ENTROPY METHODS, FUNCTIONAL INEQUALITIES AND APPLICATIONS

JEAN DOLBEAULT

ABSTRACT. Various functional inequalities are classically seen from a variational point of view in nonlinear analysis. They also have important consequences for evolution problems. For instance, entropy estimates are standard tools for relating rates of convergence towards asymptotic regimes in time-dependent equations with optimal constants of various functional inequalities. This point of view applies to linear diffusions and will be illustrated by some results on the Fokker-Planck equation based on the *carré du champ* method introduced by D. Bakry and M. Emery. In the recent years, the method has been extended from linear to nonlinear diffusions. This aspect will be illustrated by results on Gagliardo-Nirenberg-Sobolev inequalities on the sphere and on the Euclidean space. Even the evolution equations can be used as a tool for the study of detailed properties of optimal functions in inequalities and their refinements. There are also applications to other equations than pure diffusions: hypocoercivity in kinetic equations is one of them. In any case, the notion of entropy has deep roots in statistical mechanics, with applications in various areas of science ranging from mathematical physics to models in biology. A special emphasis will be put during the course on the corresponding models which offer many directions for new research developments.

PLAN OF THE COURSE

(1) history of entropy methods; heat flow, Nash and logarithmic Sobolev inequalities; the logarithmic Sobolev inequality and gaussian type interpolation inequalities by the carré du champ method; interpolation inequalities on the sphere; other classical inequalities; the large dimension limit
(2) functional inequalities on the Euclidean space; Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$ and fast diffusion flows; variational methods and entropy methods; the Sobolev inequality
(3) Some examples of applications in functional analysis and mathematical physics: 1) Legendre duality; 2) Keller duality; 3) Caffarelli-Kohn-Nirenberg inequalities; variational methods; symmetry and symmetry breaking
(4) an overview of stability results on $\mathbb{S}^d$ and on $\mathbb{R}^d$

NOTES AND REFERENCES OF THE COURSE

https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/M2-Math2026/M2-Math2026.pdf

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Date: February 1, 2026. *File:* M2-Math2026.tex.

SOME LINKS

Here is a short list of monographs and paper that are good entry points for *entropy methods* for PDEs:

• The book [2] by D. Bakry, I. Gentil and M. Ledoux presents in a mainly linear framework the results obtained by the *carré du champ* method and its extensions. It is a synthetic presentation of a very large number of works, which remains perfectly accessible with a basic training in analysis.

https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/references/BGL2014.pdf

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• In [16], A. Jüngel gave a concise presentation of entropy methods on a few well-chosen examples, which include the fast diffusion equation.

https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/references/Juengel2016.pdf

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This book is an excellent introduction to entropy methods applied to nonlinear parabolic equations.

• The book [18] by C. Villani on mass transport is an essential reference. Although very complete, it remains relatively easy to access. On the gradient flows, we can refer to [1] which is however much more technical and advanced.

https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/references/Villani2009.pdf

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• Although it is not (yet) a book, C. Schmeiser's lecture notes are available online at

https://homepage.univie.ac.at/christian.schmeiser/Entropy-course.pdf

and have the advantage of being accompanied by the full video

https://sciencesmaths-paris.fr/f/actualites-fr/entropy-methods

of the course he gave at the Institut Henri Poincaré in the fall of 2018.

• Parts of this course is taken from [14, Section 2]

https://www.ceremade.dauphine.fr/~dolbeaul/Preprints/Publi/1704.pdf

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and from the Memoir [4] (to appear as a Lectures Notes in the *Memoirs of the AMS*) on the *Stability in Gagliardo-Nirenberg-Sobolev inequalities: flows, regularity and the entropy method*

https://hal.science/hal-02887010

• Dauphine-PSL - Master 2 : *Logarithmic Sobolev Inequalities Essentials*, by Djalil Chafai and Joseph Lehec

https://djalil.chafai.net/docs/M2/chafai-lehec-m2-lsie-lecture-notes.pdf

• For a course on *symmetry and nonlinear diffusion flows*, please refer to

https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/UChile-2017/LectureNotes.pdf

• Stability results for Sobolev, logarithmic Sobolev, and related inequalities (Summer School "Direct and Inverse Problems with Applications, and Related Topics" August 19-23, 2024, Ghent, 2024). Slides: 1, 2, 3, 4, and extended summary of the course:

https://www.ceremade.dauphine.fr/ dolbeaul/Preprints/Fichiers/Dolbeault-Ghent.pdf

• For general techniques in Analysis and functional inequalities, the book *Analysis* by

Eliott H. Lieb and Michael Loss is highly recommended
https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/references/Lieb-Loss2001.pdf
• R. Frank: *The Sharp Sobolev Inequality and Its Stability: An Introduction*, see [15, Chapter 1] and the following direct link
https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/references/Frank22.pdf
• An overview on entropy methods and their applications:
https://www.ceremade.dauphine.fr/ dolbeaul/Preprints/Publi/2106.pdf
(login: Dolbeault ; password: Jean)

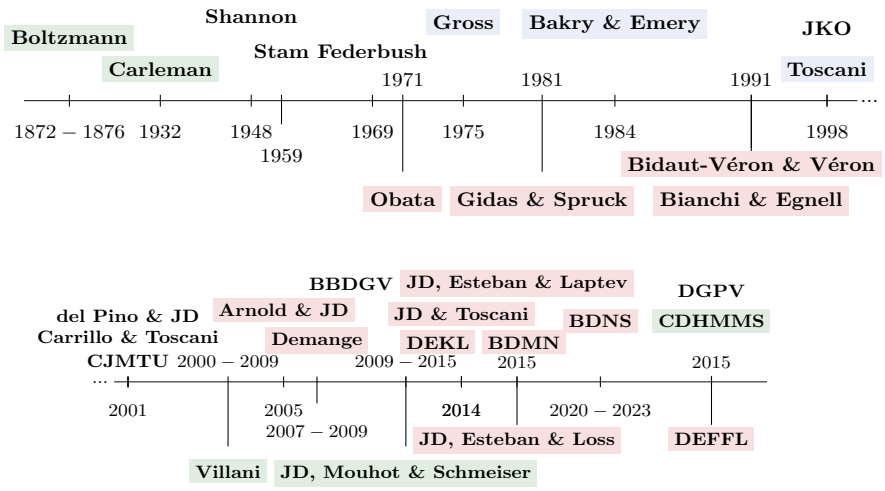
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1. Introduction to entropy methods and some classical inequalities

1.1. **Timeline and some classical inequalities.**

1.1.1. *Entropy methods.* *Entropy methods* coupled to *flows* corresponding to linear and *nonlinear* diffusion equations refer to a set of tools that has been investigated over the last 25 years, with deep roots in physics (kinetic theory of gases), probability theory (Markov processes, semi-groups and Γ_2 or ***carré du champ*** methods), information theory (Rényi entropies), differential geometry (Obata type theorems) and nonlinear analysis (elliptic and parabolic equations). Beyond the gradient flow interpretation, these methods reveal global properties of critical points and allow us to bypass some classical techniques like symmetrization. Rates of convergence are of interest but, surprisingly, much more can be learned by using nonlinear evolution equations as a tool connecting a ***nonlinear regime*** with an asymptotic, linear regime and explicit ***spectral estimates***. Remarkable successes have already been achieved concerning ***symmetry versus symmetry breaking*** issues, quantitative ***stability results***, and ***hypocoercivity methods*** for kinetic equations. The ***symmetry versus symmetry breaking*** issue in some Caffarelli-Kohn-Nirenberg inequalities has been entirely solved, quantitative ***stability results*** for classical inequalities like Sobolev and logarithmic Sobolev, in strong norms and with optimal exponents, have been proved with explicit estimates. In a similar spirit, rather systematic ***hypocoercivity methods*** for degenerate diffusions or (linear) kinetic equations are known. However, a complete and satisfactory theory of ***entropy methods based on nonlinear flows*** is not yet available.

1.1.2. *Timeline: little bit of history.* A brief history of entropy methods. Some recent directions of research. A key idea: the link between functional inequalities and rates in evolution equations.



kinetic equations and hypocoercivity

Semi-groups, linear diffusions and Markov processes

Nonlinear elliptic & parbolic PDEs, functional inequalities, symmetry, stability, *etc.*

1.1.3. *Some basic references.* Here is a short list of monographs and paper that are good entry points for *entropy methods* for PDEs:

• The book [2] by D. Bakry, I. Gentil and M. Ledoux presents in a mainly linear framework the results obtained by the *carré du champ* method and its extensions. It is a synthetic presentation of a very large number of works, which remains perfectly accessible with a basic training in analysis.

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of the course he gave at the Institut Henri Poincaré in the fall of 2018.

1.2. **Some classical functional inequalities.**

1.2.1. *Poincaré inequality.* The inequality for a bounded measure. Link with ground state estimates for the Schrödinger operator. Persson's lemma. Application to the measure $\exp(-|x|^\alpha)\,dx$ on $\mathbb{R}^d$ with $\alpha \ge 1$.

1.2.2. *Nash inequality.* Proof by interpolation. The original proof by Nash/Stein. The proof of Carlen and Loss. The variational method. Details: see Section 4 of

A variational proof of Nash's inequality

(login: Dolbeault , password: Jean)

The *Compact Support Principle* (as opposed to the *Maximum Principle*) is applied in an approximation of the Nash inequality by variational methods and explains why optimal cases are obtained by considering a Poincaré inequality on a bounded domain.

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2. Introduction to the *carré du champ* method and and some classical inequalities (continued)

2.1. **Three versions of the logarithmic Sobolev inequality.** Let $d \ge 1$, and let $d\gamma = \gamma(x)\,dx$, with $\gamma(x) = (2\pi)^{-d/2}\,\mathrm{e}^{-|x|^2/2}$, be the normalized Gaussian probability measure. The *Gaussian logarithmic Sobolev inequality* on $\mathbb{R}^d$ reads as

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma \ge \frac{1}{2}\int_{\mathbb{R}^d} |v|^2 \log|v|^2\,d\gamma \tag{1}$$

for any function $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$. Moreover, by Jensen's inequality, we know that the right-hand side of (1) is non-negative. Throughout the paper, we will consider only real-valued functions.

If v is a smooth and compactly supported function such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$, then an elementary computation shows that (1) written for v is equivalent for $u = v\sqrt{\gamma}$ to the *Euclidean logarithmic Sobolev inequality* on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \ge \frac{1}{2}\int_{\mathbb{R}^d} |u|^2 \log|u|^2\,dx + \frac{d}{4}\log\left(2\pi\,\mathrm{e}^2\right) \tag{2}$$

which, by density, holds for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_2 = 1$. However, even if u is smooth and compactly supported, it does not mean that $\int_{\mathbb{R}^d} |u|^2 \log|u|^2\,dx$ is uniformly bounded from below, whatever $\|u\|_{\mathrm{H}^1(\mathbb{R}^d,dx)}$ is.

On $(\mathbb{R}^d, dx)$, one can take advantage of scalings. For any $\lambda > 0$, let us consider

$$u_\lambda(x) := \lambda^{d/4}\,u\left(\sqrt{\lambda}\,x\right) \quad \forall\, x \in \mathbb{R}^d\,.$$

Inequality (2) applied to u_λ becomes

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \ge \frac{1}{2\lambda}\int_{\mathbb{R}^d} |u|^2 \log|u|^2\,dx + \frac{d}{4\lambda}\log\left(2\pi\,\mathrm{e}^2\,\lambda\right) \tag{3}$$

for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_2 = 1$. After optimizing on $\lambda > 0$, we obtain the *Euclidean logarithmic Sobolev inequality in scale invariant form*;

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geq \frac{\pi \, d \, \mathrm{e}}{2} \exp\left(\frac{2}{d} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx \right) \tag{4}$$

for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_2 = 1$.

2.2. **Beckner inequalities with Gaussian weight.** Inequalities which interpolate between the Gaussian Poincaré inequality and the Gaussian logarithmic Sobolev inequality. The Gaussian logarithmic Sobolev inequality is the strongest of these inequalities. The proof by convex interpolation (Latała and Oleskiewicz). Nelson's hypercontractivity method. An interpolation result between Poincaré and logarithmic Sobolev inequalities: see

pages 973-974

(login: Dolbeault , password: Jean)

The carré du champ method and some properties: see Section 2 the paper on

φ-Entropies

(login: Dolbeault , password: Jean)

2.2.1. *Sobolev inequality.* A direct proof of the optimality case (optimal constant)

The carré du champ method and some properties: see Pages 11-12 of the Memoir (login: Dolbeault , password: Jean)

3. Gagliardo-Nirenberg inequalities

3.1. **Gagliardo-Nirenberg inequalities on the sphere.**

3.1.1. *Ultraspherical entropy methods.* Stereographic projection, symmetry, cylindrical coordinates and ultraspherical operator see

Definitions and Sections 2.1–2.3

(login: Dolbeault , password: Jean)

(for definitions, see page 1349).

3.1.2. *Manifolds.* An introduction to the proofs without symmetrization and a first stability result. Generalization to compact manifolds.

3.1.3. *More general interpolation ienqualities.* Bifurcations, phase transitions, branches of optimizers: the global picture

3.2. **Gagliardo-Nirenberg inequalities on the Euclidean space.** The variational approach

Chapter 1

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and the relative entropy method

Chapter 2

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3.3. **An introduction to Caffarelli-Kohn-Nirenberg inequalities.**

4. Symmetry and stability

4.1. **Course #4: Stability in functional inequalities.** Obtaining explicit stability estimates in classical functional inequalities like the Sobolev inequality has been an essentially open question for 30 years, after the celebrated but non-constructive result [3] of G. Bianchi and H. Egnell in 1991. Recently, new methods have emerged which provide some clues on these fascinating questions. This lecture is an introduction to stability in some fundamental functional inequalities and present several methods that can be used to obtain explicit estimates entropy methods.

(1) Entropy methods on the Euclidean space [17, 4], stability results for Gagliardo-Nirenberg inequalities on the Euclidean space and extension (weights) to Caffarelli-Kohn-Nirenberg inequalities. ▷ M2-2026-1
(2) The Sobolev inequality and the non-constructive stability result of Bianchi–Egnell using concentration-compactness methods [3, 15]. Duality and stability in Hardy-Littlewood-Sobolev inequalities [8]. ▷ M2-2026-2
(3) Stability results on the sphere and on the Gaussian space seen as an infinite dimensional limit of spheres [13, 9, 12, 6, 7, 5]. ▷ M2-2026-3
(4) A constructive stability result for the Sobolev and the logarithmic Sobolev inequalities [10, 11]. ▷ M2-2026-4

Further reading and informations

▷ Lectures
http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/

▷ Preprints and papers
http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/list/
For final versions, use `Dolbeault` as login and `Jean` as password

▷ E-mail
dolbeault@ceremade.dauphine.fr

References

[1] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2005.
[2] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and geometry of Markov diffusion operators*, vol. 348 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Cham, 2014.
[3] G. Bianchi and H. Egnell, *A note on the Sobolev inequality*, J. Funct. Anal., 100 (1991), pp. 18–24.
[4] M. Bonforte, J. Dolbeault, B. Nazaret, and N. Simonov, *Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method*, Preprint arXiv: 2007.03674 and hal-03160022, to appear in *Memoirs of the AMS*, (2022).
[5] G. Brigati, J. Dolbeault, and N. Simonov, *Logarithmic Sobolev and interpolation inequalities on the sphere: Constructive stability results*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 41 (2023), pp. 1289–1321.
[6] ———, *On Gaussian interpolation inequalities*, C. R. Math. Acad. Sci. Paris, 362 (2024), pp. 21–44.

[7] ———, *Stability for the logarithmic Sobolev inequality*, J. Funct. Anal., 287 (2024), p. 110562.
[8] E. A. CARLEN, *Duality and stability for functional inequalities*, Ann. Fac. Sci. Toulouse Math. (6), 26 (2017), pp. 319–350.
[9] J. DOLBEAULT AND M. J. ESTEBAN, *Improved interpolation inequalities and stability*, Adv. Nonlinear Stud., 20 (2020), pp. 277–291.
[10] J. DOLBEAULT, M. J. ESTEBAN, A. FIGALLI, R. L. FRANK, AND M. LOSS, *Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence*, Preprint arXiv: 2209.08651 and hal-03780031, (2023).
[11] ———, *A short review on improvements and stability for some interpolation inequalities*, Preprint arXiv: 2402.08527 and hal-04456461, Proc. ICIAM 2023, (2024).
[12] J. DOLBEAULT, M. J. ESTEBAN, M. KOWALCZYK, AND M. LOSS, *Improved interpolation inequalities on the sphere*, Discrete and Continuous Dynamical Systems Series S (DCDS-S), 7 (2014), pp. 695–724.
[13] J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Interpolation inequalities on the sphere: linear vs. nonlinear flows*. hal-01206975 & arXiv:1509.09099, to appear in Annales de la Faculté des Sciences de Toulouse, 2015.
[14] J. DOLBEAULT AND X. LI, *Φ-Entropies: convexity, coercivity and hypocoercivity for Fokker-Planck and kinetic Fokker-Planck equations*, Mathematical Models and Methods in Applied Sciences, 28 (2018), pp. 2637–2666.
[15] R. L. FRANK, *The sharp Sobolev inequality and its stability: an introduction*, Lecture Notes in Mathematics (Springer), C.I.M.E. Foundation Subseries, 2348 (2024).
[16] A. JÜNGEL, *Entropy Methods for Diffusive Partial Differential Equations*, SpringerBriefs in Mathematics, Springer International Publishing, 2016.
[17] A. JÜNGEL, *Entropy methods for diffusive partial differential equations*, SpringerBriefs in Mathematics, Springer, [Cham], 2016.
[18] C. VILLANI, *Optimal transport*, vol. 338 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2009. Old and new.

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