L^2 hypocoercivity, inequalities and applications

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Outline

- **Diffusion rates and functional inequalities**
  ▷ Poincaré inequality
  ▷ Nash inequality
  ▷ Inequality for the intermediate range: weigthed Nash and Caffarelli-Kohn-Nirenberg inequalities
- L^2 **Hypocoercivity**
  ▷ Abstract setting
  ▷ The diffusion limit
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- **Decay and convergence rates for kinetic equations**
  ▷ The global picture
  ▷ Without confinement: Nash inequality
  ▷ With very weak confinement
  ▷ Without confinement and with sub-exponential local equilibria

Diffusion, rates and functional inequalities

Diffusion (Fokker-Planck) equations

If $\rho \geq 0$ is a solution of the *Fokker-Planck equation*

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \nabla \cdot (\rho \, \nabla V) \quad \text{in} \quad \mathbb{R}^d$$

with initial datum $\rho_0 \in \mathrm{L}^1(\mathbb{R}^d)$ (of mass 1), if $\mu = e^{-V}$ is the density of a probability measure such that the *Poincaré inequality*

$$\boxed{\int_{\mathbb{R}^d} |u - \bar{u}|^2 \, d\mu \leq \mathcal{C}_{\mathrm{P}} \int_{\mathbb{R}^d} |\nabla u|^2 \, d\mu \quad \forall \, u \in \mathcal{H}^1(\mathbb{R}^d, d\mu)}$$

then $u = \rho/\mu$ solves the Ornstein-Uhlenbeck equation

$$\frac{\partial u}{\partial t} = \Delta u - \nabla u \cdot \nabla V$$

and $\|u(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d, d\mu)} = \|\rho(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d, dx)} = \|\rho_0\|_{\mathrm{L}^1(\mathbb{R}^d, dx)} = \bar{u}$,

$$\frac{d}{dt} \|u(t,\cdot) - \bar{u}\|^2_{\mathrm{L}^2(\mathbb{R}^d, dx)} = -\, 2 \, \|\nabla u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d, dx)} \leq -\frac{2}{\mathcal{C}_{\mathrm{P}}} \, \|u(t,\cdot) - \bar{u}\|^2_{\mathrm{L}^2(\mathbb{R}^d, dx)}$$

$$\text{and} \quad \int_{\mathbb{R}^d} |u(t,\cdot) - \bar{u}|^2 \, d\mu \leq \int_{\mathbb{R}^d} |u_0 - \bar{u}|^2 \, d\mu \, e^{-2\, t/\mathcal{C}_{\mathrm{P}}} \quad \forall \, t \geq 0$$

The decay rate of the heat equation

If ρ is a solution of the *heat equation*

$$\frac{\partial \rho}{\partial t} = \Delta \rho \quad \text{in} \quad \mathbb{R}^d$$

with initial datum $\rho_0 \in \mathrm{L}^1(\mathbb{R}^d)$, then

$$\|\rho(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d,dx)} = \|\rho_0\|_{\mathrm{L}^1(\mathbb{R}^d,dx)}$$

$$\frac{d}{dt}\|\rho(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,dx)} = -\,2\,\|\nabla\rho(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,dx)} \le -\,\mathcal{C}\,\|\rho(t,\cdot)\|^{2+\frac{4}{d}}_{\mathrm{L}^2(\mathbb{R}^d,dx)}$$

by Nash's inequality

$$\boxed{\|u\|_2^{2+\frac{4}{d}} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|_1^{\frac{4}{d}}\,\|\nabla u\|_2^2}$$

and so

$$\|\rho(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d,dx)} \le \mathcal{C}\,\|\rho_0\|_{\mathrm{L}^2(\mathbb{R}^d,dx)}\,(1+t)^{-d/2}$$

Diffusion (Fokker-Planck) equations

Potential	$V=0$	$V(x)=\gamma\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\geq 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 1: $\partial_t u=\Delta u+\nabla\cdot(u\nabla V)$

Very weak confinement: Caffarelli-Kohn-Nirenberg

$$\frac{\partial u}{\partial t} = \Delta_x u + \nabla_x \cdot (\nabla_x V\, u) = \nabla_x \left(e^{-V}\, \nabla_x \left(e^{V}\, u\right)\right)$$

Here $x \in \mathbb{R}^d$, $d \geq 3$, and V is a potential such that $e^{-V} \notin \mathrm{L}^1(\mathbb{R}^d)$ corresponding to a *very weak confinement*

Two examples

$$V_1(x) = \gamma\, \log|x| \quad \text{and} \quad V_2(x) = \gamma\, \log\langle x\rangle$$

with $\gamma < d$ and $\langle x\rangle := \sqrt{1+|x|^2}$ for any $x \in \mathbb{R}^d$

In collaboration with Emeric Bouin and Christian Schmeiser

Potential	$V=0$	$V(x)=\gamma\,\log\|x\|$ $\gamma<d$	$V(x)=\|x\|^\alpha$ $\alpha\in(0,1)$	$V(x)=\|x\|^\alpha$ $\alpha\ge 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 2: $\partial_t u = \Delta u + \nabla\cdot(u\,\nabla V)$

Actually, this is more complicated, because the rate depends on the functional space (and of the range of the parameters)...

A first decay result (1/3)

Theorem

Assume that $d \geq 3$, $\gamma < (d-2)/2$ *and* $V = V_1$ *or* $V = V_2$
F any solution u *with initial datum* $u_0 \in \mathrm{L}^1_+ \cap \mathrm{L}^2(\mathbb{R}^d)$,

$$\|u(t,\cdot)\|_2^2 \leq \frac{\|u_0\|_2^2}{(1+c\,t)^{\frac{d}{2}}} \quad \textit{with} \quad c := \frac{4}{d}\,\min\left\{1, 1-\tfrac{2\,\gamma}{d-2}\right\}\,\mathcal{C}_{\mathrm{Nash}}^{-1}\,\frac{\|u_0\|_2^{4/d}}{\|u_0\|_1^{4/d}}$$

Here $\mathcal{C}_{\mathrm{Nash}}$ denotes the optimal constant in Nash's inequality

$$\|u\|_2^{2+\frac{4}{d}} \leq \mathcal{C}_{\mathrm{Nash}}\,\|u\|_1^{\frac{4}{d}}\,\|\nabla u\|_2^2 \quad \forall\,u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)$$

Extended range of exponents, with moments (2/3)

Theorem

Let $d \geq 1$, $0 < \gamma < d$, $V = V_1$ *or* $V = V_2$, *and* $u_0 \in L^1_+ \cap L^2(e^V)$ *with* $\left\| |x|^k u_0 \right\|_1 < \infty$ *for some* $k \geq \max\{2, \gamma/2\}$

$$\forall\, t \geq 0\,, \quad \|u(t,\cdot)\|^2_{L^2(e^V\,dx)} \leq \|u_0\|^2_{L^2(e^V\,dx)}\,(1 + c\,t)^{-\frac{d-\gamma}{2}}$$

for some c *depending on* d, γ, k, $\|u_0\|_{L^2(e^V\,dx)}$, $\|u_0\|_1$, *and* $\left\| |x|^k u_0 \right\|_1$

Extended range of exponents, self-similar variables (3/3)

$$u_\star(t,x)=\frac{c_\star}{(1+2\,t)^{\frac{d-\gamma}{2}}}\,|x|^{-\gamma}\exp\left(-\frac{|x|^2}{2\,(1+2\,t)}\right)$$

Here the initial data need to have a sufficient decay...
$c_\star$ is chosen such that $\|u_\star\|_1=\|u_0\|_1$

Theorem

Let $d\ge1$, $\gamma\in(0,d)$, $V=V_1$ assume that

$$\forall\,x\in\mathbb{R}^d\,,\quad 0\le u_0(x)\le K\,u_\star(0,x)$$

for some constant $K>1$

$$\forall\,t\ge0\,,\quad \|u(t,\cdot)-u_\star(t,\cdot)\|_p\le K\,c_\star^{1-\frac1p}\,\|u_0\|_1^{\frac1p}\left(\tfrac{e}{2\,|\gamma|}\right)^{\frac\gamma2\left(1-\frac1p\right)}(1+2\,t)^{-\zeta_p}$$

for any $p\in[1,+\infty)$, where $\zeta_p:=\frac d2\left(1-\frac1p\right)+\frac1{2\,p}\,\min\left\{2,\frac{d}{d-\gamma}\right\}$

Proofs: basic case (1/3)

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2\,dx = -\,2\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} \Delta V\,|u|^2\,dx$$

with either $V = V_1$ or $V = V_2$ and

$$\Delta V_1(x) = \gamma\,\frac{d-2}{|x|^2} \quad \text{and} \quad \Delta V_2(x) = \gamma\,\frac{d-2}{1+|x|^2} + \frac{2\,\gamma}{(1+|x|^2)^2}$$

For $\gamma \le 0$: apply Nash's inequality

$$\frac{d}{dt}\,\|u\|_2^2 \le -\,2\,\|\nabla u\|_2^2 \le -\,\frac{2}{\mathcal{C}_{\mathrm{Nash}}}\,\|u_0\|_1^{-4/d}\,\|u\|_2^{2+4/d}$$

For $0 < \gamma < (d-2)/2$: *Hardy-Nash inequalities*

Lemma

Let $d \ge 3$, $\delta < (d-2)^2/4$ *and* $\mathcal{C}_\delta = \mathcal{C}_{\mathrm{Nash}}/\left(1 - \frac{4\,\delta}{(d-2)^2}\right)$

$$\boxed{\|u\|_2^{2+\frac{4}{d}} \le \mathcal{C}_\delta\left(\|\nabla u\|_2^2 - \delta\int_{\mathbb{R}^d} \frac{u^2}{|x|^2}\,dx\right)\|u\|_1^{\frac{4}{d}} \quad \forall\,u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)}$$

Proofs: moments (2/3)

Growth of the moment

$$M_k(t) := \int_{\mathbb{R}^d} |x|^k u \, dx$$

From the equation

$$M_k' = k \left(d + k - 2 - \gamma \right) \int_{\mathbb{R}^d} u \, |x|^{k-2} \, dx \le k \left(d + k - 2 - \gamma \right) M_0^{\frac{2}{k}} \, M_k^{1 - \frac{2}{k}}$$

then use the *Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}^d} |x|^\gamma \, u^2 \, dx \le \mathcal{C} \left(\int_{\mathbb{R}^d} |x|^{-\gamma} \, \left| \nabla \left(|x|^\gamma u \right) \right|^2 dx \right)^a \left(\int_{\mathbb{R}^d} |x|^k \, |u| \, dx \right)^{2(1-a)}$$

Proofs: self-similar solutions (3/3)

The proof relies on *uniform decay estimates* + *Poincaré inequality* in self-similar variables

Proposition

Let $\gamma \in (0,d)$ *and assume that*

$$0 \le u(0,x) \le c_\star \left(\sigma + |x|^2\right)^{-\gamma/2} \exp\left(-\frac{|x|^2}{2}\right) \quad \forall\, x \in \mathbb{R}^d$$

with $\sigma = 0$ *if* $V = V_1$ *and* $\sigma = 1$ *if* $V = V_2$. *Then*

$$0 \le u(t,x) \le \frac{c_\star}{(1+2\,t)^{\frac{d-\gamma}{2}}} \left(\sigma + |x|^2\right)^{-\gamma/2} \exp\left(-\frac{|x|^2}{2\,(1+2\,t)}\right)$$

for any $x \in \mathbb{R}^d$ *and* $t \ge 0$

With sub-exponential equilibria

▷ We consider the ***homogeneous Fokker-Planck equation***

$$\partial_t g = \nabla_v \cdot \Big(F \, \nabla_v \big(F^{-1} \, g \big) \Big)$$

associated with ***sub-exponential equilibria***

$$F(v) = C_\alpha \, e^{-\langle v \rangle^\alpha} \,, \quad \alpha \in (0,1)$$

or the corresponding Ornstein-Uhlenbeck equation for $h = g/F$
– decay rates based on the weak Poincaré inequality (Kavian, Mischler)
– decay rates based on a weighted Poincaré / Hardy-Poincaré inequality

In collaboration with Emeric Bouin, Laurent Lafleche and Christian Schmeiser

Potential	$V=0$	$V(x)=\gamma\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\ge1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 3: $\partial_t u=\Delta u+\nabla\cdot(u\,\nabla V)$

Weak Poincaré inequality

$$\int_{\mathbb{R}^d} \left|h - \widetilde{h}\right|^2 d\xi \le \mathcal{C}_{\alpha,\tau} \left(\int_{\mathbb{R}^d} |\nabla h|^2 \, d\xi\right)^{\frac{\tau}{1+\tau}} \left\|h - \widetilde{h}\right\|_{L^\infty(\mathbb{R}^d)}^{\frac{2}{1+\tau}}$$

for some explicit positive constant $\mathcal{C}_{\alpha,\tau}$, $\widetilde{h} := \int_{\mathbb{R}^d} h \, d\xi$. Using

$$\frac{d}{dt} \int_{\mathbb{R}^d} \left|h(t,\cdot) - \widetilde{h}\right|^2 d\xi = -\, 2 \int_{\mathbb{R}^d} |\nabla_v h|^2 \, d\xi$$

where $h = g/F$ and $d\xi = F \, dv$ + Hölder's inequality

$$\int_{\mathbb{R}^d} \left|h - \widetilde{h}\right|^2 d\xi \le \left(\int_{\mathbb{R}^d} \left|h - \widetilde{h}\right|^2 \langle v\rangle^{-\beta} \, d\xi\right)^{\frac{\tau}{\tau+1}} \left(\int_{\mathbb{R}^d} \left\|h - \widetilde{h}\right\|_{L^\infty(\mathbb{R}^d)}^2 \langle v\rangle^{\beta\,\tau} \, d\xi\right)^{\frac{1}{1+\tau}}$$

with $(\tau+1)/\tau = \beta/\eta$, then for with $\mathcal{M} = \sup_{s\in(0,t)} \left\|h(s,\cdot) - \widetilde{h}\right\|_{L^\infty(\mathbb{R}^d)}^{2/\tau}$

$$\int_{\mathbb{R}^d} \left|h(t,\cdot) - \widetilde{h}\right|^2 d\xi \le \left(\left(\int_{\mathbb{R}^d} \left|h(0,\cdot) - \widetilde{h}\right|^2 d\xi\right)^{-\frac{1}{\tau}} + \frac{2\,\tau^{-1}}{\mathcal{C}_{\alpha,\tau}^{1+1/\tau}\,\mathcal{M}}\, t\right)^{-\tau}$$

Weighted Poincaré inequality

There exists a constant $\mathcal{C} > 0$ such that

$$\boxed{\int_{\mathbb{R}^d} |\nabla h|^2 \, F \, \mathrm{d}v \ge \mathcal{C} \int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \langle v \rangle^{-\beta} \, F \, \mathrm{d}v}$$

with $\beta = 2\,(1-\alpha)$, $\widetilde{h} := \int_{\mathbb{R}^d} h\,F\,\mathrm{d}v$ and $F(v) = C_\alpha \, e^{-\langle v \rangle^\alpha}$ and $\alpha \in (0,1)$

Written in terms of $g = h\,F$, the inequality becomes

$$\boxed{\int_{\mathbb{R}^d} \left| \nabla_v \big(F^{-1} \, g \big) \right|^2 \, F^2 \, \mathrm{d}\mu \ge \mathcal{C} \int_{\mathbb{R}^d} |g - \overline{g}|^2 \, \langle v \rangle^{-2\,(1-\alpha)} \, \mathrm{d}\mu}$$

where $\mathrm{d}\mu = F\,\mathrm{d}v$ and $\overline{g} := \left(\int_{\mathbb{R}^d} g \, \mathrm{d}v \right) F$

$$\frac{d}{dt}\int_{\mathbb{R}^d}|h(t,v)|^2\,\langle v\rangle^k\,F\,\mathrm{d}v+\,2\int_{\mathbb{R}^d}|\nabla_v h|^2\,\langle v\rangle^k\,F\,\mathrm{d}v$$
$$=-\int_{\mathbb{R}^d}\nabla_v(h^2)\cdot\left(\nabla_v\langle v\rangle^k\right)F\,\mathrm{d}v$$

With $\ell=2-\alpha$, $a\in\mathbb{R}$, $b\in(0,+\infty)$

$$\nabla_v\cdot\left(F\,\nabla_v\langle v\rangle^k\right)=\frac{k}{\langle v\rangle^4}\left(d+(k+d-2)\,|v|^2-\alpha\,\langle v\rangle^\alpha\,|v|^2\right)\le a-b\,\langle v\rangle^{-\ell}$$

Proposition (Weighted L^2 norm)

There exists a constant $\mathcal{K}_k>0$ such that, if h solves the Ornstein-Uhlenbeck equation, then

$$\forall\,t\ge0\quad\|h(t,\cdot)\|_{L^2(\langle v\rangle^k\,\mathrm{d}\xi)}\le\mathcal{K}_k\,\left\|h^{\mathrm{in}}\right\|_{L^2(\langle v\rangle^k\,\mathrm{d}\xi)}$$

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left|h(t,\cdot)-\widetilde{h}\right|^2 \mathrm{d}\xi = -\,2\int_{\mathbb{R}^d}|\nabla_v h|^2\,\mathrm{d}\xi \le -\,2\,\mathcal{C}\int_{\mathbb{R}^d}\left|h-\widetilde{h}\right|^2\,\langle v\rangle^{-\beta}\,\mathrm{d}\xi$$

+ Hölder

Theorem

Assume that $\alpha\in(0,1)$. Let $g^{\rm in}\in \mathrm{L}^1_+(\mathrm{d}\mu)\cap\mathrm{L}^2(\langle v\rangle^k\mathrm{d}\mu)$ for some $k>0$ and consider the solution g to the homogeneous Fokker-Planck equation with initial datum $g^{\rm in}$. If $\overline{g}=\left(\int_{\mathbb{R}^d} g\,\mathrm{d}v\right)F$, then

$$\int_{\mathbb{R}^d}|g(t,\cdot)-\overline{g}|^2\,\mathrm{d}\mu \le \left(\left(\int_{\mathbb{R}^d}\left|g^{\rm in}-\overline{g}\right|^2\mathrm{d}\mu\right)^{-\beta/k}+\frac{2\,\beta\,\mathcal{C}}{k\,\mathcal{K}^{\beta/k}}\,t\right)^{-k/\beta}$$

with $\beta=2\,(1-\alpha)$ and $\mathcal{K}:=\mathcal{K}_k^2\,\left\|g^{\rm in}\right\|^2_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}\mu)}+\Theta_k\left(\int_{\mathbb{R}^d} g^{\rm in}\,\mathrm{d}v\right)^2$

L^2 Hypocoercivity

▷ Abstract statement

▷ Diffusion limit

▷ The extension to the non-compact case

Collaboration with C. Mouhot and C. Schmeiser
+ E. Bouin, S. Mischler

An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$

$$\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^* \mathsf{T}\Pi\big)^{-1} (\mathsf{T}\Pi)^*$$

* denotes the adjoint with respect to $\langle \cdot, \cdot \rangle$

Π is the orthogonal projection onto the null space of L

The assumptions

λ_m, λ_M, and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity:*

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \|(1-\Pi)F\|^2 \tag{H1}$$

▷ *macroscopic coercivity:*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2 \tag{H2}$$

▷ *parabolic macroscopic dynamics:*

$$\Pi \mathsf{T} \Pi\, F = 0 \tag{H3}$$

▷ *bounded auxiliary operators:*

$$\|\mathsf{AT}(1-\Pi)F\| + \|\mathsf{AL}F\| \leq C_M \|(1-\Pi)F\| \tag{H4}$$

The estimate

$$\frac{1}{2}\,\frac{d}{dt}\|F\|^2 = \langle \mathsf{L}F, F\rangle \leq -\,\lambda_m \|(1-\Pi)F\|^2$$

is not enough to conclude that $\|F(t,\cdot)\|^2$ decays exponentially

Equivalence and entropy decay

For some $\delta > 0$ to be determined later, the L^2 entropy / Lyapunov functional is defined by

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

so that $\langle \mathsf{AT}\Pi F, F\rangle \sim \|\Pi F\|^2$ and

$$\begin{aligned} -\frac{d}{dt}\mathsf{H}[F] &=: \mathsf{D}[F] \\ &= -\,\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT}\Pi F, F\rangle \\ &\quad -\,\delta\,\mathrm{Re}\langle \mathsf{TA}F, F\rangle + \delta\,\mathrm{Re}\langle \mathsf{AT}(1-\Pi)F, F\rangle - \delta\,\mathrm{Re}\langle \mathsf{AL}F, F\rangle \end{aligned}$$

▷ *entropy decay rate:* for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$\lambda\,\mathsf{H}[F] \le \mathsf{D}[F]$$

▷ *norm equivalence* of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \frac{2+\delta}{4}\,\|F\|^2$$

Exponential decay of the entropy

$$\lambda = \frac{\lambda_M}{3\,(1+\lambda_M)}\,\min\left\{1,\lambda_m,\frac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\},\ \delta = \frac12\,\min\left\{1,\lambda_m,\frac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\}$$

$$h_1(\delta,\lambda) := (\delta\,C_M)^2 - 4\left(\lambda_m - \delta - \frac{2+\delta}{4}\,\lambda\right)\left(\frac{\delta\,\lambda_M}{1+\lambda_M} - \frac{2+\delta}{4}\,\lambda\right)$$

Theorem

Let L *and* T *be closed linear operators (respectively Hermitian and anti-Hermitian) on* $\mathcal{H}$*. Under* (H1)–(H4)*, for any* $t \ge 0$

$$\mathsf{H}[F(t,\cdot)] \le \mathsf{H}[F_0]\,e^{-\lambda_\star t}$$

where $\lambda_\star$ *is characterized by*

$$\lambda_\star := \sup\left\{\lambda>0\,:\,\exists\,\delta>0\ s.t.\ h_1(\delta,\lambda)=0\,,\ \lambda_m - \delta - \tfrac14\,(2+\delta)\,\lambda > 0\right\}$$

Sketch of the proof

- Since $\mathsf{AT}\Pi = \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*\mathsf{T}\Pi$, from (H1) and (H2)

$$-\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT}\Pi F, F\rangle \ge \lambda_m\,\|(1-\Pi)F\|^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,\|\Pi F\|^2$$

- By (H4), we know that

$$|\mathrm{Re}\langle \mathsf{AT}(1-\Pi)F, F\rangle + \mathrm{Re}\langle \mathsf{AL}F, F\rangle| \le C_M\,\|\Pi F\|\,\|(1-\Pi)F\|$$

- The equation $G = \mathsf{A}F$ is equivalent to $(\mathsf{T}\Pi)^*F = G + (\mathsf{T}\Pi)^*\,\mathsf{T}\Pi\,G$

$$\langle \mathsf{TA}F, F\rangle = \langle G, (\mathsf{T}\Pi)^*\,F\rangle = \|G\|^2 + \|\mathsf{T}\Pi G\|^2 = \|\mathsf{A}F\|^2 + \|\mathsf{TA}F\|^2$$

$$\langle G, (\mathsf{T}\Pi)^*\,F\rangle \le \|\mathsf{TA}F\|\,\|(1-\Pi)F\| \le \frac{1}{2\,\mu}\,\|\mathsf{TA}F\|^2 + \frac{\mu}{2}\,\|(1-\Pi)F\|^2$$

$$\|\mathsf{A}F\| \le \frac{1}{2}\,\|(1-\Pi)F\|\,,\ \|\mathsf{TA}F\| \le \|(1-\Pi)F\|\,,\ |\langle \mathsf{TA}F, F\rangle| \le \|(1-\Pi)F\|^2$$

- With $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$

$$\mathsf{D}[F] - \lambda\,\mathsf{H}[F] \ge (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{2+\delta}{4}\,\lambda\,(X^2+Y^2)$$

Hypocoercivity

Corollary

For any $\delta \in (0,2)$, if $\lambda(\delta)$ is the largest positive root of $h_1(\delta,\lambda) = 0$ for which $\lambda_m - \delta - \frac{1}{4}(2+\delta)\,\lambda > 0$, then for any solution F of the evolution equation

$$\|F(t)\|^2 \le \frac{2+\delta}{2-\delta}\, e^{-\lambda(\delta)\,t}\, \|F(0)\|^2 \quad \forall\, t \ge 0$$

From the norm equivalence of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \frac{2+\delta}{4}\,\|F\|^2$$

We use $\frac{2-\delta}{4}\,\|F_0\|^2 \le \mathsf{H}[F_0]$ so that $\lambda_\star \ge \sup_{\delta\in(0,2)} \lambda(\delta)$

Formal macroscopic (diffusion) limit

Scaled evolution equation

$$\varepsilon \frac{dF}{dt} + \mathsf{T} F = \frac{1}{\varepsilon} \mathsf{L} F$$

on the Hilbert space $\mathcal{H}$. $F_\varepsilon = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$

$$\begin{aligned} \varepsilon^{-1}: &\qquad \mathsf{L} F_0 = 0\,, \\ \varepsilon^{0}: &\qquad \mathsf{T} F_0 = \mathsf{L} F_1\,, \\ \varepsilon^{1}: &\qquad \tfrac{dF_0}{dt} + \mathsf{T} F_1 = \mathsf{L} F_2 \end{aligned}$$

The first equation reads as $u = F_0 = \Pi F_0$
The second equation is simply solved by $F_1 = -\,(\mathsf{T}\Pi)\,F_0$
After projection, the third equation is

$$\tfrac{d}{dt}\,(\Pi F_0) - \,\Pi \mathsf{T}\,(\mathsf{T}\Pi)\,F_0 = \Pi \mathsf{L} F_2 = 0$$

$$\partial_t u + (\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)\,u = 0$$

is such that $\frac{d}{dt}\|u\|^2 = -\,2\,\|(\mathsf{T}\Pi)\,u\|^2 \le -\,2\,\lambda_M\,\|u\|^2$

The compact case

(H1) *Regularity & Normalization:* $V \in W^{2,\infty}_{\rm loc}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} e^{-V}\,dx = 1$

(H2) *Spectral gap / Poincaré inequality:* for some $\Lambda > 0$, $\forall\, u \in H^1(e^{-V}dx)$ such that $\int_{\mathbb{R}^d} u\, e^{-V}\,dx = 0$

$$\boxed{\int_{\mathbb{R}^d} |u|^2\, e^{-V}\,dx \le \Lambda \int_{\mathbb{R}^d} |\nabla_x u|^2\, e^{-V}\,dx}$$

(H3) *Pointwise conditions:*
there exists $c_0 > 0$, $c_1 > 0$ and $\theta \in (0,1)$ s.t.

$$\Delta V \le \frac{\theta}{2}\,|\nabla_x V(x)|^2 + c_0\,, \quad |\nabla_x^2 V(x)| \le c_1\,(1 + |\nabla_x V(x)|)\;\forall\, x \in \mathbb{R}^d$$

(H4) *Growth condition:* $\int_{\mathbb{R}^d} |\nabla_x V|^2\, e^{-V}\,dx < \infty$

Theorem (D., Mouhot, Schmeiser)

Let L *be either a Fokker-Planck operator or a linear relaxation operator with a local equilibrium* $F(v) = (2\pi)^{-d/2}\,\exp(-|v|^2/2)$*. If* f *solves*

$$\partial_t f + v\cdot\nabla_x f - \nabla_x V\cdot\nabla_v f = \mathsf{L} f$$

then

$$\forall\, t \ge 0\,, \quad \|f(t) - F\|^2 \le (1+\eta)\,\|f_0 - F\|^2\, e^{-\lambda t}$$

Decay and convergence rates for kinetic equations

What can we do when at least one of the coercivity conditions is missing ? microscopic coercivity (H1) *or macroscopic coercivity* (H2)

In collaboration with Emeric Bouin, Stéphane Mischler, Clément Mouhot, Christian Schmeiser + Laurent Lafleche

Some non-compact cases

- The global picture
▷ by what can we replace the Poincaré inequalities ?

- Nash's inequality and a decay rate when $V = 0$

- Very weak confinement: Caffarelli-Kohn-Nirenberg inequalities and moments

- With sub-exponential equilibria: weighted Poincaré / Hardy-Poincaré

Some references

- *Some entries in the literature which will not be considered in this lecture*

▷ *Weak Poincaré inequality*: (Röckner & Wang, 2001), (Kavian, Mischler), (Cao, PhD thesis), (Hu, Wang, 2019) + (Ben-Artzi, Einav) for recent spectral considerations

▷ *Weighted Nash inequalities*: (Bakry, Bolley, Gentil, Maheux, 2012), (Wang, 2000, 2002, 2010)

▷ Related topic:
fractional diffusion (Cattiaux, Puel, Fournier, Tardif,...)

- *Our strategy: rely on the estimates of the diffusion limits*

The global picture

- Depending on the local equilibria and on the external potential (H1) and (H2) (which are Poincaré type inequalities) can be replaced by other functional inequalities:

▷ *microscopic coercivity* (H1)

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \, \|(1-\Pi)F\|^2$$

$\Longrightarrow$ ***weak Poincaré inequalities*** or ***Hardy-Poincaré inequalities***

▷ *macroscopic coercivity* (H2)

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \, \|\Pi F\|^2$$

$\Longrightarrow$ ***Nash inequality***, ***weighted Nash*** or ***Caffarelli-Kohn-Nirenberg inequalities***

- This can be done at the level of the *diffusion equation* (homogeneous case) or at the level of the *kinetic equation* (non-homogeneous case)

Kinetic Fokker-Planck equations

B = Bouin, L = Lafleche, M = Mouhot, MM = Mischler, Mouhot
S = Schmeiser

Potential	$V = 0$	$V(x) = \gamma \log \lvert x \rvert$ $\gamma < d$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \in (0,1)$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \ge 1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v) = e^{-\langle v \rangle^\beta}$, $\beta \ge 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b < 1$, $\beta = 2$ convergence	DMS, Mischler- Mouhot $e^{-\lambda t}$ convergence
$F(v) = e^{-\langle v \rangle^\beta}$, $\beta \in (0,1)$	BDLS: $t^{-\zeta}$, $\zeta =$ $\min\left\{\frac{d}{2}, \frac{k}{\beta}\right\}$ decay			
$F(v) = \langle v \rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 1: $\partial_t f + v \cdot \nabla_x f = F\, \nabla_v\big(F^{-1}\, \nabla_v f\big)$. Notation: $\langle v \rangle = \sqrt{1 + |v|^2}$

A result based on Nash's inequality

$$\boxed{\partial_t f + v\cdot\nabla_x f = \mathsf{L} f\,,\quad (t,x,v)\in\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d}$$

$$\mathsf{D}[f] = -\frac{d}{dt}\mathsf{H}[f] \geq \mathsf{a}\left(\|(1-\Pi)f\|^2 + 2\,\langle \mathsf{AT}\Pi f, f\rangle\right)$$

We observe that

$$\begin{aligned}\mathsf{A}^* f &= \mathsf{T}\Pi\left(1+(\mathsf{T}\Pi)^*\mathsf{T}\Pi\right)^{-1} f\\ &= \mathsf{T}\left(1+(\mathsf{T}\Pi)^*\mathsf{T}\Pi\right)^{-1}\Pi f = M\,\mathsf{T}\,u_f = v\,M\cdot\nabla_x u_f\end{aligned}$$

if u_f is the solution in $\mathrm{H}^1(\mathbb{R}^d)$ of $u_f - \Theta\,\Delta u_f = \rho_f$, and

$$\|u_f(t,\cdot)\|_{\mathrm{L}^1(dx)} = \|\rho_f(t,\cdot)\|_{\mathrm{L}^1(dx)} = \|f_0\|_{\mathrm{L}^1(dx\,dv)}$$

$$\|u_f\|^2_{\mathrm{L}^2(dx)} \leq \|\rho_f\|^2_{\mathrm{L}^2(dx)}\,,\quad \|\nabla_x u_f\|^2_{\mathrm{L}^2(dx)} \leq \frac{1}{\Theta}\,\langle \mathsf{AT}\Pi f, f\rangle$$

$$\|\rho_f\|^2_{\mathrm{L}^2(dx)} = \|\Pi f\|^2 \leq \|u_f\|^2_{\mathrm{L}^2(dx)} + 2\,\langle \mathsf{AT}\Pi f, f\rangle$$

Nash's inequality

$$\|u\|^2_{L^2(dx)} \le \mathcal{C}_{\rm Nash}\,\|u\|^{\frac{4}{d+2}}_{L^1(dx)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{L^2(dx)}\quad \forall\,u\in L^1\cap H^1(\mathbb{R}^d)$$

Use $\|\Pi f\|^2 \le \Phi^{-1}\big(2\,\langle \mathsf{AT\Pi} f, f\rangle\big)$ with $\Phi^{-1}(y) := y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}}$ to get

$$\|(1-\Pi)f\|^2 + 2\,\langle \mathsf{AT\Pi} f, f\rangle \ge \Phi(\|f\|^2) \ge \Phi\big(\tfrac{2}{1+\delta}\,\mathsf{H}[f]\big)$$

$$\mathsf{D}[f(t,\cdot)] = -\frac{d}{dt}\mathsf{H}[f(t,\cdot)] \ge \mathsf{a}\,\Phi\big(\tfrac{2}{1+\delta}\,\mathsf{H}[f(t,\cdot)]\big)$$

As $s\to 0_+$, $\Phi(s)\sim s^{1+\frac{d}{2}}$ + Grönwall: $\mathsf{H}[f(t,\cdot)]\sim t^{-d/2}$ as $t\to+\infty$

Algebraic decay rates in $\mathbb{R}^d$

$V = 0$: On the whole Euclidean space, we can define the entropy

$$\mathsf{H}[f] := \tfrac{1}{2}\,\|f\|^2_{\mathrm{L}^2(dx\,d\gamma)} + \delta\,\langle \mathsf{A} f, f\rangle_{dx\,d\gamma}$$

Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|^2_{\mathrm{L}^2(dx)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(dx)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(dx)}$$

proves that $\mathsf{H}[f] \le C\left(\mathsf{H}[f_0] + \|f_0\|^2_{\mathrm{L}^1(dx\,dv)}\right)(1+t)^{-\frac{d}{2}}$

Theorem

There exists a constant $C > 0$ such that, for any $t \ge 0$

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma)} \le C\left(\|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma)} + \|f_0\|^2_{\mathrm{L}^2(d\gamma;\,\mathrm{L}^1(dx))}\right)(1+t)^{-\frac{d}{2}}$$

- Factorization / enlargement of the space (Gualdani, Mischler, Mouhot) allows to consider weights with polynomial growth
- Zero moments of order larger than 1 and mode-by-mode analysis (Fourrier): faster decay rates (BDMMS)

Kinetic Fokker-Planck equation, very weak confinement

Let us consider the kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \mathsf{L} f$$

where $\mathsf{L} f$ is one of the two following collision operators

(a) a Fokker-Planck operator

$$\mathsf{L} f = \nabla_v \cdot \Big(F \, \nabla_v \left(F^{-1} \, f \right) \Big)$$

(b) a scattering collision operator

$$\mathsf{L} f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \, \big(f(v') \, F(\cdot) - f(\cdot) \, F(v') \big) \, dv'$$

$$\boxed{V(x) \sim \gamma \log |x| \,, \quad \gamma \in (0, d)}$$

Potential	$V=0$	$V(x)=\gamma\,\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\ge1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v)=e^{-\langle v\rangle^\beta}$, $\beta\ge1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b<1$, $\beta=2$ convergence	DMS, Mischler- Mouhot $e^{-\lambda t}$ convergence
$F(v)=e^{-\langle v\rangle^\beta}$, $\beta\in(0,1)$	BDLS: $t^{-\zeta}$, $\zeta=$ $\min\left\{\frac d2,\frac k\beta\right\}$ decay			
$F(v)=\langle v\rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 2: $\partial_t f+v\cdot\nabla_x f=F\,\nabla_v\big(F^{-1}\,\nabla_v f\big)$. Notation: $\langle v\rangle=\sqrt{1+|v|^2}$

Decay rates

$$\forall\,(x,v)\in\mathbb{R}^d\times\mathbb{R}^d\,,\quad \mathcal{F}(x,v)=M(v)\,e^{-V(x)}\,,\quad M(v)=(2\pi)^{-\frac{d}{2}}\,e^{-\frac12\,|v|^2}$$

$$\textbf{(H1)}\quad 1\le\sigma(v,v')\le\overline{\sigma}\,,\quad\forall\,v\,,\,v'\in\mathbb{R}^d\,,\quad\text{for some}\quad\overline{\sigma}\ge1$$

$$\textbf{(H2)}\quad\int_{\mathbb{R}^d}\big(\sigma(v,v')-\sigma(v',v)\big)\,M(v')\,dv'=0\quad\forall\,v\in\mathbb{R}^d$$

+ *Caffarelli-Kohn-Nirenberg inequalities*

Theorem

Let $d\ge1$, $V=V_2$ *with* $\gamma\in[0,d)$, $k>\max\{2,\gamma/2\}$ *and* $f_0\in\mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)$ *such that*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^k\,f_0\,dx\,dv+\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^k\,f_0\,dx\,dv<+\infty$$

If (H1)–(H2) *hold, then there exists* $C>0$ *such that*

$$\forall\,t\ge0\,,\quad\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)}\le C\,(1+t)^{-\frac{d-\gamma}{2}}$$

Kinetic Fokker-Planck, no confinement and sub-exponential local equilibria

- the *Fokker-Planck* operator

$$\mathsf{L}_1 f = \nabla_v \cdot \Big(F\, \nabla_v \big(F^{-1}\, f \big) \Big)$$

- the *scattering* collision operator

$$\mathsf{L}_2 f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \Big(f(v')\, F(\cdot) - f(\cdot)\, F(v') \Big)\, \mathrm{d}v'$$

under assumptions (H1)–(H2)

$$V = 0 \qquad \boxed{F(v) = e^{-\langle v \rangle^\beta} \quad \beta \in (0,1)}$$

Potential	$V=0$	$V(x)=\gamma\,\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\ge1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v)=e^{-\langle v\rangle^\beta}$, $\beta\ge1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b<1$, $\beta=2$ convergence	DMS, Mischler- Mouhot $e^{-\lambda t}$ convergence
$F(v)=e^{-\langle v\rangle^\beta}$, $\beta\in(0,1)$	BDLS: $t^{-\zeta}$, $\zeta=$ $\min\left\{\frac d2,\frac k\beta\right\}$ decay			
$F(v)=\langle v\rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 3: $\partial_t f+v\cdot\nabla_x f=F\,\nabla_v\big(F^{-1}\,\nabla_v f\big)$. Notation: $\langle v\rangle=\sqrt{1+|v|^2}$

The decay rate with sub-exponential local equilibria

Theorem

Let $\alpha \in (0,1)$, $\beta > 0$, $k > 0$ and let $F(v) = C_\alpha\, e^{-\langle v \rangle^\alpha}$. Assume that either $\mathsf{L} = \mathsf{L}_1$ and $\beta = 2\,(1-\alpha)$, or $\mathsf{L} = \mathsf{L}_2$ + Assumptions. There exists a numerical constant $\mathcal{C} > 0$ such that any solution f of

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad f(0,\cdot,\cdot) = f^{\rm in} \in \mathrm{L}^2(\langle v \rangle^k \mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^1_+(\mathrm{d}x\,\mathrm{d}v)$$

satisfies

$$\forall\, t \ge 0\,, \quad \|f(t,\cdot,\cdot)\|^2 = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \big|f(t,x,v)\big|^2\,\mathrm{d}x\,\mathrm{d}\mu \le \mathcal{C}\,\frac{\big\|f^{\rm in}\big\|^2}{(1+\kappa\,t)^{\zeta}}$$

with rate $\zeta = \min\{d/2, k/\beta\}$, for some positive κ which is an explicit function of the two quotients, $\big\|f^{\rm in}\big\| \,/\, \big\|f^{\rm in}\big\|_k$ and $\big\|f^{\rm in}\big\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)} \,/\, \big\|f^{\rm in}\big\|$

Proof (1/2)

$$\begin{aligned}\mathsf{D}[f] := &- \langle \mathsf{L} f, f\rangle + \delta\,\langle \mathsf{AT\Pi} f, \Pi f\rangle \\ &+ \delta\,\langle \mathsf{AT}(\mathrm{Id}-\Pi) f, \Pi f\rangle - \delta\,\langle \mathsf{TA}(\mathrm{Id}-\Pi) f, (\mathrm{Id}-\Pi) f\rangle \\ &- \delta\,\langle \mathsf{AL}(\mathrm{Id}-\Pi) f, \Pi f\rangle\end{aligned}$$

- ***microscopic coercivity***. If $\mathsf{L} = \mathsf{L}_1$, we rely on the ***weighted Poincaré inequality***

$$\boxed{\langle \mathsf{L} f, f\rangle \le -\,\mathcal{C}\,\|(\mathrm{Id}-\Pi) f\|^2_{-\beta}}$$

If $\mathsf{L} = \mathsf{L}_2$, we assume that there exists a constant $\mathcal{C} > 0$ such that

$$\int_{\mathbb{R}^d} \big|h - \widetilde{h}\big|^2\,\langle v\rangle^{-\beta}\,F\,\mathrm{d}v \le \mathcal{C} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \sigma(v,v')\,\big|h' - h\big|^2\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'$$

- ***Weighted*** L^2 ***norms*** Let $k > 0$, $f^{\mathrm{in}} \in \mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$ a solution. $\exists\,\mathcal{K}_k > 1$ such that

$$\forall\, t \ge 0\,,\quad \|f(t,\cdot,\cdot)\|_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)} \le \mathcal{K}_k\,\big\|f^{\mathrm{in}}\big\|_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)}$$

Proof (2/2)

$$\mathsf{H}_\delta[f] := \frac{1}{2}\,\|f\|^2 + \delta\,\langle \mathsf{A}f, f\rangle\,, \quad \frac{d}{dt}\mathsf{H}_\delta[f] = -\,\mathsf{D}[f]$$

- There exists $\kappa > 0$ such that $\forall\, f \in \mathrm{L}^2\left(\langle v\rangle^{-\beta}\,\mathrm{d}x\,\mathrm{d}\mu\right) \cap \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$,

$$\mathsf{D}[f] \geq \kappa\left(\|(\mathrm{Id}-\Pi)f\|_{-\beta}^2 + \langle \mathsf{AT\Pi}f, \Pi f\rangle\right)$$

- For any $f \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^2(\mathrm{d}x\,\mathrm{d}v)$,

$$\langle \mathsf{AT\Pi}f, \Pi f\rangle \geq \Phi\left(\|\Pi f\|^2\right)$$

$$\Phi^{-1}(y) := 2\,y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}}\,, \quad \mathsf{c} = \Theta\,\mathcal{C}_{\mathrm{Nash}}^{-\frac{d+2}{d}}\,\|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}^{-\frac{4}{d}}$$

- For any $f \in \mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$,

$$\|(\mathrm{Id}-\Pi)f\|_{-\beta}^2 \geq \Psi\left(\|(\mathrm{Id}-\Pi)f\|^2\right)$$

$$\Psi(y) := C_0\,y^{1+\beta/k}\,, \quad C_0 := \left(\mathcal{K}_k\,\left(1+\Theta_k\right)\|f^{\mathrm{in}}\|_k\right)^{-\frac{2\,\beta}{k}}$$

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Thank you for your attention !