many) binary MRF(s)
  - Get global minimizer

Outline

- α) Notations
- *A*) Total Variation minimization with convex fidelity
- *B*) Levelable energies
- *C*) Markov Random Fields with Convex Priors
- *D*) $L^1 + TV$ on the FLST-tree

Notations

- Discretization

$$\begin{array}{cclc} s & \in & S & \text{finite discrete grid} \\ u_s & \in & [0, L-1] & \text{finite number of gray-levels} \\ s \sim t & \rightarrow & (s,t) & \text{neighbors} \;\rightarrow\; \text{cliques (C-connectivity)} \end{array}$$

- Level sets

$$u^{\lambda} = \{s \in S | \mathbb{1}_{u_s \leq \lambda}\}$$

- We consider level sets as variables

A) Total Variation minimisation with convex fidelity

- Convex problem
- Image restoration
- Reformulation through level sets
- Polynomial algorithm
- Results

TV: Reformulation through level sets

- Total Variation

$$\underbrace{\int_\Omega |\nabla u| = \int_{\mathbb{R}} P(u^\lambda)}_{\text{Co-area formula}} = \sum_{\lambda=0}^{L-2} P(u^\lambda) = \sum_{\lambda=0}^{L-2} \underbrace{\sum_{(s,t)} w_{st}\, |u_s^\lambda - u_t^\lambda|}_{R_{st}(u_s^\lambda, u_t^\lambda)}$$

- Data fidelity

$$D(u_s,\ v_s) = \sum_{\lambda=0}^{L-2} \underbrace{(D(\lambda+1, v_s) - D(\lambda, v_s))(1 - u_s^\lambda)}_{D^\lambda(u_s^\lambda, v_s)} + D(0, v_s)$$

$$\rightarrow E(u|v) = \sum_{\lambda=0}^{L-2} \underbrace{\left(R_{st}(u_s^\lambda, u_t^\lambda) + D^\lambda(u_s^\lambda, v_s)\right)}_{\text{binary MRF}} + C = \sum_{\lambda=0}^{L-2} E^\lambda(u^\lambda, v)$$

TV: Independent minimization and reconstruction

- Minimize (MAP) independently each binary MRF
  - $E(u|v) \to E(\{u\}^\lambda, v)$
  - Family of minimizers: $\{\hat{u}^\lambda\}_{\lambda=0\ldots(L-2)}$
- Reconstruction: $\hat{u}_s = \inf\{\lambda | \mathbb{1}_{\hat{u}_s^\lambda} = 1\}$ provided that

$$u_s^\lambda \leq u_s^\mu \quad \forall \lambda < \mu \quad \forall s \quad (monotony)$$

monotone lemma

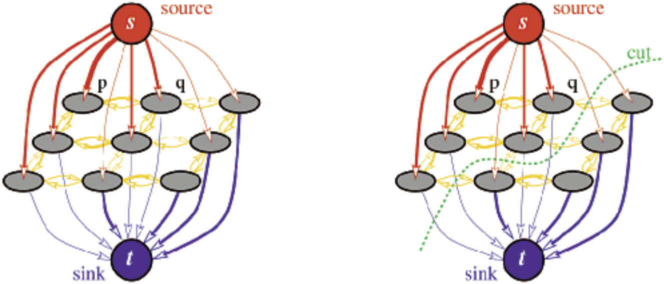
If $E(u_s \mid \{u_t\}_{t\sim s},\ v_s) = \sum_{\lambda=0}^{L-2} (\ \Delta\phi_s(\lambda)\ u_s^\lambda\ +\ \chi_s(\lambda)\ )$ *where*

$\Delta\phi_s(\lambda) \nearrow$ *of* λ *and* $\chi_s(\lambda)$ *is* independent *of* u_s^λ,

$\Rightarrow$ *Then monotony is preserved.*

"convex+TV" models satisfies lemma's conditions

- How to minimize a binary Markovian energy
- Build a graph such that its minimum cost cut yields an optimal labelling



  - construction of the graph: [*Kolmogorov and Zabih*, PAMI 2004]
  - minimum cost cut algorithm: [*Boykov and Kolmogorov*, PAMI 2004]
  - in practice quasi-linear (w.r.t number of pixels)

TV: Graph construction conditions

- Regularity conditions described in [*Kolmogorov and Zabih*, PAMI 2004]
- Binary Markovian energy with pairwise interaction

$$E(x_1, \ldots, x_n) = \sum_i E^i(x_i) + \sum_{i<j} E^{i,j}(x_i, x_j)$$

- $E^i(x_i)$: always regular
- $E^{i,j}(x_i, x_j)$: regular iff submodular, i.e.

$$E^{i,j}(0,0) + E^{i,j}(1,1) \leq E^{i,j}(1,0) + E^{i,j}(0,1)$$

- TV case: $\sum_{st} w_{st} |u_s - u_t|$

$$0 \leq w_{st} \quad \text{ok}$$

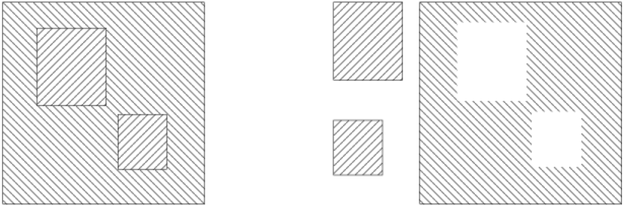
TV: Minimization algorithm

- Decomposition through level sets (recall)

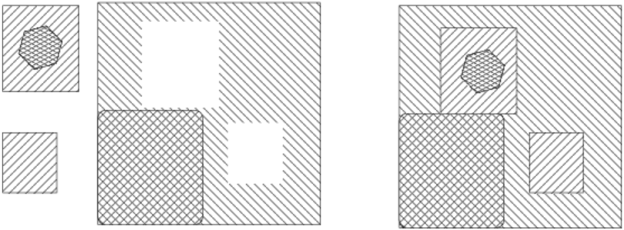
$$E(u|v) = \sum_{\lambda=0}^{L-2} E^{\lambda}(u^{\lambda}|v)$$

- Direct approach $\Longrightarrow$ $(L-1)$ minimum cost cuts per pixel
- A divide-and-conquer algorithm with dichotomy
  - decompose into independent subproblems
  - solve each subproblem
  - recompose the solution

- **Decomposition**: Solve for a level λ ; connected components



- **Solving sub-problems and recomposition**



- Thresholding : dichotomy on $[0, L-1] \Longrightarrow \log_2(L)$ minimum cost cuts per pixel

TV: Results

TV: Results, additive Gaussian noise

$\mu = 0, \sigma = 12$

restored image ($\beta = 23, 5$)

TV: Results, additive Gaussian noise

$\mu = 0, \sigma = 20$

restored image ($\beta = 44, 5$)

TV: Results, time




TV: Results, time

- size (512×512) ; L^2; time in seconds

Image	$\beta = 2$	$\beta = 5$	$\beta = 10$	$\beta = 20$	$\beta = 30$	$\beta = 40$
Lena	2,07	2,24	2,53	3,04	3,40	3,75
Aerien	2.13	2.24	2.45	2.75	3.06	3.28
Barbara	2.07	2.26	2.51	2.87	3.22	3.50

- size (256×256) ; L^2; time in seconds

Image	$\beta = 2$	$\beta = 5$	$\beta = 10$	$\beta = 20$	$\beta = 30$	$\beta = 40$
Lena	0,51	0,54	0,60	0,72	0,8	0,87
Aerien	0,55	0,57	0,61	0,67	0,74	0,78
Barbara	0,53	0,55	0,60	0,69	0,75	0,80
Girl	0,52	0,55	0,64	0,75	0,85	0,92

Experiments performed on a Pentium 4 3 GHz

TV: Partial conclusion

- Exact solution for ”convex+TV” models
- Reformulation through level sets
- Polynomial algorithm
- Complexity $\log_2(L) \cdot T(n, (C+2)n)$
- Similar algorithm proposed by
  - Chambolle [Chambolle CMAP 2005]
  - Hochbaum [Hochbaum ACM 2001]

where $T(n, m)$ is the time required to performed a minimum cost cut on a graph of n nodes and m edges.

Energies nivelées

A) Total Variation minimization with convex fidelity

B) Generalization to levelable energies (includes ”convex+TV”)

- Definition and caracterization
- Results
- Links with mathematical morphology

- **Idea**: Generalization of the decomposition on levels ets
- **Goal**: caracterize the class of energies such that

$$E(u|v) = \sum_{\lambda=0}^{L-2} E^{\lambda}(u^{\lambda} \mid v) + C$$

Définition

A function is levelable iff

$$f(x, y \ldots) = \sum_{\lambda=0}^{L-1} \psi(\lambda, \mathbb{1}_{\lambda<x}, \mathbb{1}_{\lambda<y} \ldots)$$

Levelable energies: Definition et caracterization

- every function of a single variable is levelable.
- global criteria / local criteria

Proposition

The total energy is levelable

$$\Leftrightarrow$$

Every energy associated to a clique is a levelable function

Proposition

$U(x,y) = U(y,x)$, *is levelable iff*

$$\begin{aligned} U(x,y) &= S(\max(x,\ y)) - S(\min(x,\ y)) + D(x) + D(y) \\ &= f(\max(x,\ y)) - g(\min(x,\ y)) \ , \end{aligned}$$

Levelable energies: Definition et caracterization

Proposition

Assumptions:

1. $U(x,\ y) = U(y,\ x)$ *levelable*
2. $\forall y \in [0, L-1]$, $U(x,y)$ *reaches its minimum for* $x = y$.

Then $U(x,\ y) = |S(x) - S(y)| + D(x) + D(y)$
with S, $S + D$, $S - D$ $\nearrow$

- Goal reached

$$E(u|v) = \sum_{\lambda=0}^{L-2} \left\{ \underbrace{\sum_{(s,t)} \overbrace{R_{st}(\lambda)}^{\geq 0} \mid u_s^\lambda - u_t^\lambda \mid + \sum_s \delta(\lambda, v_s)\,(1 - u_s^\lambda)}_{E^\lambda(u^\lambda, v)} \right\} + C$$

Levelable energies: Exact minimisation

- New equivalent energy (i.e, same solutions)

$$E(\{u\}^{\lambda}|v) = \sum_{\lambda=0}^{L-2} E^{\lambda}(u^{\lambda}|v) + \sum_{s} \underbrace{\alpha H(u_s^{\lambda} - u_s^{\lambda+1})}_{monotony}$$

where $H(\cdot)$ is the Heaviside function

- Graph representation (submodularity)

$$H(0,0) - H(1,1) \leq H(1,0) + H(0,1)$$

$$0 \leq 1 \;\; ok$$

Levelable energies: impulsive noise restoration, TV


20% corrupted pixels

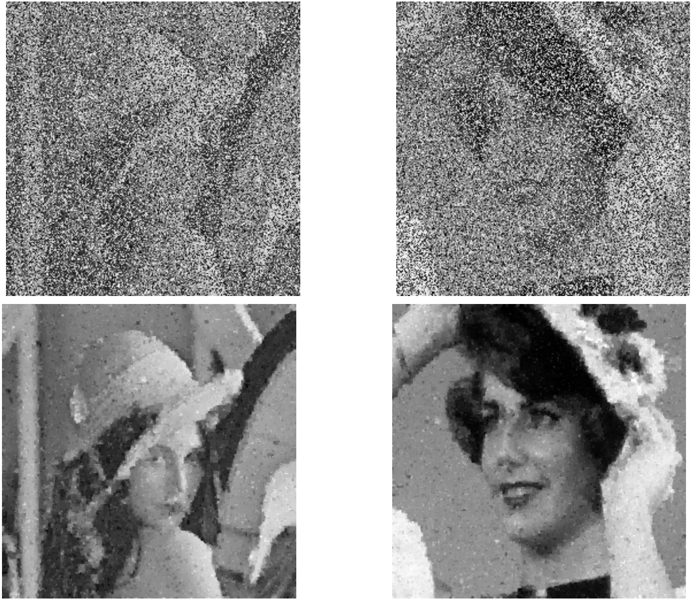

$\beta = 0,20$

Levelable energies: impulsive noise restoration, TV

40% corrupted pixels

$\beta = 0,20$

Levelable energies: impulsive noise restoration, TV



70% corrupted pixels

$\beta = 0,40$

levelable energies: Time complexity

- Complexity is pseudo-polynomial

$$T(nL, nCL)$$

- To be polynomial $L \to \log_2(L)$

- Other available minimization algorithm [*Ishikawa*, PAMI 2003]
- size of images 256 $\times$ 256 ; time in seconds (Pentium 4 3GHz)

Image	p	Levelable	Ishikawa	ratio
Lena	0,20	114,78	425,14	3,70
Lena	0,40	159,14	633,09	3.98
Lena	0,70	252,67	1203.22	4.76
Girl	0,20	114,09	469,44	4.11
Girl	0,40	171,68	648,72	3.78
Girl	0,70	272,72	1553,66	5.70

Links with mathematical morphology

$L^1 + TV$ is invariant with change of contrast

Definition and lemma

• *A continuous and non-decreasing function* $h : \mathbb{R} \mapsto \mathbb{R}$ *, is called a continuous change of contrast.*
•*A filter* $\mathcal{T}$ *is invariant w.r.t. a change of contrast iff it satisfies:*

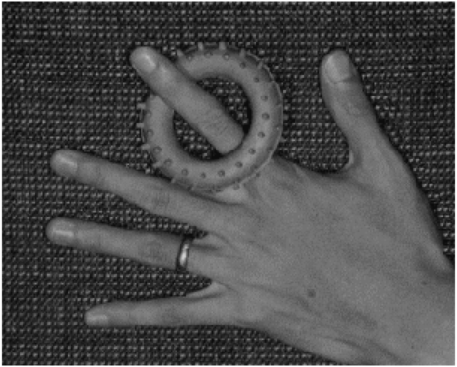
$$h(\mathcal{T}(u)) = \mathcal{T}(h(u)) \ ,$$

where u is an image and h a change of contrast.

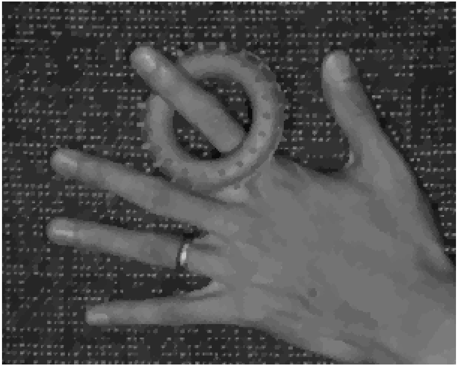
IF $\hat{u}$ minimizer for $E^{L^1+TV}(\cdot|v)$

then $h(\hat{u})$ minimizer for $E^{L^1+TV}(\cdot|h(v))$

Levelable energies: Mathematical morphology



original



$\beta = 1,5$

Levelable energies: Mathematical morphology

$\beta = 2,5$

$\beta = 3,0$

Levelable energies: Partial conclusion

- Exact minimization for ”any + levelable”
- Includes non-convex energies
- Pseudo-polynomial complexity : $T(nL, CnL)$
- However $U(x, y) = (x - y)^2$ is not levelable

MRFs: Convex priors

A) Total Variation minimization with convex fidelity

B) Generalization to levelable energies

C) Generalization to Markovian energies with convex priors

- Convex priors
- Convex MRFs

MRF with convex priors

- Convex priors

$$E(u|v) = \sum_s f_s(u_s, v_s) + \sum_{t \sim s} g_{st}(|u_s - u_t|)$$

- Reformulation

$$g(k,l) = \sum_{\mu=0}^{L-2} \sum_{\lambda=0}^{L-2} \underbrace{G(\lambda,\mu)}_{\leq 0 \text{ since } g \text{ convex}} \quad (1 - k^\lambda)(1 - l^\mu) \quad + \dots$$

- Regularity

$$G(\lambda,\mu) = 2g(\lambda - \mu) - g(\lambda - \mu + 1) - g(\lambda - \mu - 1) \leq 0 \quad \textit{ok}$$

Convex MRFs: Proximity theorem

- Fidelity and priors are convex
- Norm L^∞ on images

Proposition

Let u be an image such that

$$E(u|v) > \min_u E(u|v) \ .$$

There exists $\hat{u}$ a global minimizer for $E(\cdot|v)$ and $\delta \in \{-1, 0, 1\}^{|S|}$ such that

$$E(u|v) > E(u+\delta|v) \ .$$

Besides we have

$$\|(u+\delta) - \hat{u}\|_\infty = \|u - \hat{u}\|_\infty - 1 \ .$$

Convex MRFs: Discrete steepest descent

- $\forall s \rightarrow u_s = u_s + b_s d$
  b_s binary variable
  d = moving direction

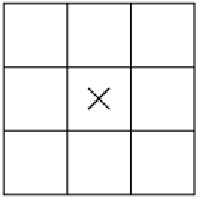
$$E(\{b_s\}|v) = \sum_s f_s(u_s + b_s d), v_s) + \sum_{t \sim s} g_{st}(u_s - u_t + b_s d - b_t d)$$

$\rightarrow$ submodular

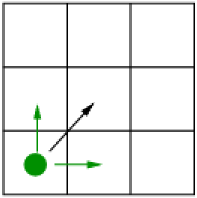
- How computing a discrete steepest descent with only one direction ?

Convex MRFs: Discrete steepest descent

- Use two minimum cost cuts



1



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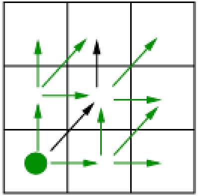

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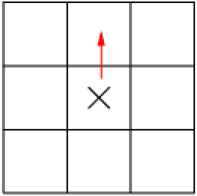


4

Convex MRFs: Discrete steepest descent

- Requires $\frac{L}{2}$ steepest descents
- Pseudo polynomial complexity: $L \cdot T(n, (C+2)n)$
- Similar to the approach of
  - Bioucas Dias *et al.* [Bioucas 05 ibpria]
  - Murota [Murota SIAM book 2003]
- This is Primal algorithm of Kolmogorov [Kolmogorov 05 TR]

Only proofs are different

- How to speedup ?

Convex MRFs: Discrete steepest descent (Scaling)

- Scaling of a function (with a non negative integer)

$$f^n(x) = f(nx) \ .$$

- Scaled convex MRFs remains convex

$$E^n(u^n|v) = \sum_s f_s^n(u_s^n, v_s) + g_{st}^n(u_s^n - u_t^n) \ ,$$

- Heuristics:
  - Minimize once with geometrically decreasing steps: $d = 2^k$, 2^{k-1}, $\ldots$, 2^0
  - Then minimize with step 1 until convergence
  - In practice quasi $log_2 L$ steepest descents for image restoration models

Convex MRFs: Results

$\mu = 0, \sigma = 12$

$\beta = 15$, $|\nabla \cdot|^{1,2}$

$\mu = 0, \sigma = 20$

$\beta = 30$, $|\nabla \cdot|^{1,2}$

Convex MRFs: Results

TV

$|\nabla \cdot|^{1,2}$

Convex MRFs: Time results

Experiments performed on a Pentium 4 3 GHz

Image (256^2)	$\beta = 5$	$\beta = 10$	$\beta = 20$	$\beta = 30$	$\beta = 40$	$\beta = 50$
girl (h)	1,61	1,86	2,26	2,57	2,82	2,98
lena (h)	1,62	1,88	2,24	2,49	2,71	2,90
barbara (h)	1,58	1,80	2,12	2,38	2,58	2,75

Image (512^2)	$\beta = 5$	$\beta = 10$	$\beta = 20$	$\beta = 30$	$\beta = 40$	$\beta = 50$
lena (h)	6,22	7,34	8,94	10,14	11,21	12,19
aérien (h)	5,93	6,84	8,10	9,02	9,77	10,46
barbara (h)	6,05	7,01	8,54	9,62	10,63	11,43

Image (512^2)	$\beta = 5$	$\beta = 10$	$\beta = 20$	$\beta = 30$	$\beta = 40$	$\beta = 50$
lena (h)	6,22	7,34	8,94	10,14	11,21	12,19
lena (1)	101,59	121,00	145,21	163,01	177,0	189,67

(1) $\rightarrow$ algorithm with a step of 1

(h) $\rightarrow$ algorithm with heuristic

For $L = 256 \rightarrow$ ratio $\simeq 15$

Convex MRFs: Partial conclusion

- ”any+ convex” models
  - Includes non-convex energies
  - Exact minimization
  - Pseudo polynomial complexity : $T(nL, CnL^2)$
- ”convex + convex” models
  - Exact minimization
  - Pseudo polynomial complexity : $L \cdot T(n, (C+2)n)$
  - With scaling heuristic: tends to be quasi $2\log_2 L \cdot T(n, (C+2)n)$ in practice for image restoration models

$L^1 + TV$ on the FLST-tree

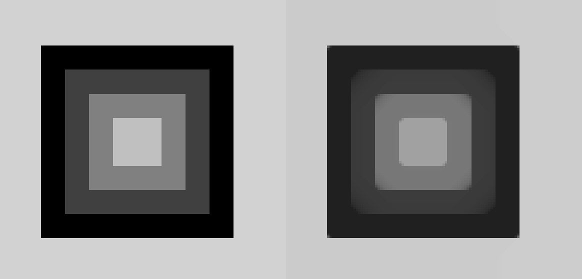
A) Total Variation minimization with convex fidelity

B) Levelable energies

C) Markov Random Fields with Convex Priors

D) $L^1 + TV$ on the FLST-tree

L^2 + TV, contrast and contours



- Loss of contrast $\rightarrow$ use L^1
- How to preserve contours ?

Fast Level Set Transform tree

- Level Sets
  $L^\lambda(u) = \{x \in \Omega | u(x) \leq \lambda\}\ , U^\lambda(u) = \{x \in \Omega | u(x) > \lambda\}$
- Inclusion property

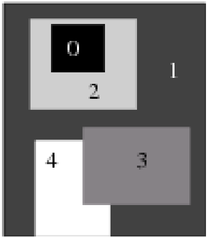
$$U^\lambda(u) \subset U^\mu(u)\ \forall \lambda \geq \mu$$

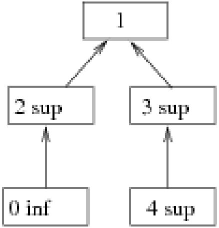
$$L^\lambda(u) \subset L^\mu(u)\ \forall \lambda \leq \mu$$

- Induce a tree [Salembier *et al.* ITIP 98] :
  - conected components of lower sets
  - $L^\lambda \rightarrow$ "Min-Tree" $\rightarrow$ dark objects on light background
  - $U^\lambda \rightarrow$ "Max-Tree" $\rightarrow$ light object on dark background
- "Fast Level Set Transform" (FLST) [Monasse *et al* ITIP 2000]
  - Merge the 2 trees into a single one
  - Need of a criteria : Holes

Fast Level Set Transform Tree

- Shapes = connected components of level sets whose holes have been filed.
- Definition of the tree
  - 1 node = 1 shape
  - Parent = smallest form which contains it
  - children = included forms
- Decomposition of the image into forms S_1 ... S_n





Fast Level Set Transform Tree

- $L^1 + TV$ on the FLST tree
  equivalent to
  $L^1 + TV$ + edge preservation
- Attributes associated to each node
  - gray level u_i
  - area for data fidelity $\rightarrow |D_i|$
  - perimeter for TV (co-aire formula) $\rightarrow P_i$
- Data fidelity:

$$\sum_{i=1}^{N} |D_i||u_i - v_i|$$

- Total Variation:

$$\sum_{i=1}^{N-1} P_i|u_i - u_i^p|$$

$L^1 + TV$ sur l'arbre de la FLST

- Finally

$$E^{L^1+TV}(u|v) = \sum_{i=1}^{N} |D_i||u_i - v_i| + \beta \sum_{i=1}^{N-1} P_i|u_i - u_i^p|$$

- Sites: nodes the tree
- Neighborhoods $\rightarrow$ parents et children
- pairwise interactions
- MAP of the Markovian energy$L^1 + TV$
  $\rightarrow L^1 + TV$ algorithm of the first part

Résultats

Original image

$\beta = 3$

Résultats

$\beta = 15$ borders of the resul

5 regions

Résultats

Original image

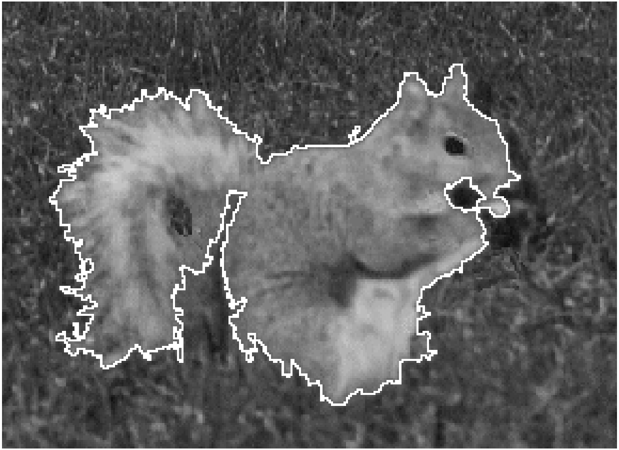
Pertinent contours ?

Résultats

$\beta = 1$

Résultats

$\beta = 2$



borders of the result ($\beta = 10$) superimposed on the original image

2 regions

Résultats, temps de calcul

Image	FLST	Minimization
Lena (256x256)	0.18	0.11
Lena (512x512)	1.09	1.04
Woman (522x232)	0.39	0.06
Squirrel (209x288)	0.24	0.19

FLST tree computed with the implementation available in Megawave

(ENS de Cachan)

Partial conclusion

- Contrast preservation (since it is morphological)
- Edge preservation
- Fast
- Good simplification for future segmentaion

Conclusion

- Exact optimization for
  - Convex + TV (polynomial)
  - Convex + Convex (pseudo-polynomial)
  - Any + Levelable (pseudo-polynomial)
  - Any + Convex (pseudo-polynomial)
- $L^1 + TV$ is invariant with respect to changes of contrast