Denoising Tensors via Lie Group Flows

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Abstract. The need to regularize tensor fields arise recently in various applications. We treat in this paper tensors that belong to matrix Lie groups. We formulate the problem of these $SO(N)$ flows in terms of the principal chiral model (PCM) action. This action is defined over a Lie group manifold. By minimizing the PCM action with respect to the group element, we obtain the equations of motion for the group element (or the corresponding connection). Then, by writing the gradient descent equations we obtain the PDE for the Lie group flows. We use these flows to regularize in particular the group of N-dimensional orthogonal matrices with determinant one i.e. SO(N). This type of regularization preserves their properties (i.e., the orthogonality and the determinant). A special numerical scheme that preserves the Lie group structure is used. However, these flows regularize the tensor field isotropically and therefore discontinuities are not preserved. We modify the functional and thereby the gradient descent PDEs in order to obtain an anisotropic tensor field regularization. We demonstrate our formalism with various examples.

1 Introduction

For more than a decade PDE's are widely used to tackle many image processing problems such as image restoration, segmentation, image enhancing and much more. Especially interesting are the nonlinear PDE's which in the context of image restoration has been proved to have remarkable denoising, deblurring as well as edges preserving properties. We will mention some of these works such as the pioneering work by Perona and Malik [19] on image denoising, the work by Osher and Rudin [16] on image enhancement and many others which are discussed extensively in [1, 31, 11, 23]. Some of the image processing problems may be formulated in terms of Lagrangian actions where the variation of the Lagrangian leads to the equations of motion. The gradient descent equations then defines the PDE's that we wish to apply to images in order to obtain the desired result [15, 21] (i.e, segmentation, denoising, etc).

Earlier studies dealt with scalar valued images. It was later generalized to vector-valued images (see for example [35, 2, 24, 22, 29] and references therein). Works on constrained regularization of vector-valued image were treated in the literature as well in [20, 28, 3, 26, 12].

In the last years new methods which consider tensor-valued images have emerged. In these new methods at each point of the two (or three) dimensional image space a tensor is attached rather than a scalar or a vector. This tensor field might be noisy and therefore one has to regularize it in order to extract its original texture. Moreover, the tensor field has certain properties that we wish to preserve along the flow (e.g., orthogonality, unit norm, etc) and it might lie on a non-flat manifold. In order to regularize these fields and preserve their original properties one has to adopt new methods, both analytical and numerical.

A solution to the problem of orthogonal tensor field regularization was proposed by Deriche et al. [30, 5]. In their formalism, the orthogonality of the tensor field is preserved by adding a constraint term to the *unconstraint* gradient descent equation using Lagrange multipliers. The constraint term preserve the orthonormality of the vector basis along the flow. The unconstraint gradient descent equation was obtained by minimizing the unconstrained ϕ functional. Different methods for regularization of tensor fields were proposed recently by [32, 4, 18, 17, 13]. Their work is mainly in relation with the DT-MRI application.

In this work we suggest a novel and natural framework to the problem of tensor field regularization. We assume here that the tensor at each point is a Lie group element and construct a regularization flow that respects the group's structure. The constrained gradient descent equation will be derived directly from a Lagrangian without any additional constraint e.g. without a lagrange multiplier. The functional is defined directly on the Lie group manifold. In order to solve the PDE numerically such that the Lie group field evolves on the group manifold we use the Lie group integrating methods introduced in [8, 10]. Our main example is the $SO(N)$ group which is of relevant in DT-MRI yet the formalism is of general applicability.

The plan of the paper is as follows: In section 2 we will give some mathematical preliminaries that will be used in this work. In section 3 we present the generalized Principal Chiral Model (PCM) action and derive the gradient descent equations. The gradient descent equations for this action will define the PDE flow on the group manifold. In section 4 we describe how to implement the flow to evolve on group manifold in general and on $SO(N)$ in particular. We will present and use modern Lie-group numerical integration methods. Results are given in section 5 where we demonstrate regularization of noisy three-dimensional orthogonal tensor field. Finally, concluding remarks are given in section 6.

2 A bit about Lie groups

For our discussion it is essential to introduce some of the basic definitions considering Lie groups and Lie algebra.

Definition 1 *A* Lie group *is a group $\mathcal{G}$ which is a differentiable manifold equipped with smooth product $\mathcal{G} \times \mathcal{G} \mapsto \mathcal{G}$.*

Definition 2 *The* Lie algebra $\mathfrak{g}$ *of Lie group G is defined as the linear vector space of all tangent vectors to G at the identity. This tangent space is denoted $T_I\mathcal{G}$.*

Definition 3 *A real matrix Lie group is a smooth subset $\mathcal{G} \subseteq \mathrm{I\!R}^{N\times N}$ closed under matrix product and matrix inversion. The identity matrix is denoted $I \in \mathcal{G}$.*

Definition 4 *A Lie algebra of a matrix Lie group is a linear subspace $\mathfrak{g} \subseteq \mathrm{I\!R}^{N\times N}$ equipped with the operation $\mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ which is the Lie bracket (the commutator) $[A, B] = AB - BA$. This operation is bilinear, skew-symmetric ($[A, B] = -[B, A]$), and satisfies the Jacobi identity*

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0. \tag{1}$$

Definition 5 *The elements which span the Lie algebra space are called the generators of the Lie group or the infinitesimal operators of the group. Let t_a, t_b and t_c be the generators of the Lie group, then their algebra is close under the commutator operation*

$$[t_a, t_b] = f^c_{ab} t_c, \tag{2}$$

where f^c_{ab} are the structure constants of the group and are antisymmetric in their lower indices $f^c_{ab} = -f^c_{ba}$.

We will demonstrate our study on the special orthogonal matrix Lie group, $SO(N)$. Its elements are $N \times N$ orthogonal matrices with determinant one. This group is a subgroup of $O(N)$ which is the orthogonal group and its elements are $N \times N$ orthogonal matrices. The Lie algebra of $SO(N)$ and $O(N)$ is denoted $\mathfrak{so}(n)$ and consists of $N \times N$ skew-symmetric matrices. $O(N)$ and $SO(N)$ are special cases of quadratic Lie group which takes the form

$$\mathcal{G} = \{X | X^T P X = P\}, \tag{3}$$

where P is a constant matrix (for $O(N)$ and $SO(N)$, P is identity matrix). The corresponding Lie algebra is given by $\mathfrak{g} = \{A | PA + A^T P = 0\}$.

In order to map elements of the Lie algebra into the Lie group one may use the following maps

Definition 6 *The exponential mapping $expm : \mathfrak{g} \mapsto \mathcal{G}$ is defined as*

$$expm(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}, \tag{4}$$

where $expm(0) = I$. Note that for A which is sufficiently near $o \in \mathfrak{g}$ the exponential mapping has a smooth inverse given by the matrix logarithm $logm : \mathcal{G} \mapsto \mathfrak{g}$.

For quadratic groups one may also use the Cayley mapping

Definition 7 *The Cayley mapping $Cay : \mathfrak{g} \mapsto \mathcal{G}$ is defined as*

$$Cay_\rho(x) = (I - \rho x)^{-1}(I + \rho x), \tag{5}$$

where ρ is a non-negative constant. When $\rho = 1/2$ the Cayley map is a special case of the Padé approximant to the exponential, $Cay_{1/2}(x) = expm(x) + O(x^3)$.

Definition 8 *The inverse of the Cayley mapping* $invcay : \mathcal{G} \mapsto \mathfrak{g}$ *is defined as*

$$invcay_\rho(X) = \frac{1}{\rho}(I + X)^{-1}(X - I). \tag{6}$$

Note that if X has an eigenvalue -1, this transform is undefined.

Definition 9 *The adjoint representation, Ad, and its derivative, ad, are given by the formulae*

$$Ad_A(B) = BAB^{-1}, \tag{7}$$

$$ad_A(B) = [A, B] = AB - BA. \tag{8}$$

3 The Generalized Principal Chiral Model

The principal chiral models (PCMs) which are known also as the sigma models arise in many branches of physics (e.g., classical and quantum physics, condensed matter, high-energy physics, etc...). These models are known to be integrable [33, 34, 7]. We consider a variation of the sigma models which is the generalized principal chiral model (GPCM) and is given by the action [26, 9]

$$\mathcal{L} = \int d^2x\, \eta^{\mu\nu} H_{ab}(g)(g^{-1}\partial_\mu g)^a (g^{-1}\partial_\nu g)^b, \tag{9}$$

where g takes values in the Lie group $\mathcal{G}$, η is the spatial metric and $H_{ab}(g)$ is invertible symmetric $dim\mathcal{G} \times dim\mathcal{G}$ matrix such that

$$H_{ab}(g) = H(g)K_{ab}, \tag{10}$$

where K_{ab} is the bilinear Killing form

$$K_{ab} = Tr(t_a t_b), \;\; t_a, t_b \in \mathfrak{g}. \tag{11}$$

The bilinear form is considered as the metric over the Lie group manifold.

Since we are interested in tensor fields which are attached to two-dimensional flat image space, we will take the metric $\eta^{\mu\nu}$ to be the Euclidean metric $\eta^{\mu\nu} = \delta^{\mu\nu}$. The integration is taken over the two-dimensional image space. The term $A_\mu = g^{-1}\partial_\mu g$ is known as the flat-connection and also as the Yang-Mills gauge field. The flat-connection is an element of the Lie algebra and therefore it may be represented in terms of the generators of the Lie algebra such that

$$A_\mu = g^{-1}\partial_\mu g = A^a_\mu t_a. \tag{12}$$

Also, it obeys the Bianchi identity

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0. \tag{13}$$

In order to obtain the equations of motion we vary the GPCM action with respect to $g^{-1}\delta g$ to obtain

$$-H^{ad}\frac{\delta\mathcal{L}}{\delta\rho^d} = \partial_\mu A^{\mu a} + \Gamma^a_{bc} A^b_\mu A^{\mu c} = 0, \tag{14}$$

where we have used the relation

$$\delta A^a_\mu = \partial_\nu \delta\rho^a - f^a_{bc} A^b_\nu \delta\rho^c, \tag{15}$$

and where $\delta\rho = g^{-1}\delta g$. The connection Γ^a_{bc} is a sum of two parts

$$\Gamma^a_{bc} = S^a_{bc} + \gamma^a_{bc}, \tag{16}$$

where S^a_{bc} is defined as

$$\begin{aligned} S^a_{bc} &= \frac{1}{2}(F^a_{bc} + F^a_{cb}), \\ F^a_{bc} &= (H^{-1})^{ap} f^q_{pb} H_{qc}. \end{aligned} \tag{17}$$

The second part are the Christoffel symbols for the metric H_{ab}

$$\gamma^a_{bc} = \frac{1}{2}(H^{-1})^{ad}(\partial_b H_{cd} + \partial_c H_{bd} - \partial_d H_{bc}). \tag{18}$$

Taking H_{ab} to be constant on the group manifold (i.e., $H_{ab} = K_{ab}$) we have $\gamma^a_{bc} = 0$. The bilinear form over the $SO(N)$ group manifold, for example, is negative definite and is given by $K_{ab} = Tr(t_a t_b) = -2\delta_{ab}$. Plugging H_{ab} into Eq. (17) we have,

$$F^a_{bc} = 2(f^a_{bc} + f^a_{cb}). \tag{19}$$

However, since the structure constants are antisymmetric in their indices (i.e., $f^a_{bc} = -f^a_{cb}$), $F^a_{bc} = 0$ and we are left with the equation of motion

$$\partial_\mu A^{\mu a} = 0. \tag{20}$$

Contracting this equation with the group generators t_a from the right we have

$$\partial_\mu A^{\mu a} t_a = \partial_\mu A^\mu = 0. \tag{21}$$

Since $A^\mu = g^{-1}\partial^\mu g$ we may write the equation of motion in the following form

$$\partial_\mu(g^{-1}\partial^\mu g) = 0. \tag{22}$$

In order to write the gradient descent equations we have to remember that the term $\partial_\mu(g^{-1}\partial^\mu g)$ is in the Lie algebra and therefore the left hand side (LHS) of the gradient descent equation should contain a term which is also in the Lie algebra. Therefore, we suggest the following expression

$$\begin{aligned} g^{-1}\frac{\partial g}{\partial t} &= \partial_\mu(g^{-1}\partial^\mu g) \\ &= \partial_x(g^{-1}\partial_x g) + \partial_y(g^{-1}\partial_y g). \end{aligned} \tag{23}$$

where $\left(g^{-1}\frac{\partial g}{\partial t}\right) \in \mathfrak{g}$.

Multiplying by g from the left of both sides we have

$$\frac{\partial g}{\partial t} = g\partial_\mu(g^{-1}\partial^\mu g). \tag{24}$$

This equation has the form of an orthogonal *ODE* flow

$$\frac{\partial g}{\partial t} = ga, \tag{25}$$

where $g \in \mathcal{G}$ and $a \in \mathfrak{g}$. The numerical solutions of this type of equation were discussed in [10, 8]. However, since a and g in our case depend on the spatial coordinates as well as the time, our equation is an orthogonal *PDE* flow. Note that $\frac{\partial g}{\partial t} \in T_g\mathcal{G}$ and the RHS (right hand side) lies also in $T_g\mathcal{G}$ since it is a left-trivialization form of the tangent written as ga. In [10, 8] the tangent is written in its right-trivialization form $ag \in T_g\mathcal{G}$. A *different* orthogonal PDE flow which has the same form was discussed in [30, 5].

4 Implementation

The implementation of Eq. (24) is not straightforward. In order to get the desired results, the flow has to evolve on the group manifold. This means that the group element g has to preserve its properties (i.e., orthogonality and unit determinant) for every time t. Since the group manifold is not a linear space, we cannot use classical PDEs integration schemes since the group structure will not be preserved along the flow. For the same reason we cannot use finite-difference schemes in order to evaluate the spatial derivatives. Therefore, the first challenge is to find a scheme which enables to evaluate the spatial derivative such as $\partial_\mu g \in T_g\mathcal{G}$ and $g^{-1}\partial_\mu g \in \mathfrak{g}$ for any number of iterations.

This goal is achieved by using the exponential mapping in order to express the Lie group element in terms of the Lie algebra. Then, the spatial derivative of the group element reads $\partial_\mu exp(a)$ where now we have to evaluate the derivative of the exponent. For the scalar case $a \in \mathbb{R}$ and for Abelian groups (where $a, \tilde{a} \in \mathfrak{g}$ commute) the formula for the derivative of the exponent is $\frac{d}{dx}exp(a(x)) = a'(x)exp(a(x))$. However, this formula does not hold for non-Abelian groups such as $SO(N)$ with $N > 2$ since $[a, \tilde{a}] \neq 0$. Therefore, one should apply a different formula.

The correct formula may be written in terms of the *dexp* functionsuch that

$$\frac{\partial}{\partial x}exp(a(x,t)) = dexp_{a(x,t)}a'(x,t)exp(a(x,t)), \tag{26}$$

where a' is the derivative with respect to the spatial coordinate. The *dexp* function is defined as a power series as follows

$$dexp_A B = B + \frac{1}{2!}[A, B] + \frac{1}{3!}[A, [A, B]] + \frac{1}{4!}[A, [A, [A, B]]] + \ldots = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}ad_A^k B. \tag{27}$$

As we have mentioned earlier, the derivative of the exponential mapping should lie in $T_{g(x,t)}\mathcal{G}$. In the right-trivialization form, the tangent may be written as $a(x,t)g(x,t)$ which is exactly the expression in the RHS of Eq. (26). However, since in Eq. (24) the tangent is written in its left-trivialized form $g(x,t)a(x,t) \in T_{g(x,t)}\mathcal{G}$, we should use the left-trivialized version of Eq. (26) which takes the form [10]

$$\frac{\partial}{\partial x} exp(a(x,t)) = exp(a(x,t)) dexp_{-a(x,t)} a'(x,t), \tag{28}$$

where the sign of the commutators in the *dexp* series has been changed by adding a minus sign. Finally, we multiply this equation from the left by $g^{-1} = exp(-a(x,t))$ to obtain

$$g^{-1}\partial_\mu g = dexp_{-a(x,t)} \partial_\mu a(x,t) \; . \tag{29}$$

Then, the flow reads

$$\frac{\partial g(x,t)}{\partial t} = g(x,t)\partial_\mu \left[dexp_{-a(x,t)}(\partial^\mu a(x,t)) \right] . \tag{30}$$

Since the Lie-algebra is a linear space, the partial derivative of a may be evaluated using e.g. the forward finite difference scheme

$$\begin{aligned} \frac{\partial a}{\partial x} &\approx \frac{a(x+h,y) - a(x,y)}{h}, \\ \frac{\partial a}{\partial y} &\approx \frac{a(x,y+h) - a(x,y)}{h}, \end{aligned} \tag{31}$$

where h is the grid size. The partial derivative of the *dexp* function will be evaluated using the backward finite difference scheme. The values of the Lie-algebra elements on the grid will be calculated using the *logm* operator such that

$$logm : g(x,y,t) \mapsto a(x,y,t). \tag{32}$$

In order that the proposed flow evolves on the group manifold we use methods of Lie group integration mainly due to Iserles et al. [10]. We apply the simplest time integration operator which is the Lie-group version of the forward Euler operator. It reads

$$g_{n+1} = \phi(dt\, a(g_n, t_n)) g_n, \tag{33}$$

where dt is the time step, a is the element of the algebra and $\phi : \mathfrak{g} \mapsto \mathcal{G}$. For our 'left-trivialized' flow we may use the following forward Euler operator

$$g_{n+1} = g_n \phi(dt\, a(g_n, t_n)). \tag{34}$$

Therefore, our time step operator reads

$$\begin{aligned} g_{n+1} &= g_n \, exp\left(\partial_\mu \, dexp_{-a(x,t)}(\partial^\mu a(x,t)) \right) \\ &= g_n \, exp\left(\partial_\mu \, dexp_{-log\, g(x,t)}(\partial^\mu log\, g(x,t)) \right) . \end{aligned} \tag{35}$$

Although on each iteration we have to calculate the *dexp* power series, this calculation is almost immediate since this power series converges very fast. The calculation of the first eight terms is accurate enough where the norm of the eighth element if already of order 10^{-6}.

These calculations may be also be done via the Cayley mapping where the *dcay* function will replace the *dexp* function. However, despite of the Cayley mapping advantages (fast calculations), we have found that it is not a suitable choice for our algorithm. The main reason is that we have to use the *invcay* function instead of the *logm*. As we have pointed out in definition 8, the *invcay* mapping is undefined when $X \in \mathcal{G}$ has an eigenvalue -1. Since some elements of $SO(N)$ do have an eigenvalue -1, this causes the algorithm to be unstable numerically and to diverge.

5 Experiments

We demonstrate in Fig. 1 the isotropic regularization of an orthogonal tensor field using our proposed orthogonal PDE flow. We have built a synthetic tensor field of $SO(3)$ matrices which represents 3D rotations. We have created a discontinuity such that the tensor field is divided into two homogenous regions where each region corresponds to a different 3D rotation. The orthogonal matrices are represented in terms of the three column vectors where for $SO(N)$ matrix, these vectors form an N-dimensional orthonormal vector basis. A Gaussian noise has been added to the original field as normally distributed random rotations around the axes. We have applied Eq. (24) to the noisy field for 100 iterations and with a time step of $dt = 0.1$. As expected, the result of the regularization process is a smooth averaged tensor field where the discontinuity has not been preserved.

5.1 Anisotropic regularization

It is clear that Eq. (24) has to be modified in order to obtain an anisotropic regularization of the tensor field. It is well known due to the work by Perona and Malik that this goal may be achieved by replacing the diffusion constant by a spatially dependent function which is a function of the image gradient. This function has to be smooth and monotonically decreasing with $c(0) = 1$ and $c(+\infty) = 0$ whereas it controls the amount of local regularization. We will adopt this attitude. Since in our proposed model the operator which measures the gradients over the tensor field is $g^{-1}\partial_\mu g$, a suitable choice for this function will be

$$c(x, y, t) = \exp(-||g^{-1}(x, y, t)\partial_\mu g(x, y, t)||^2/k^2), \tag{36}$$

where k is the threshold. The flow then takes the form

$$\frac{\partial g}{\partial t} = g\partial_\mu(c(x, y, t)g^{-1}\partial^\mu g). \tag{37}$$

We have tested the modified equation on the same noisy tensor field which is presented in Fig. 1b. The result of the anisotropic regularization is presented

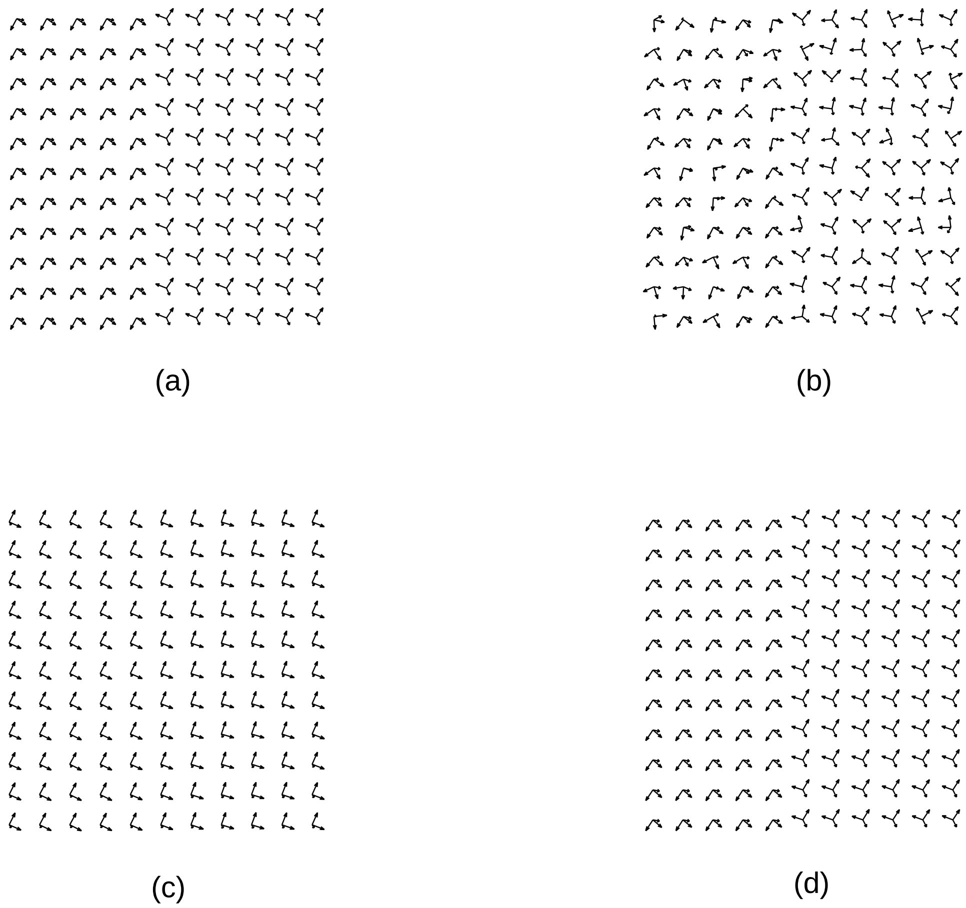

Fig. 1. (a) Original orthogonal tensor field. (b) Noisy field. (c) The result of applying the isotropic orthogonal PDE flow. (d) The results of applying the anisotropic orthogonal PDE flow for 50 iterations and time step $dt = 0.1$. The original tensor field has been recovered.

in Fig. 1d. One can see that at the end of the process the original tensor field has been recovered where the discontinuity has been preserved. In both cases, the isotropic and the anisotropic, the properties of the matrices (orthogonality and determinant one) has been preserved. The threshold k has been set by hand where we have found that its value has to be around one.

The distance between the regularized tensor field and the original tensor field was approximated using the MSE criterion. Let H be the original tensor field and $\hat{H}$ the regularized tensor field, then

$$MSE(\hat{H} - H) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} |\hat{H}_{m,n} - H_{m,n}|^2, \tag{38}$$

where MN is the size of the grid. For the example which is presented in Fig. 1 we have $MSE = 0.0057$ which means that the regularized tensor field is very close to the original one. We have repeated the same experiment for the weighting function

$$c(x, y, t) = \frac{1}{1 + \left(\frac{||g^{-1}\partial_\mu g||}{k}\right)^2}. \tag{39}$$

We have set the threshold to a value of $k = 0.4$ where the results in this case were as good as in the previous case with $MSE = 0.006$.

6 Summary

In this work we proposed a novel framework to tackle the problem of regularizing of Lie group tensor fields in general and the $SO(N)$ group in particular. This was obtained using a PDE flow which was derived directly from a minimization of the GPCM action. Since this action is defined over Lie-group manifold which is a constrained manifold, we arrived at the constrained flow without any additional operations. We have applied the proposed flow to a three-dimensional orthogonal tensor field in order to regularize it. Then, we have modified the flow à la Perona and Malik in order to obtain an anisotropic regularization of the tensor field. This framework is general where it can be applied to any dimension directly and without any additional complexities.

This work may be extended to many directions. We would like to apply this framework to the regularization problem of DT-MRI data sets. This framework may also be integrated with recent level-set frameworks [14, 27, **?**] in order to consider regularization of tensor fields which are attached to non-flat manifolds. Also, other Lie-group manifolds rather than $SO(N)$ may be considered. All of these challenging problems as well as many others are under investigation.

References

1. Aubert, G. and Kornprobst, P.: Mathematical Problems in Image Processing. Springer-Verlag, New-York (2002).

2. Blomgren, P. and Chan, T. F.: Color TV: Total Variation Methods for Restoration of Vector Valued Images. IEEE Trans. on Image processing **7** (1998) 304–309.
3. Chan, T. and Shen, J.: Variational Restoration of nonflat Image Features: Models and Algorithms. SIAMJ Appl. Math. **61** (2000) 1338–1361.
4. Chen, Y., Guo, W., Zeng, Q., Yan, X., Huang, F., Zhang, H., He, G., Vemuri, B. C., Liu, Y.: Estimation, Smoothing, and Characterization of Apparent Diffusion Coefficient Profiles from High Angular Resolution DWI. Proceedings of the 2004 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR 2004) **1** (2004) 588–593.
5. Chefd'hotel, C., Tschumperlé, D., Deriche, R. and Faugeras, O.: Regularizing Flows for Constrained Matrix-Valued Images. Journal of Mathematical Imaging and Vision **20** (2004) 147–162.
6. Gallier, J. and Xu, D.: Computing Exponentials of Skew Symmetric Matrices and Logarithms of Orthogonal Matrices. Internation Journal of Robotics and Automation **18** (2003) 10–20.
7. Gürses, M. and Karasu, A.: Integrable Sigma Model. International Journal of Modern Physics A **6** (1990) 487–500.
8. Hairer, E., Lubich, C. and Wanner, G.: Geometric Numerical Integration. Springer-Verlag, Berlin Heidelberg (2002).
9. Hlavatý, L. and Šnobl, L.: Principal Chiral Models on non-Semisimple Group. J. Phys. A **34** (2001) 7795–7809.
10. Iserles, A., Munthe-Kaas, H. Z., Nørsett, S. P. and Zanna, A.: Lie-Group Methods. Acta Numerica **9** (2000) 215–365.
11. Kimmel, R.: Numerical Geometry of Images: Theory, Algorithms, and Applications, Springer-Verlag, New York (2004).
12. Kimmel, R. and Sochen, N.: Orientation Diffusion or How to Comb a Porcupine. Journal of Visual Communication and Image Representation **13** (2001) 238–248.
13. Lenglet, C., Rousson, M., Deriche, R. and Faugeras, O.: Statistics on Multivariate Normal Distributions: A Geometric Approach and its Application to Diffusion Tensor MRI. Technical Report 5242, INRIA, June (2004).
14. Memoli, F., Sapiro, G. and Osher, S.: Solving Variational Problems and Partial Differential Equations Mapping into General Target Manifolds. J. Comput. Phys. **195** (2004) 263–292.
15. Mumford, D. and Shah, J.: Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems. Communications on Pure and Applied Mathematics **42** (1989) 577–684.
16. Osher, S. and Rudin, L. I.: Feature-Oriented Image Enhancement using Shock Filters. SIAM Journal of Numerical Analysis **27** (1990) 919–940.
17. Pennec, X., Fillard, P. and Ayache, N.: A Riemmannian Framework for Tensor Computing. International Journal of Computer Vision (2005), to appear.
18. Pennec, X.: Probabilities and Statistics on Riemannian Manifolds: A Geometric approach. Research Report 5093, INRIA, January 2004.
19. Perona, P. and Malik, J.: Scale-Space and Edge Detection using Anisotropic Diffusion. IEEE Transactions on Pattern Analysis and Machine Intelligence **12** (1990) 629–639.
20. Perona, P.: Orientation Diffusion. IEEE Trunsactions on Image Processing **7** (1998) 457–467.
21. Rudin, L., Osher S. and Fatemi, E.: Nonlinear Total Variation Based Noise Removal Algorithms. Physica D **60** (1992) 259–268.
22. Sochen, N., Kimmel, R. and Malladi, R.: From High Energy Physics to Low Level Vision. Scale-Space Theories in Computer Vision (1997) 236–247.

23. Sapiro, G.: Geometric Partial Differential Equations and Image Analysis, Cambridge University Press, New York (2001).
24. Sapiro, G. and Ringach, D.: Anisotropic Diffusion of Multivalued Images with Application to Color Filtering. IEEE Trans. On Image Processing **5** (1996) 1582–1586.
25. Shafrir, D. Sochen, N., Deriche, R.: Regularization of Mappings between Implicit Manifolds of Arbitrary Dimension and Codimension, VLSM '05: Proceedings of the 3rd IEEE Workshop on Variational, Geometric and Level-set methods in Computer Vision 2005.
26. Sochen, N.: Integrable Generalized Principal Chiral Models. Phys. Lett. B **391** (1997) 374–380.
27. Sochen, N., Deriche, R. and Lopez-Perez, L.: The Beltrami Flow over Implicit Manifolds, ICCV '03: Proceedings of the Ninth IEEE International Conference on Computer Vision (2003) 832-839.
28. Tang, B., Sapiro, G. and Caselles, V.: Direction Diffusion. International Conference on Computer Vision (1999).
29. Tschumperlé, D.: PhD thesis: PDE's Based Regularization of Multivalued Images and Applications. Universite de Nice-Sophia, Antipolis (2002).
30. Tschumperlé, D. and Deriche, R.: Orthonormal Vector Sets Regularization with PDE's and Application. International Journal of Computer Vision **50** (2002) 237–252.
31. Weickert, J.: Anisotropic Diffusion in Image Processing, Teubner-Verlag, Stuttgart (1998).
32. Weickert, j., Feddern, C., Welk, M., Burgeth, B. and Brox, T.: PDEs for Tensor Image Processing. In: Weickert, J. and Hagen, H. (Eds.): Visualization and Processing of Tensor Fields. Springer, Berlin (2005), to appear.
33. Wiegmann, P. B.: On the Theory of nonabelian Goldstone Bosons in Two Dimesnions: Exact Solution of the $SU(N) \otimes SU(N)$ nonlinear σ Model". Physics Letters B **141** (1984) 217–222.
34. Wiegmann, P. B.: Exact Factorized S-Matrix of the Chiral Field in Two Dimension". Physics Letters B **142** (1984) 173–176.
35. Whitaker, R. and Gerig, G.: Vector-Valued Diffusion. In: B. M. ter Haar Romeny (ed.): Geometry Driven Diffusion in Computer Vision, Kluwer Academic Publishers, The Netherlands, (1994) 93–134.