A random matrix approach to many-body localization

François Huveneers (Paris Dauphine)

Collaboration with Wojciech De Roeck (KU Leuven)

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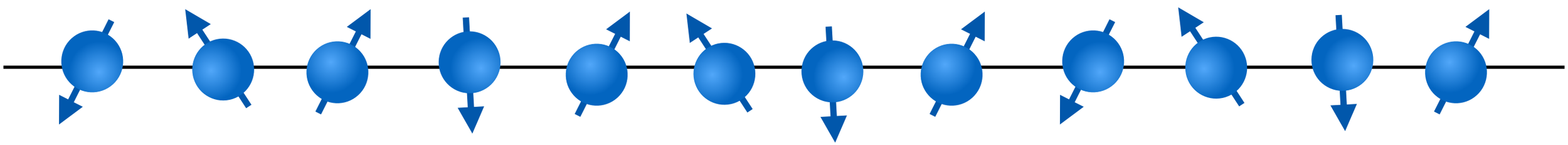
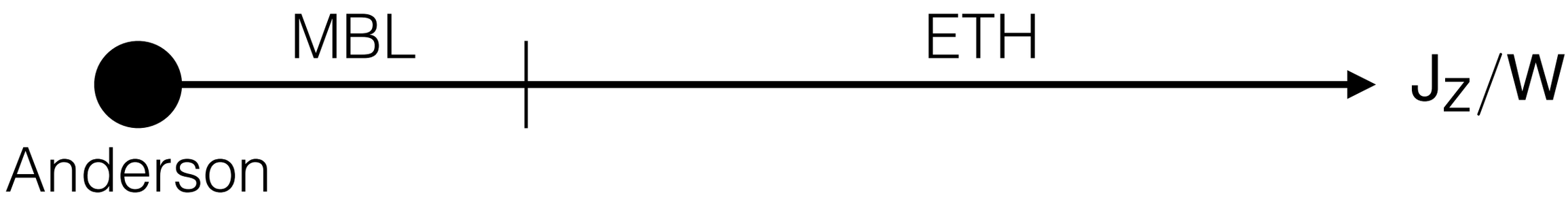
Localized vs Ergodic

Spin chain in d=1, e.g. XXZ chain:

$$H = \sum_{i=1}^{L} h_i S_i^z + J_\perp \sum_{i=1}^{L-1} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + J_z \sum_{i=1}^{L-1} S_i^z S_{i+1}^z$$





(**W**: disorder strength)

Many-body localization (MBL)

- Many-body analog of Anderson localization
- Strong breakdown of ergodicity
- No thermalization

E.g.: quantum quench:

$$E(T_L) \qquad\qquad E(T_R)$$

wait…

$$E(T_L) + \mathcal{O}(\varepsilon) \qquad\qquad E(T_R) + \mathcal{O}(\varepsilon)$$

Perturbation around Anderson insulator: $J_z/W \ll 1$

See Gornyi, Mirlin, Polyakov ('05), Basko, Aleiner, Altshuler ('06), Oganesyan, Huse ('07), Imbrie ('12), …

Ergodic material (ETH)

Transition expected as J/W > critical value

Eigenstate Thermalization Hypothesis:
each eigenstate is thermal

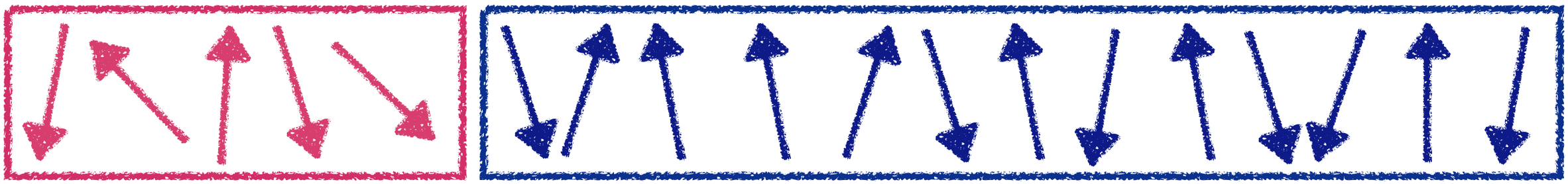
$$\langle E|O_i|E\rangle = \langle O_i\rangle_{T(E)} + \mathcal{O}(e^{-cL})$$

Connection with Random Matrices via Berry's conjecture:
Eigenstates 'ressemble' eigenstates of a RM

More details and refinements later

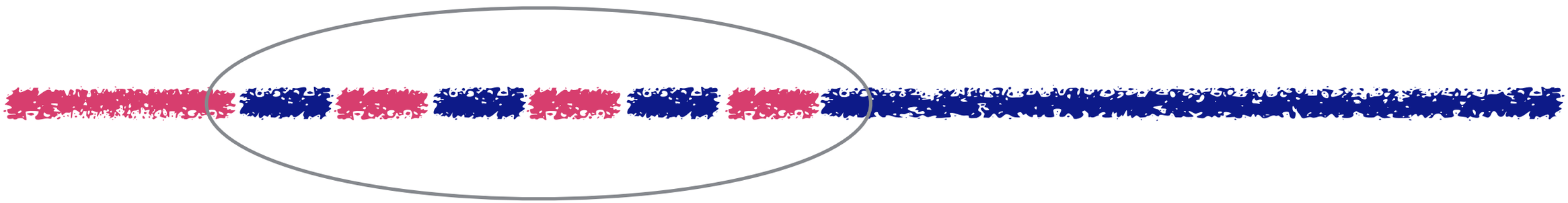
Small bath connected to disordered spins

Finite bath + localized spins



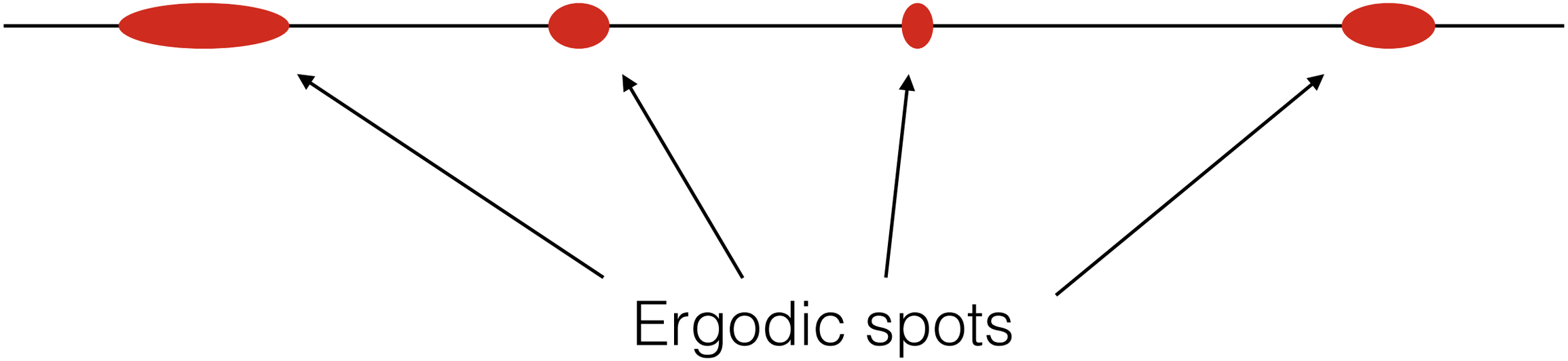
ETH MBL

Aim: Find a description of the **interface region**



Natural set-up

Disorder fluctuations create small baths



Stability issue:
Could the interface region be of the order of the full volume?

Avalanche scenario

(too) naive scenario:

Ergodic spot thermalizes the near spins…

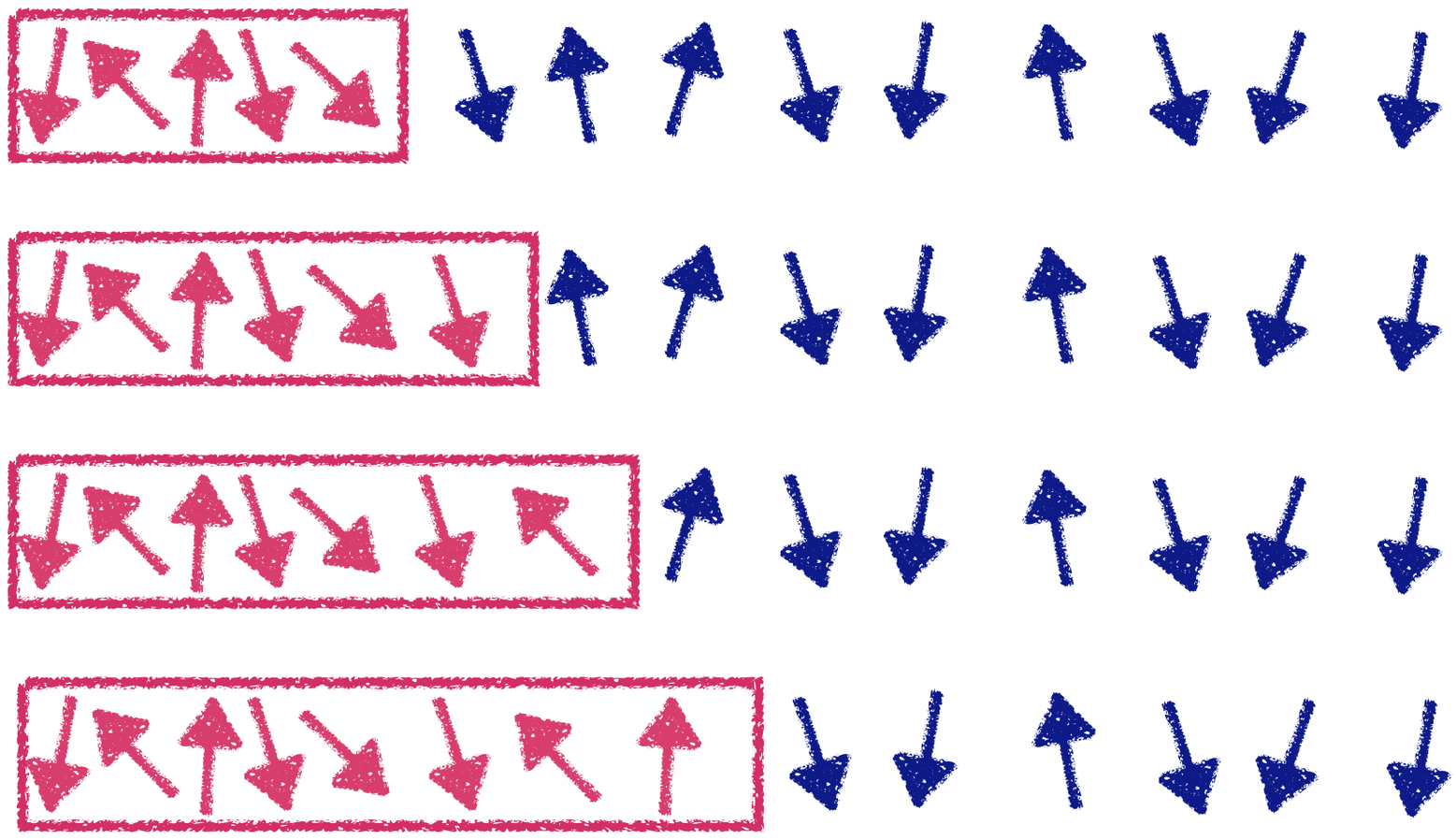
… and becomes a larger spot

Eventually, the full material becomes thermal!

- d=1: ‘good’ models: Fallacy! — see Imbrie (’12)
- d>1: open question

Our approach

Couple spins one by one to a small bath



Random Matrix + Energy conservation

1 spin coupled to a bath

Single spin coupled to an ergodic Hamiltonian:

$$H = H_B + H_{BS} + H_S$$

$$H_B = \text{Ergodic Ham}, \quad H_{BS} = g\,\sigma_B^{(x)}\sigma_S^{(x)}, \quad H_S = h\,\sigma_S^{(z)}$$

How do uncoupled states hybridize to form the eigenstates?

$$|E\rangle = \sum_{b,s} \phi_E(b,s)|b,s\rangle$$

?

I. Local operators in the bath (ETH)

A local operator is characterized by 3 energy scales:

$$\langle \mathsf{b}'|\sigma_{\mathsf{B}}|\mathsf{b}\rangle \;=\; \sqrt{\delta} \;\times\; \sqrt{\mathsf{v}(\omega)} \;\times\; \eta_{\mathsf{b},\mathsf{b}'}, \qquad \omega = \mathsf{E}(\mathsf{b}') - \mathsf{E}(\mathsf{b})$$

Level spacing
(global scale)

Structure factor
(scale: energy per site)

‘Random’ phases
(scale: level spacing)

$$\int \mathrm{d}\omega\, \mathsf{v}(\omega) \;\sim\; 1, \qquad \text{local interactions:}\;\; \mathsf{v}(\omega) \;\sim\; \mathrm{e}^{-|\omega|/\varepsilon}$$

II. Condition for thermalization

The Bath is not ‘perfect’: strong enough coupling needed

$$\mathcal{G} := \frac{\text{Matrix element}}{\text{Level spacing}} = \frac{\langle b', s'|H_{BS}|b, s\rangle}{\delta/2}$$

$$\sim \frac{g \times \sqrt{v(\omega)} \times \sqrt{\delta}}{\delta/2}, \qquad \omega = E_{b'} + E_{s'} - E_b - E_s$$

$\mathcal{G} \ll 1$: product structure approximatively preserved

$\mathcal{G} \gg 1$: hybridization, the spin gets ‘thermalized’

III. Energy conservation

States only hybridize within some energy window

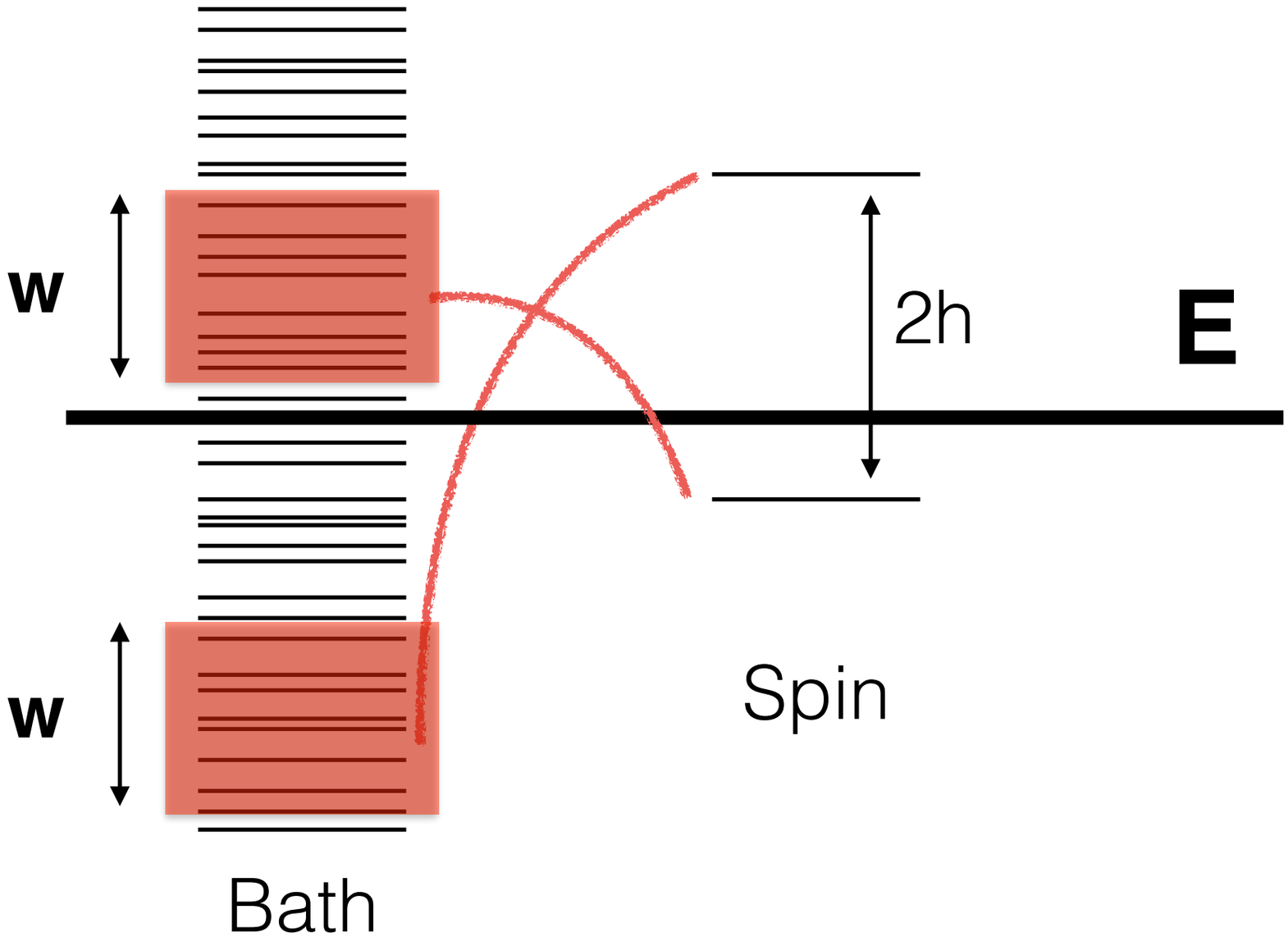
$$|E_B + E_S - E| \lesssim w$$

Fermi Golden Rule:

$$w \sim g^2 \nu(2h)$$

Upper bound on **w** can be justified through Feshbach formula

III. Energy conservation (pictorially)



Cartoon for the eigenstates of H

$$|\mathsf{E}\rangle \;=\; \sum_{\mathsf{b},\mathsf{s}} \phi_{\mathsf{E}}(\mathsf{b},\mathsf{s})|\mathsf{b},\mathsf{s}\rangle$$

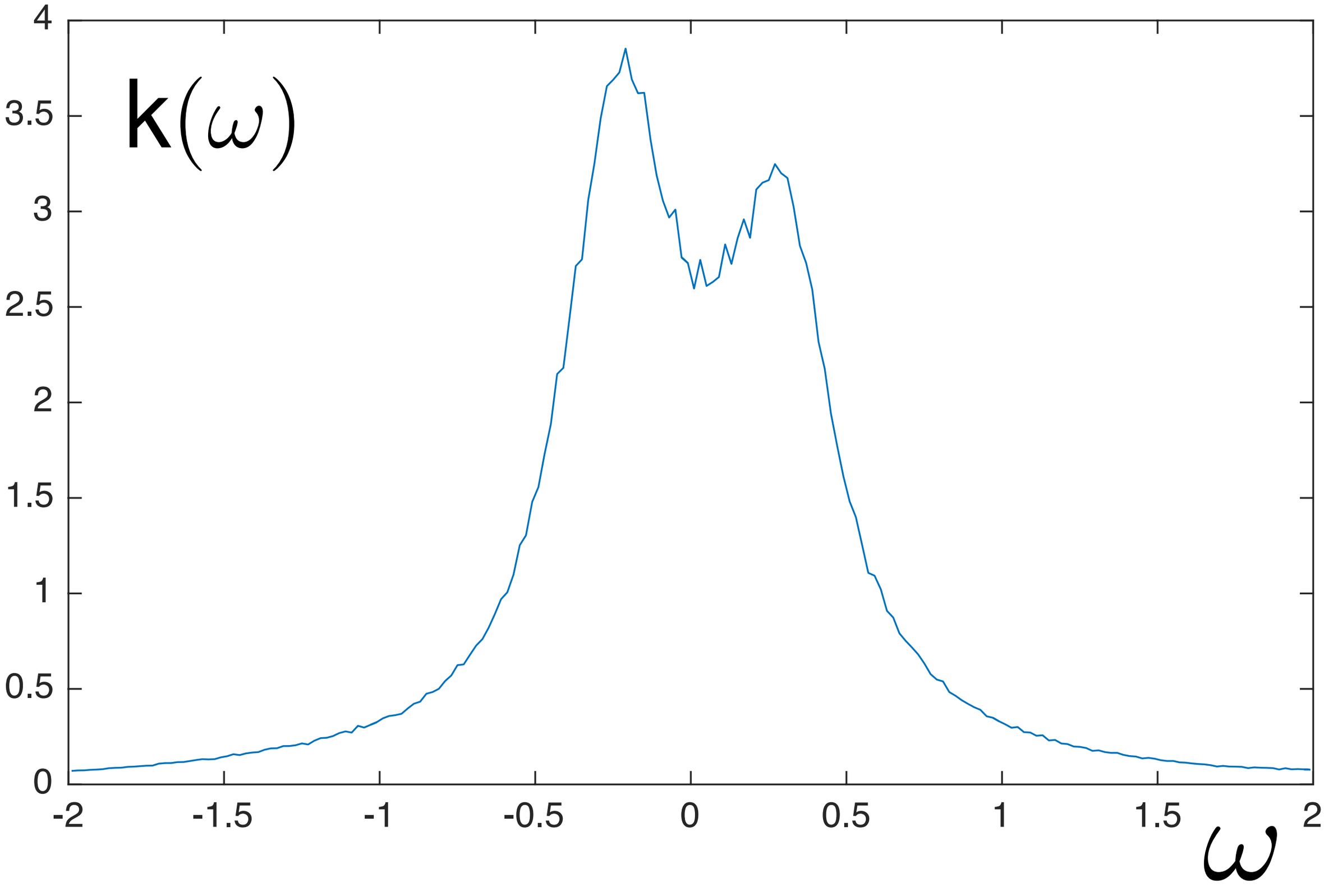
$$\phi_{\mathsf{E}}(\mathsf{b},\mathsf{s}) \;=\; \sqrt{\delta/2} \;\times\; \sqrt{\mathsf{k}(\omega)} \;\times\; \eta_{\mathsf{E}}(\mathsf{b},\mathsf{s})$$

Full level spacing — Hybridization function — ‘Random’ phase

$$\omega \;=\; \mathsf{E}_{\mathsf{B}} + \mathsf{E}_{\mathsf{S}} - \mathsf{E} \qquad\qquad \mathsf{k}(\omega) \;\sim\; \frac{1}{\mathsf{w}\left(1 + (\omega/\mathsf{w})^2\right)}$$

Some refinements will be needed later…

E.g.: bath with 11 spins



New structure factors

- Crucial to iterate the procedure
- We distinguish between bath / spin operators

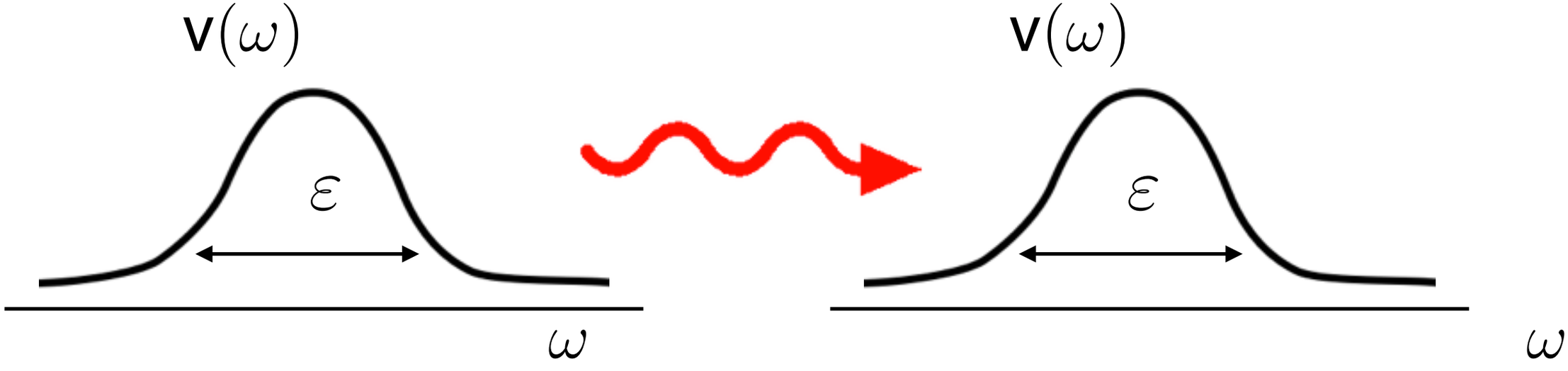
Common to both: ETH on the **full** Hilbert space:

$$\langle E'|\sigma_i|E\rangle = \sqrt{\delta/2} \times \sqrt{v(\omega)} \times \eta_{E,E'}, \qquad \omega = E' - E$$

different for $i \in B$ or $i = S$

Bath operator

In first approximation, structure factors do not change



!!! Dimension is **twice** as big as without extra spin:

The bath is now ‘closer to being an ideal bath’

Bath operator (refinement)

This cannot be precisely true… Structure factors have a dynamical meaning:

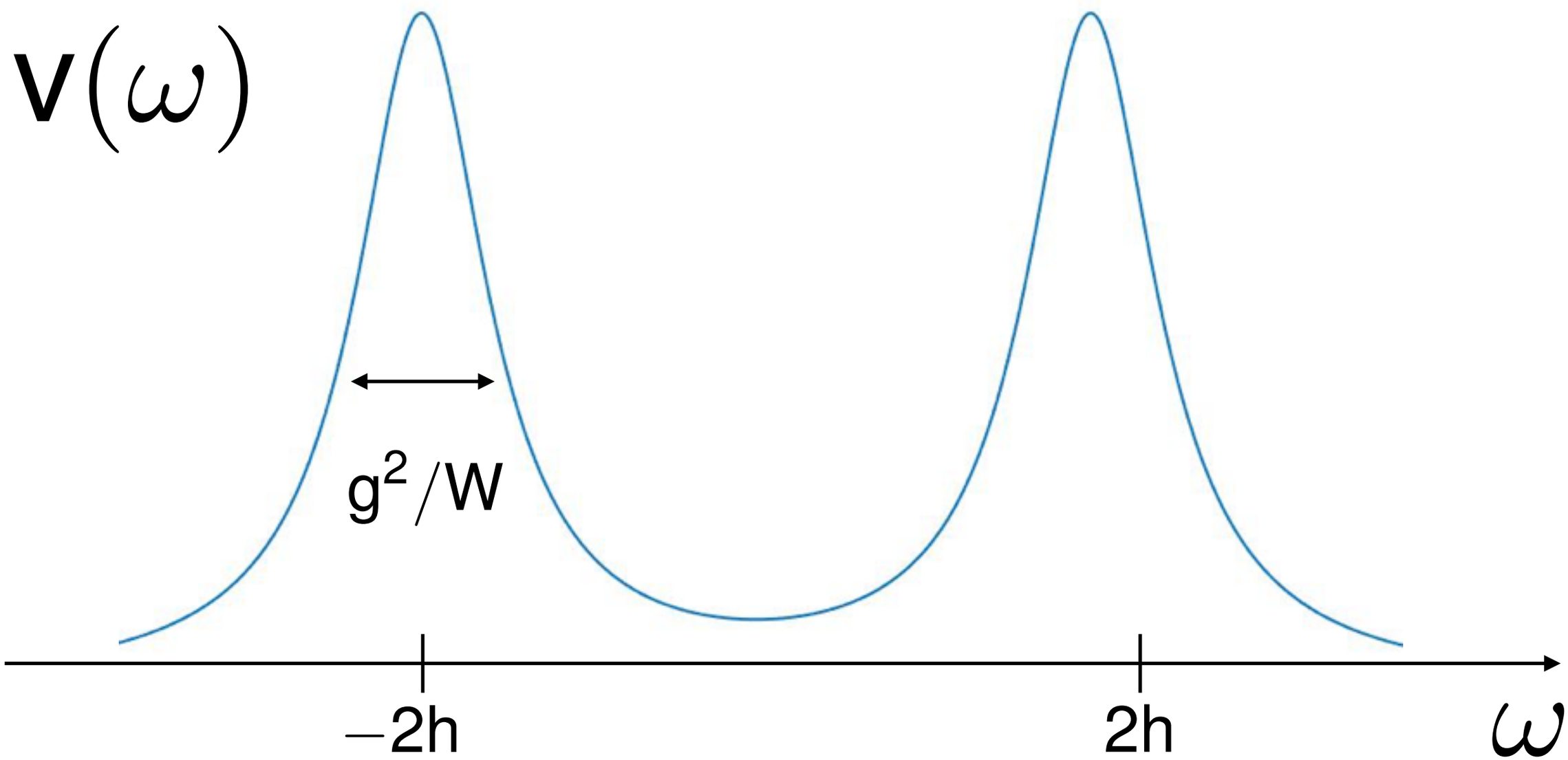
$$\mathbf{v}(\omega) \;=\; \int \mathbf{dt}\, \mathbf{e}^{-\mathrm{i}\omega \mathrm{t}} \langle \sigma_{\mathrm{i}}(\mathbf{t}) \sigma_{\mathrm{i}} \rangle_{\beta(\mathbf{E})}$$

Convergence to equilibrium of a bath operator is (weakly) affected by the (weakly) coupled spin.

The back-reaction from the spin to the bath can be estimated. It leads to a ‘small’ correction. Debated issue!

Spin operator

The structure factor exhibits strong frequency dependence



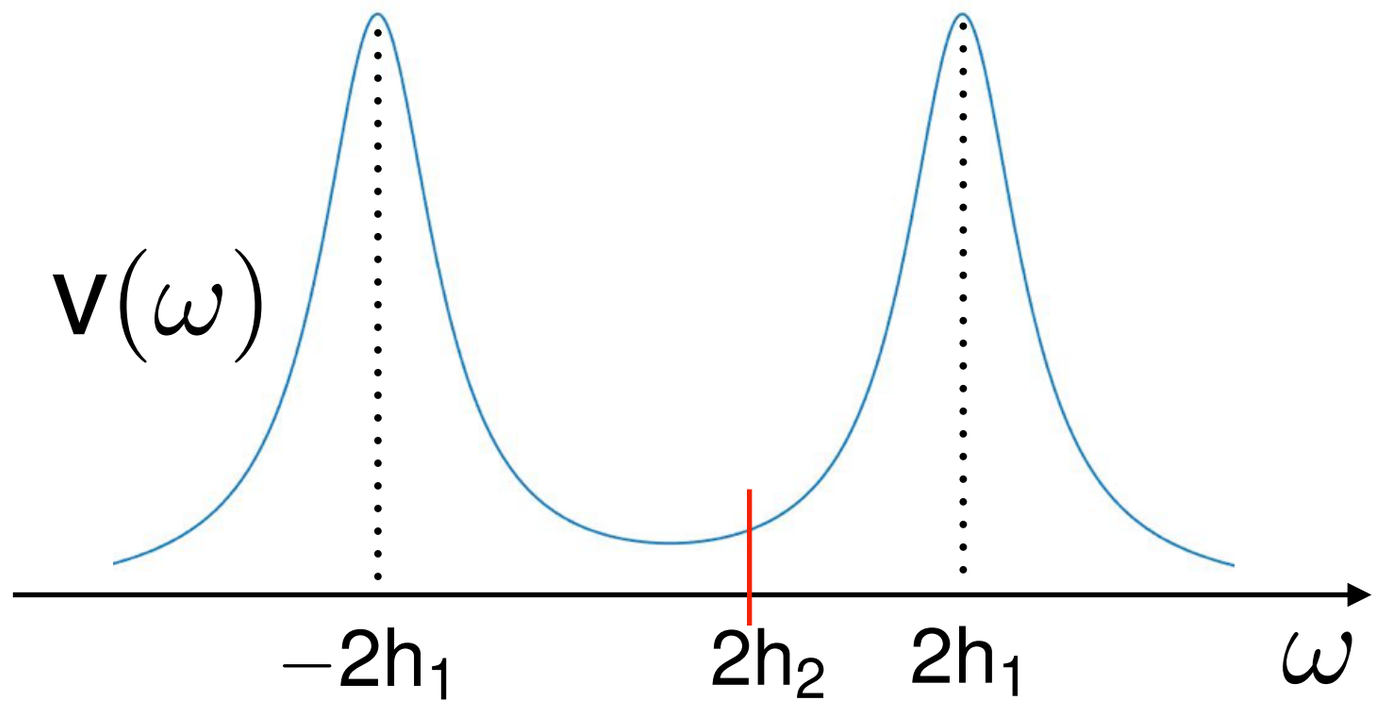
Coupling a second spin

New Bath + spin_2 = Bath + spin_1 + spin_2

Coupling is suppressed due to frequency mismatch:

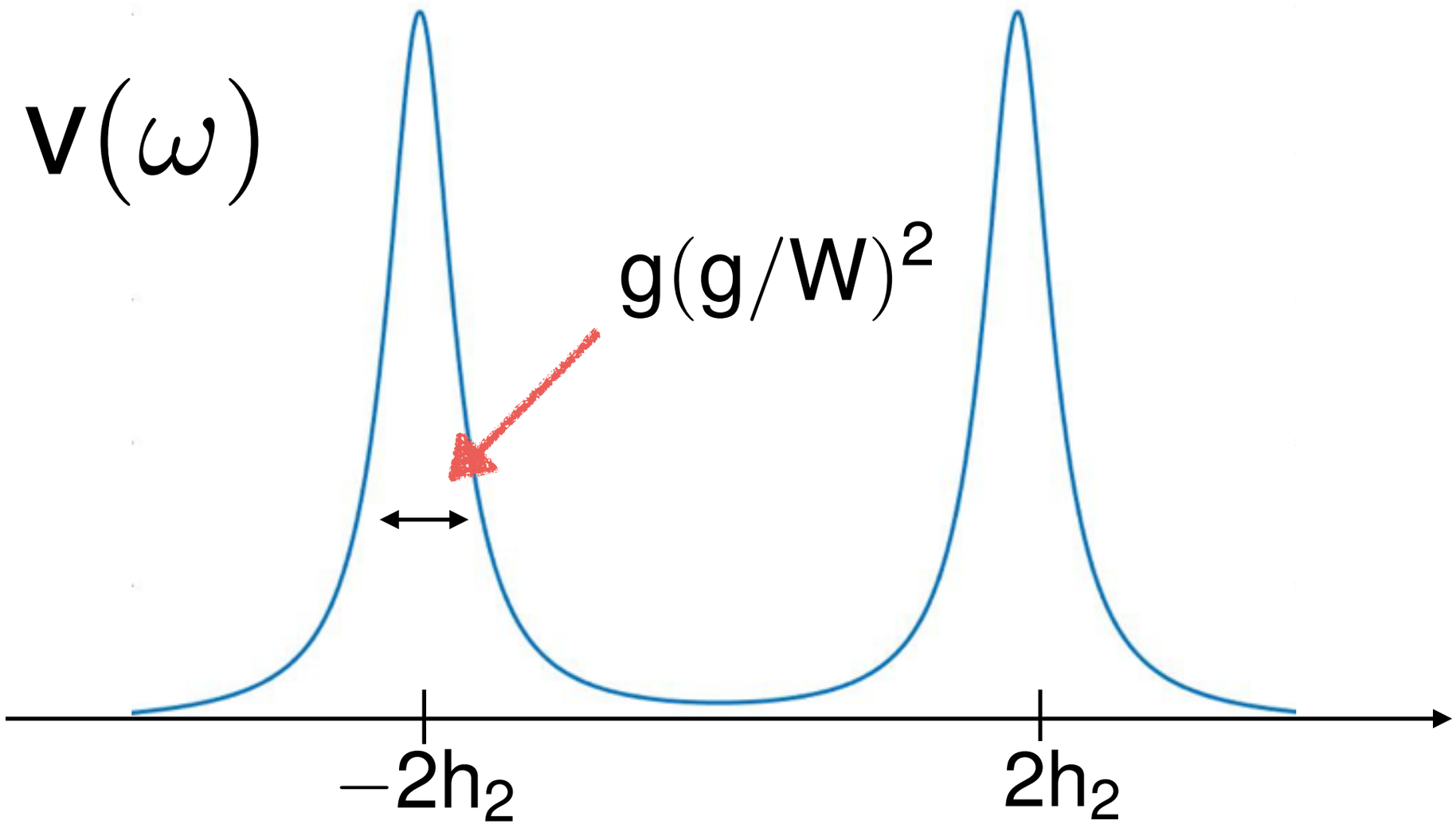
$$|E(b') + E(s_2) - E| \lesssim w$$

$$w \sim g^2 v(2h_2) \quad (\mathbf{v}\text{: structure factor of spin 1})$$



Structure factor of the second spin

Much more narrow than that of the first spin:



(from estimates on the tails of the structure factor of spin 1)

How far does this go (d=1)?

This process stops when the width of the structure factors becomes of the order of the level spacing

With **n** extra spins: width: $(g/W)^n =: e^{-n/\xi}$

level spacing: $2^{-(\ell_B+n)}$

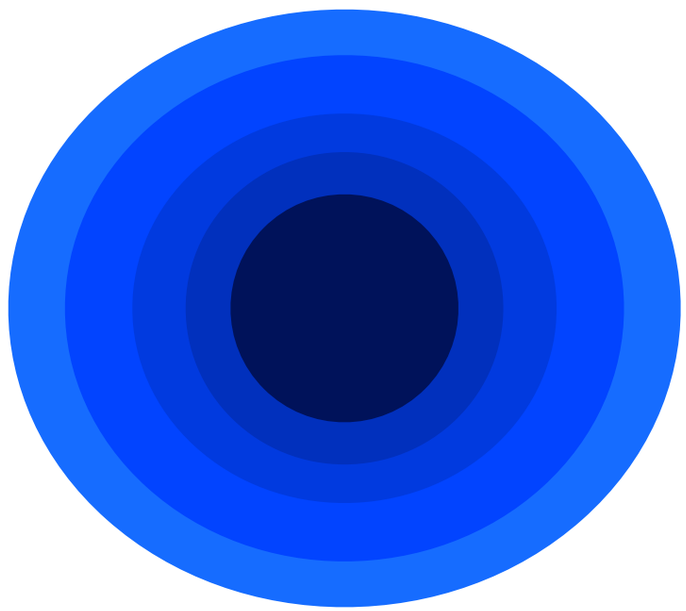
The size of the boundary region is thus given by

$$n = \frac{\log 2}{\xi^{-1} - \log 2} \times \ell_B$$

provided that $\xi < (\log 2)^{-1}$

And for d>1 ?

We imagine coupling layers of spins:



When **n** layers are coupled:

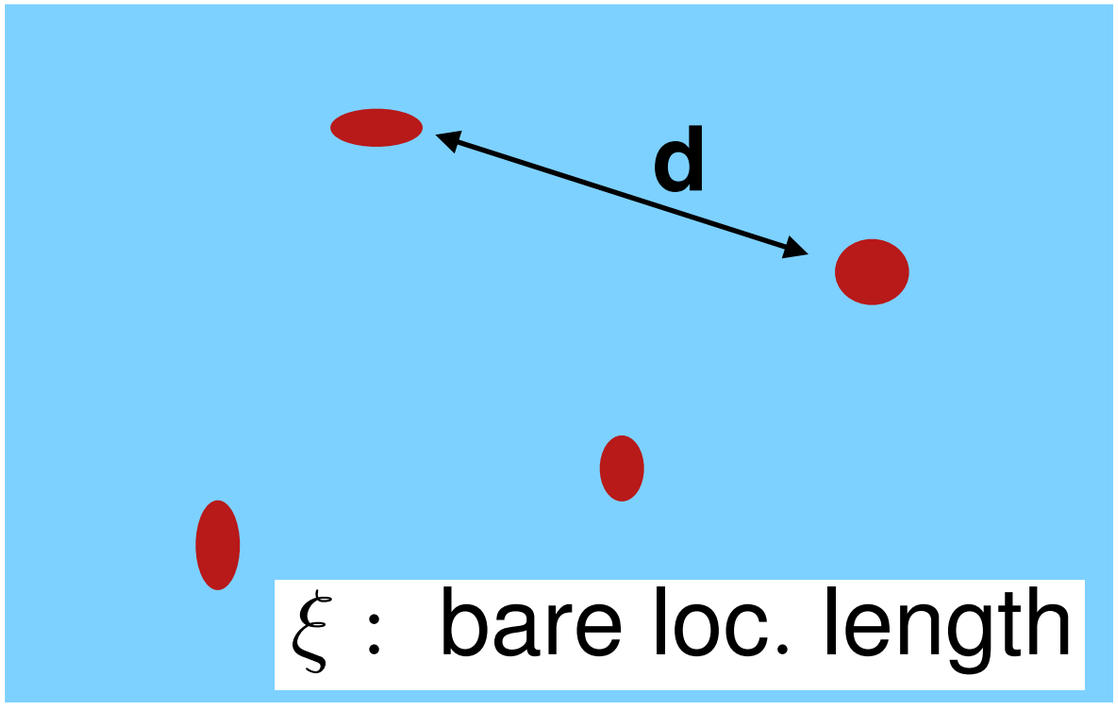
Width of the structure factors: $(g/W)^n =: e^{-n/\xi}$

Level spacing: $2^{-(\ell_B+n)^2}$ ← $2 = d$

If $\ell_B > \dfrac{1}{4\xi \log 2}$, the hybridization process never stops!

See also Nandkishore et al., PRB 90, 2014

Very long time scales for thermalization



$$t_{Therm} \sim e^{d/\xi} \qquad d \sim e^{\xi^{-3}}$$

Transport due to this mechanism is astronomically reduced as the bare localization length is sent to 0!

Concrete measurement: IPRs

- Peaks in the structure factors reduce the dimension on the space on which a local operator acts effectively
- IPR: parameter to quantify this *effective dimension*

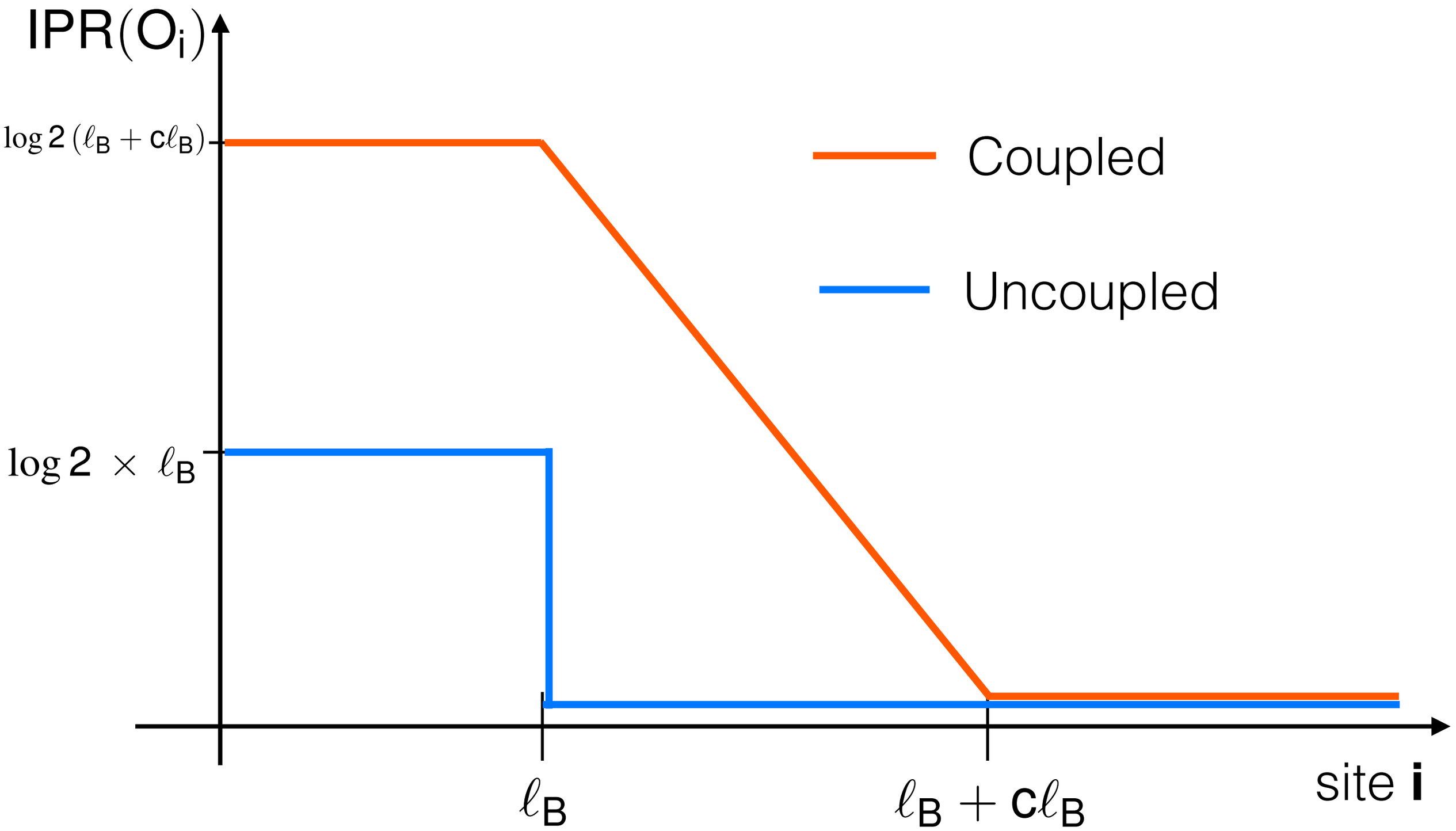
Fix an energy **E**:

$$\mathrm{IPR}(O_i) \;:=\; -\log \sum_{E'} |\langle E'|O_i|E\rangle|^4$$

Ergodic: $\mathrm{IPR}(O_i) \sim L$

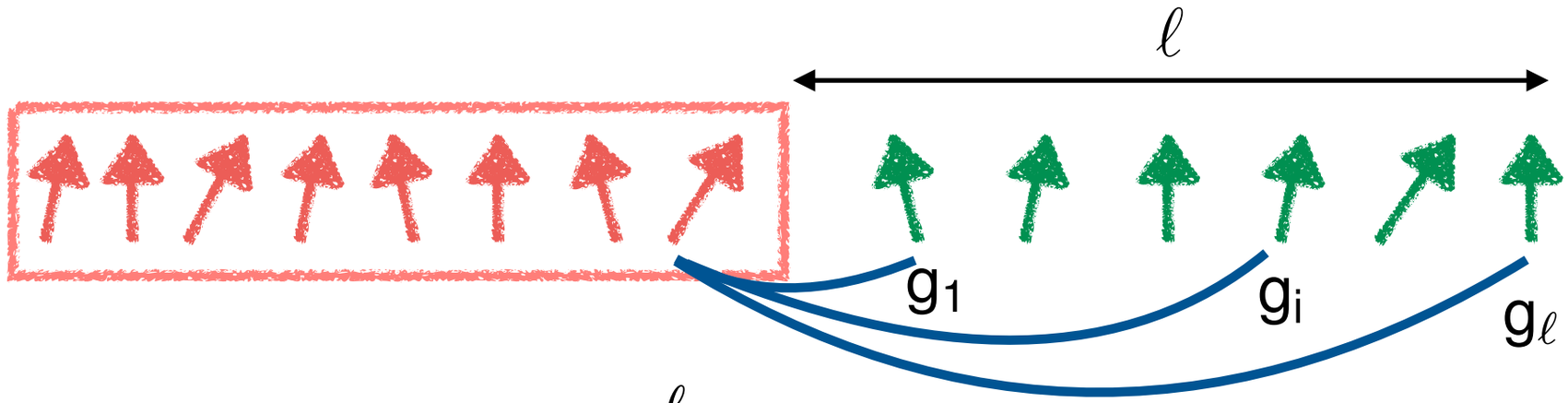
MBL: $\mathrm{IPR}(O_i) \sim 1$

Prediction for IPRs



Numerics for IPRs

Localized spins already diagonalized (LIOMs): No interaction among themselves, only bath-spin interaction via tails



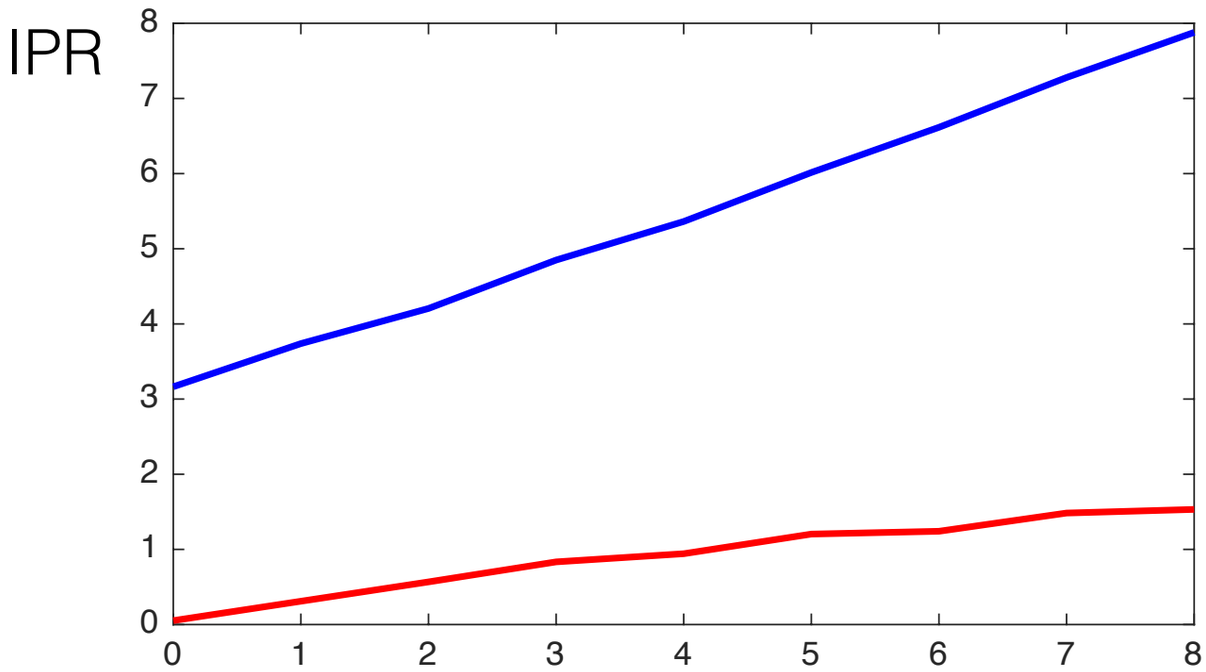
$$H = H_B + \sum_{i=1}^{\ell} \left(g_i \, \sigma_B^{(x)} \sigma_i^{(x)} + h_i \, \sigma_i^{(z)} \right)$$

H_B : Bath hamiltonian (RM or ergodic)

$g_i = 0.2 \times (0.75)^i$

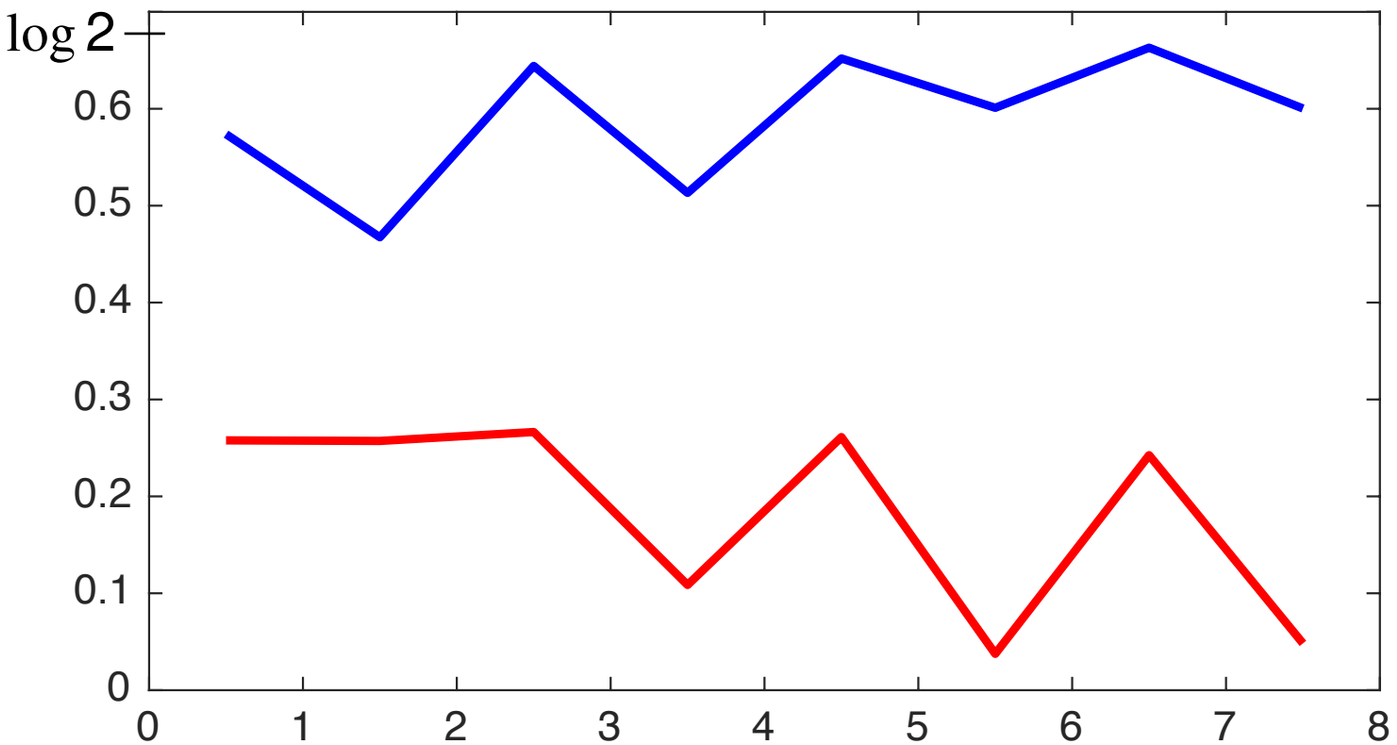
$0.5 < h_i < 1.5$ (random)

IPRs for operators in the bath



IPR of $\sigma_0^{(x)}$ for a 6 spins RM bath

IPR of $\sigma_0^{(x)}$ for a 1 spin bath



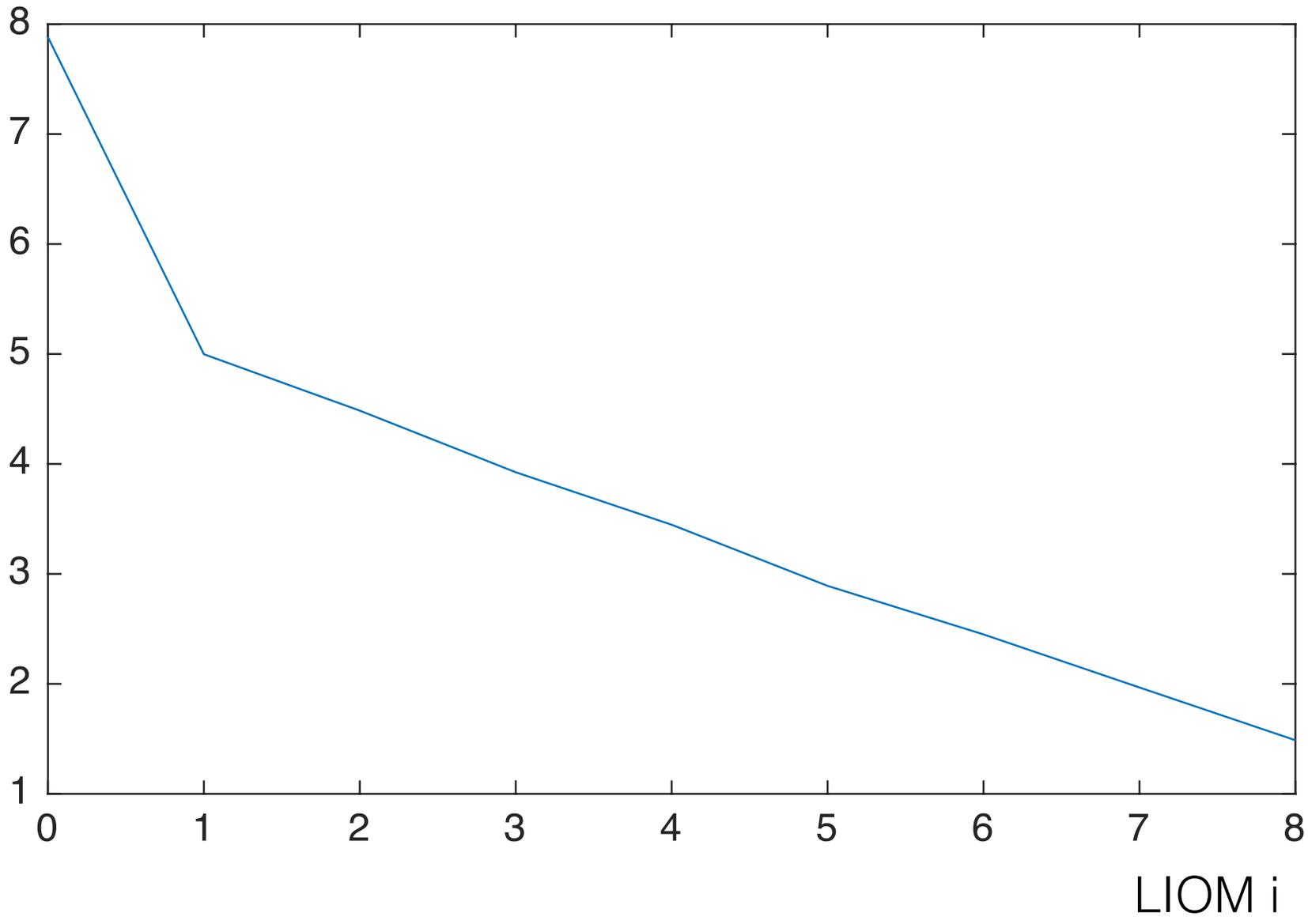
Slopes of the above curves

Theoretical prediction:

$\log 2$

IPRs for spin operators

IPR of $\sigma_i^{(x)}$



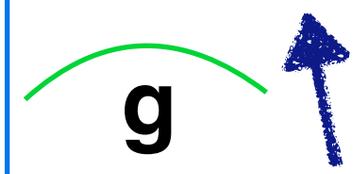
Theory predicts linear decrease of IPR outside the bath

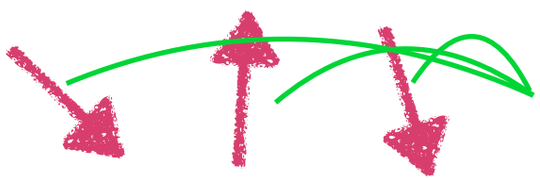
Universal features of thermalization

Thermalization of an external spin by three different baths:

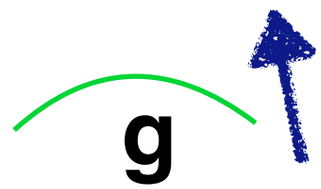
random matrix $\quad \dim = 2^9$

9 spins bath





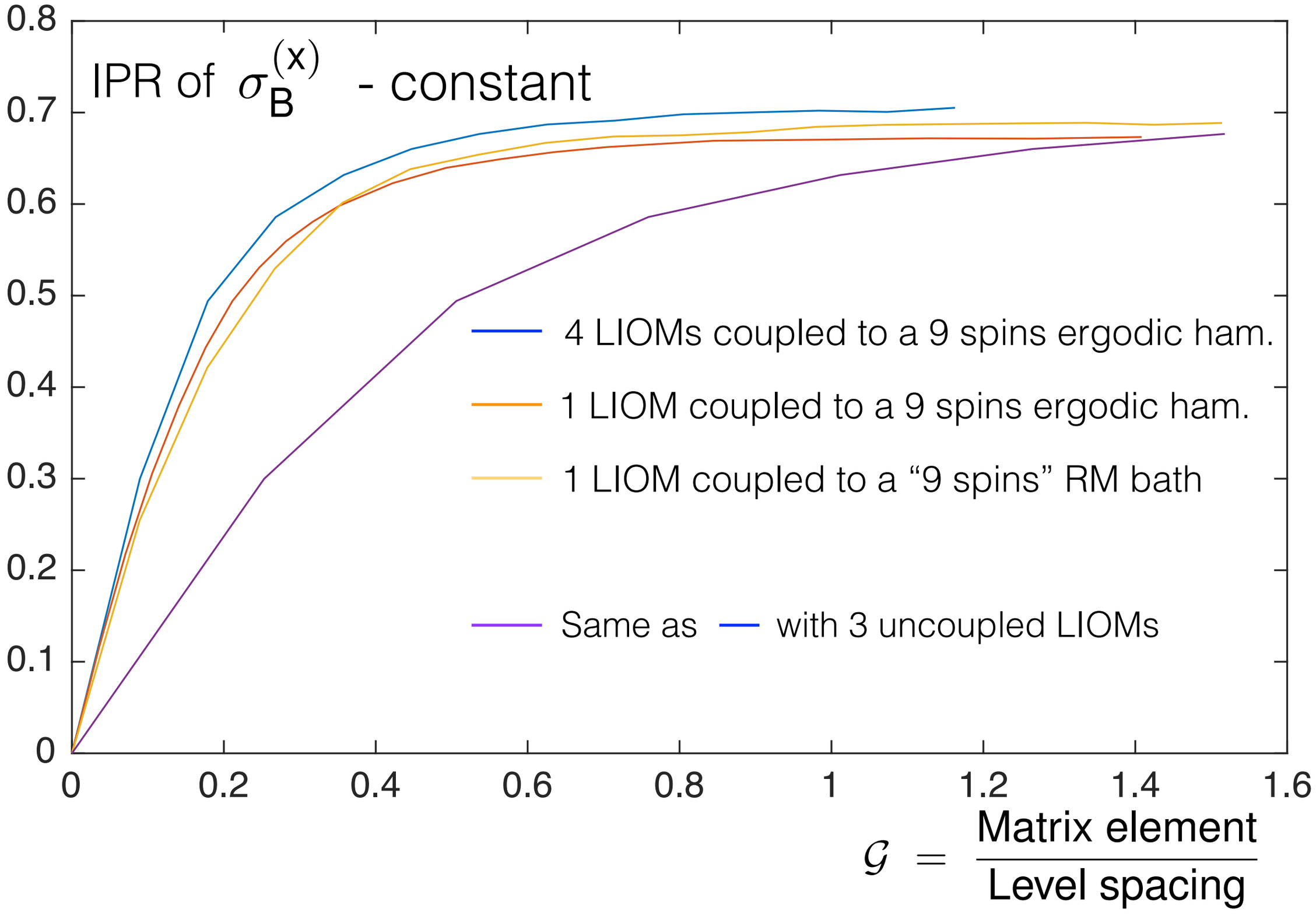
9 spins bath



To compare them, replace **g**
by the dimensionless parameter:

$$\mathcal{G} = \frac{\text{Matrix element}}{\text{Level spacing}}$$

Universal feature of thermalization



Conclusions

- Qualitative description of the boundary region bath / MBL
- Quantitative description through a simple parameter (IPR)
- Universal features of thermalization
- (In)stability of MBL in d>1 or with ‘long range’ hopping ?

See W. De Roeck, F. Huveneers, arXiv:1608.01815