Random walk in 1d diffusive random environment

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discussions with F. Simenhaus, M. Salvi and F. Völlering

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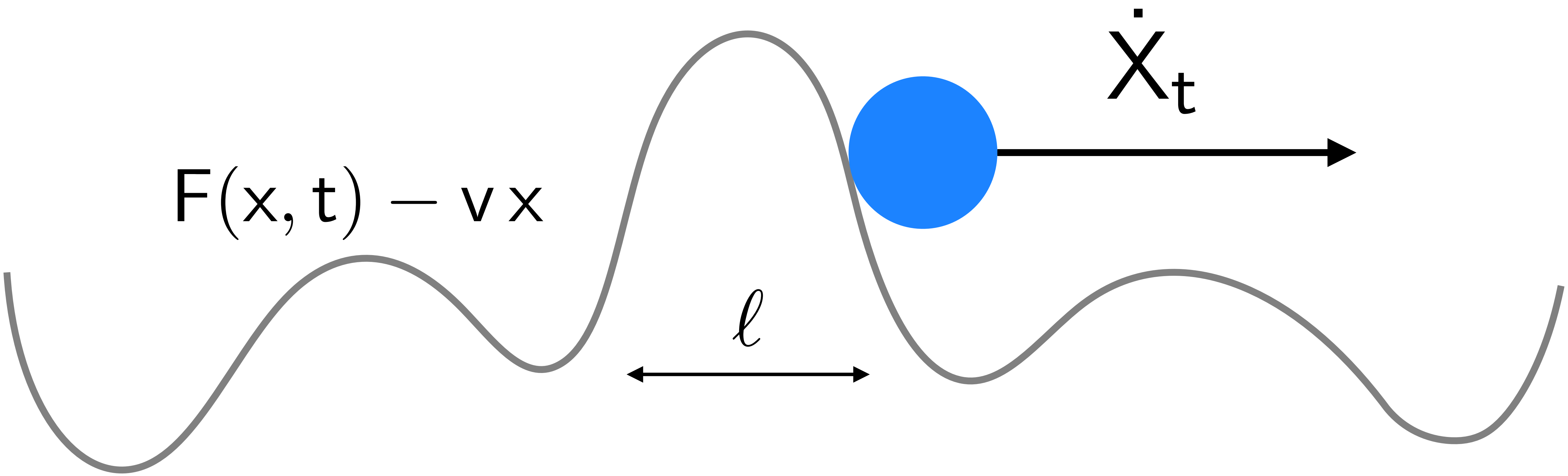
Passive particle = walker
on a fluctuating surface

$$\dot{X}_t = \lambda\big(-\partial_x F(X_t, t) + v \big) \qquad (\lambda > 0)$$

fluctuation

drift $(0 \leq v \ll 1)$



the fluctuating surface F(x,t)

F(x,t) of Edwards-Wilkinson type:

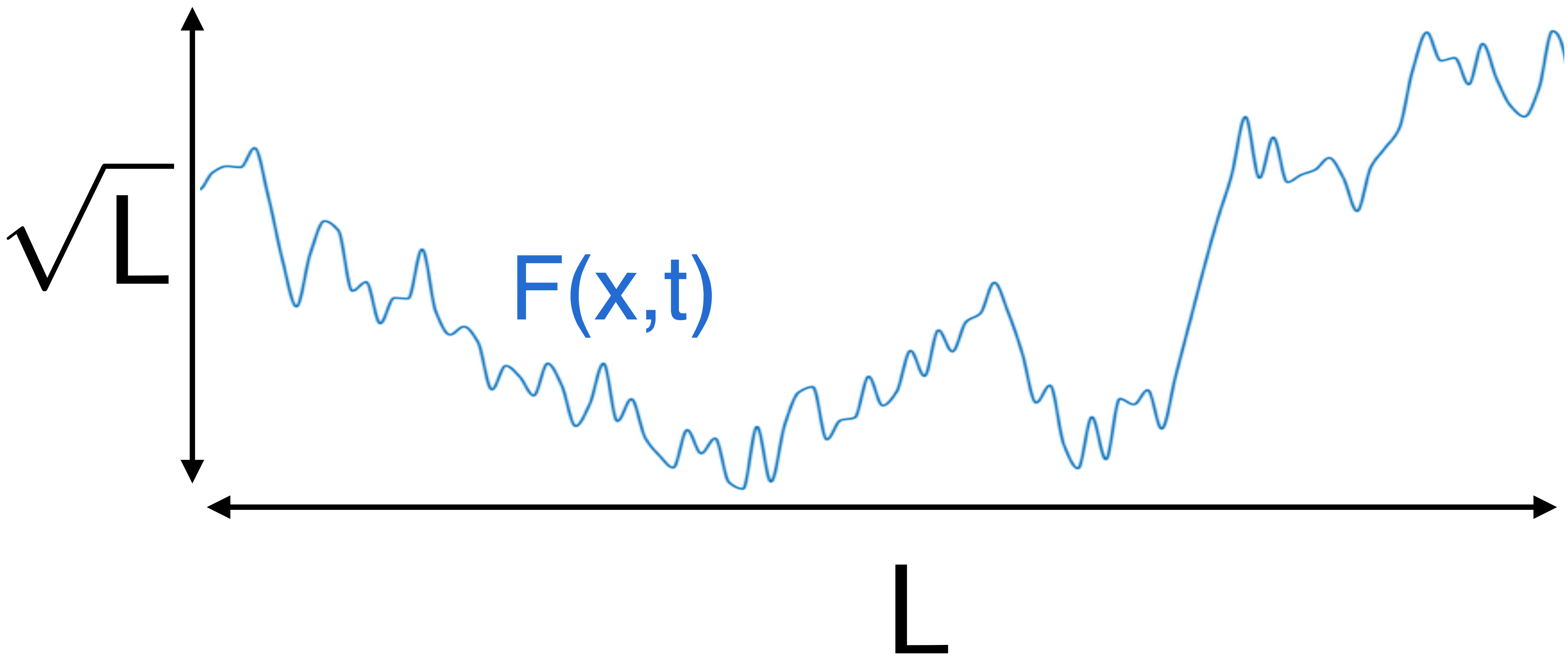
- $\langle \partial_x F(x,t) \rangle = 0$
- $\partial_t F \; = \; D\, \partial_x^2 F + \xi$ (D: diffusion constant)
- $\xi = \xi(x,t)$ white noise in time,
  correlation length ℓ in space ($\ell = 1$ here)

Remark:

- Two rates: $\lambda\,, D$
- Dimensionless control parameters: $\lambda / D\,, v$

Main characteristics of **F(x,t)**

Relaxation time ~ L^2 / D



Formal definition of **F(x,t)**

$$-\partial_x F(x,t) \;=\; \int_{\mathbb{R}} dk \left(\sqrt{\ell}\, e^{-(k\ell)^2}\right) \left(A_k(t)\, e^{ikx} \,+\, cc\right) \qquad (\ell = 1)$$

$$\langle A_k(s) A^*_{k'}(t) \rangle \;=\; \delta(k - k')\, e^{-Dk^2|t-s|} \qquad \text{(gaussian)}$$

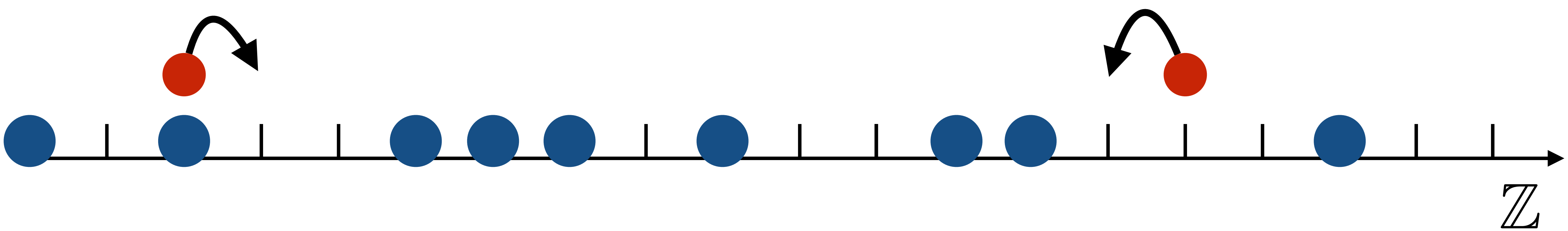
The field is a superposition of independent modes:

mode **k**: wavelength: $1/k$ relaxation time: $1/Dk^2$

both useful way to think, and used in the numerics

Similar behavior expected for similar models

e.g. popular lattice model (see Avena and Thomann, J. Stat. Phys. 2012):



$-\partial_x \mathsf{F} + \mathsf{v}$ ⟷ SSEP at density $\rho = 1/2 + \mathsf{v}$

diffusion constant **D** ⟷ SSEP with rate **D**

coupling strength λ ⟷ jump rate λ of the walker

Randomness on the walker (molecular mobility) could be added too

What is interesting about it?

1) Diffusive environment come out naturally in the presence of single conserved quantity

2) Predicting the behavior of the walker is puzzling!

$$\left\langle \partial_x F(x,s)\, \partial_x F(y,t) \right\rangle \sim \frac{e^{-\frac{(x-y)^2}{D|t-s|}}}{(D|t-s|)^{1/2}}$$

Slow decay of correlations $\Rightarrow$ memory effects

$\Rightarrow$ **? ? ?**

(Gopalakrishnan, Phys Rev. E, 2004; Avena and Thomann, J. Stat. Phys. 2012)

Main questions

1) Does the walker reach a NESS at $v > 0$ and/or $v = 0$?
If yes, how fast?

NESS: non-equilibrium stationary state,
invariant measure for the environment seen by the walker.

2) For $v > 0$, let $V(v) = \lim_{t\to\infty} X_t^v / t$

$$V(v) \sim v^{\alpha} \quad \text{as} \quad v \to 0, \quad \alpha\,?$$

3) For v=0,

$$\langle X_t^2 \rangle \sim t^{\beta} \quad \text{as} \quad t \to \infty, \quad \beta\,?$$

4) Do the answers depend on the value of λ/D ?

These questions are correlated

Much can be learned by studying the response of a system to an external perturbation.

E.g. linear response relates diffusion in equilibrium to fluxes out of equilibrium.

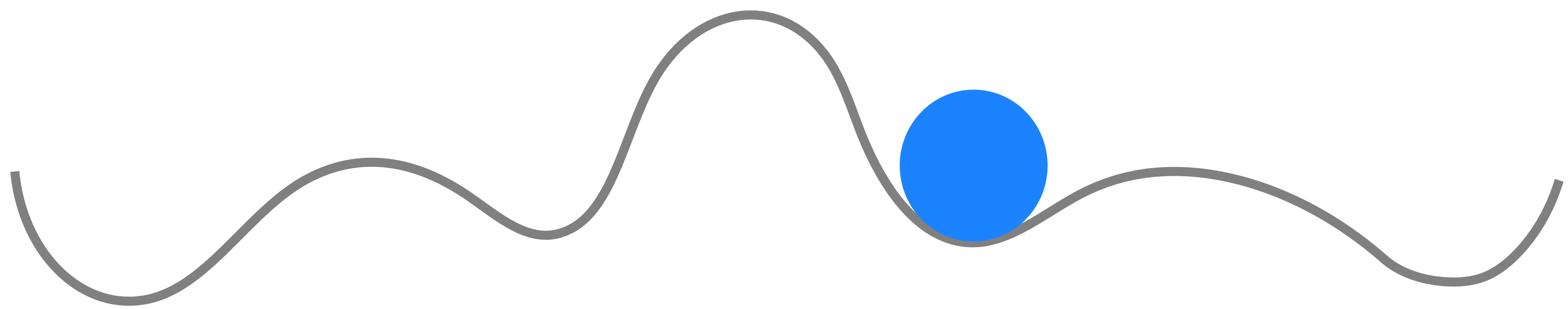
Generalization to systems out of equilibrium usually relies on the existence of a NESS.

Two distinct regimes

(Gopalakrishnan, Phys Rev. E, 2004)

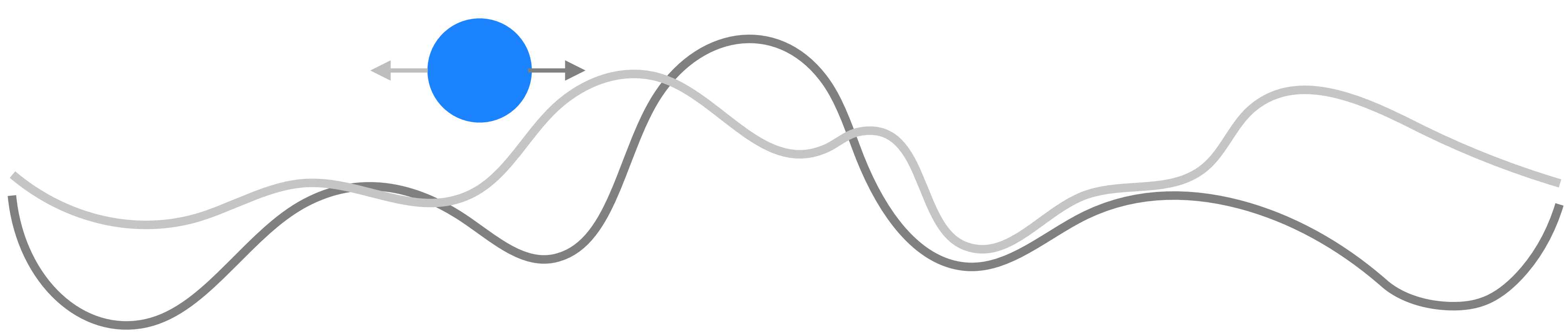
$\lambda/D \gg 1$: quasi-static or trapped

the walker sticks to the minima of F(x,t) - vx



$\lambda/D \ll 1$: homogenized regime

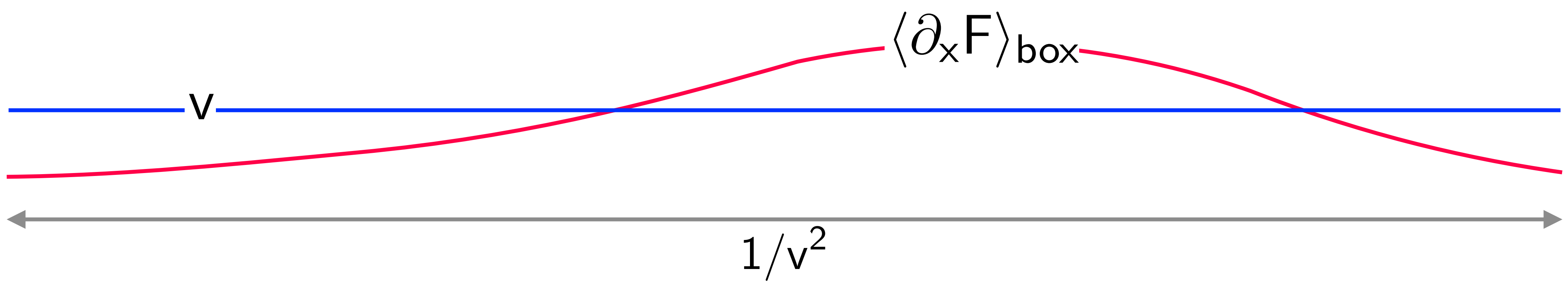
the walker evolves slower than the environment



1) Trapped regime $\lambda/D \gg 1$

Drift $V(v)$ for $v \to 0$

Spatial average of $\partial_x F$ in a box of size $1/v^2$: v



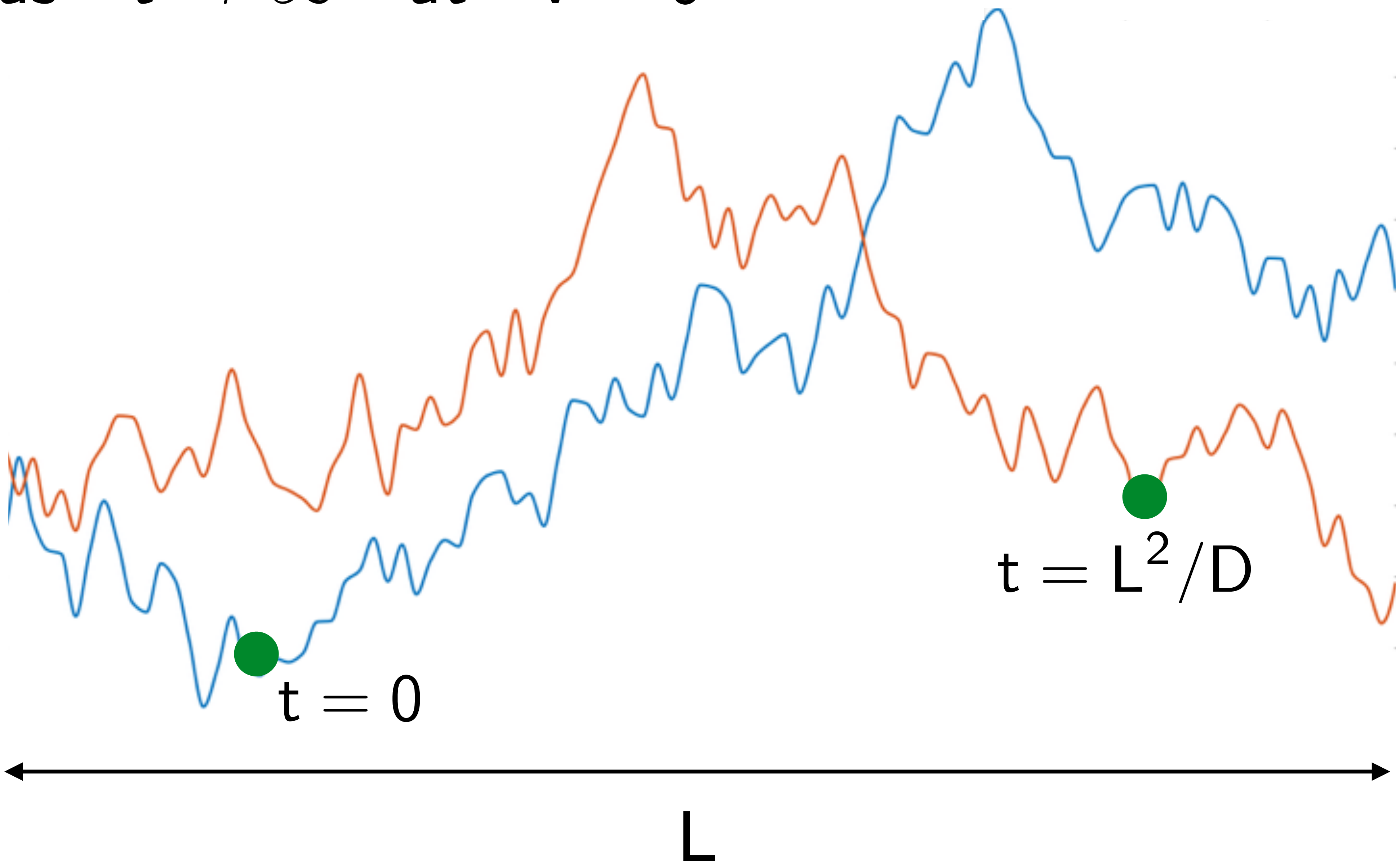
Relaxation time in this box ~ time to pass across : $1/Dv^4$

hence

$$V(v) = \frac{\text{space}}{\text{time}} = \frac{1/v^2}{1/Dv^4} = D^{-1}\, v^2$$

1) Trapped regime $\lambda/D \gg 1$

$\langle X_t^2 \rangle$ as $t \to \infty$ at $v = 0$



hence

$$\langle X_t^2 \rangle = Dt$$

2) Homogenized regime $\lambda/D \ll 1$

Self-consistent approximation:

$$X_t = \lambda \int_0^t ds \left(-\partial_x F(X_s, s) + v \right)$$

$$X_t = \lambda \int_0^t ds \left(-\partial_x F(Y_s, s) + v \right)$$

- Y_t is independent from the environment
- X_t and Y_t have same law

can make sense if the walker is indeed not trapped

2) Homogenized regime $\lambda/D \ll 1$

Solving for the 1st moment at **v > 0**:

$$\langle X_t \rangle = \lambda v t \quad \Rightarrow \quad \boxed{V(v) = \lambda v}$$

Solving (gaussian approx.) for the second moment at **v = 0**:

$$\langle X_t^2 \rangle = \lambda^2 \int_0^t \int_0^t ds ds' \langle \partial_x F(Y_s, s) \partial_x F(Y_{s'}, s') \rangle$$

$$\Rightarrow \quad \ldots \quad \Rightarrow \quad \boxed{\langle X_t^2 \rangle = (\lambda t)^{4/3}}$$

see F. Völlering, in prep.

Validity of the self-consistent approximation

- For the SCA to be valid, the NESS should exist and "ressemble" the equilibrium state of the environment

- Dynamical point of view: fluctuations at all scales should be strong enough to overcome the trapping

SCA yields the same super-diffusive exponent for Sinai walk in a static environment. We know $\langle X_T^2 \rangle \sim (\log T)^4$

Numerical results

1) time to stationarity at $v > 0$. Useful both by itself and to tune numerical experiments.

2) $V(v)$ as $v \to 0$

3) $\langle X_t^2 \rangle$ as $t \to \infty$ at $v = 0$

Numerics reveal that

1) the walker is always trapped (quasi-static regime)
2) possible important finite size effects
3) possible logarithmic like corrections wrt naive predictions (to be discussed later)

Time **T** to stationarity at **v > 0**

- Consider $-\partial_x F + v$ and decompose

$$-\partial_x F(x,t) = \int_{|k|\leq (cv)^2} dk(\dots) + \int_{|k|>(cv)^2} dk(\dots)$$

amplitude: **cv << v.**
Neglect this term!

get refreshed
in a time $1/Dv^4$

For generic observables: $\boxed{T \sim 1/Dv^4}$

- Thus **v > 0** = infra-red cut-off

- We expect no NESS at v = 0

Numerics: convergence of V(v)

We test this scaling on

$$V(v,t) = \frac{\langle X_t \rangle}{t} \to V(v) \quad \text{as} \quad t \to \infty$$

Given $v > 0$, let

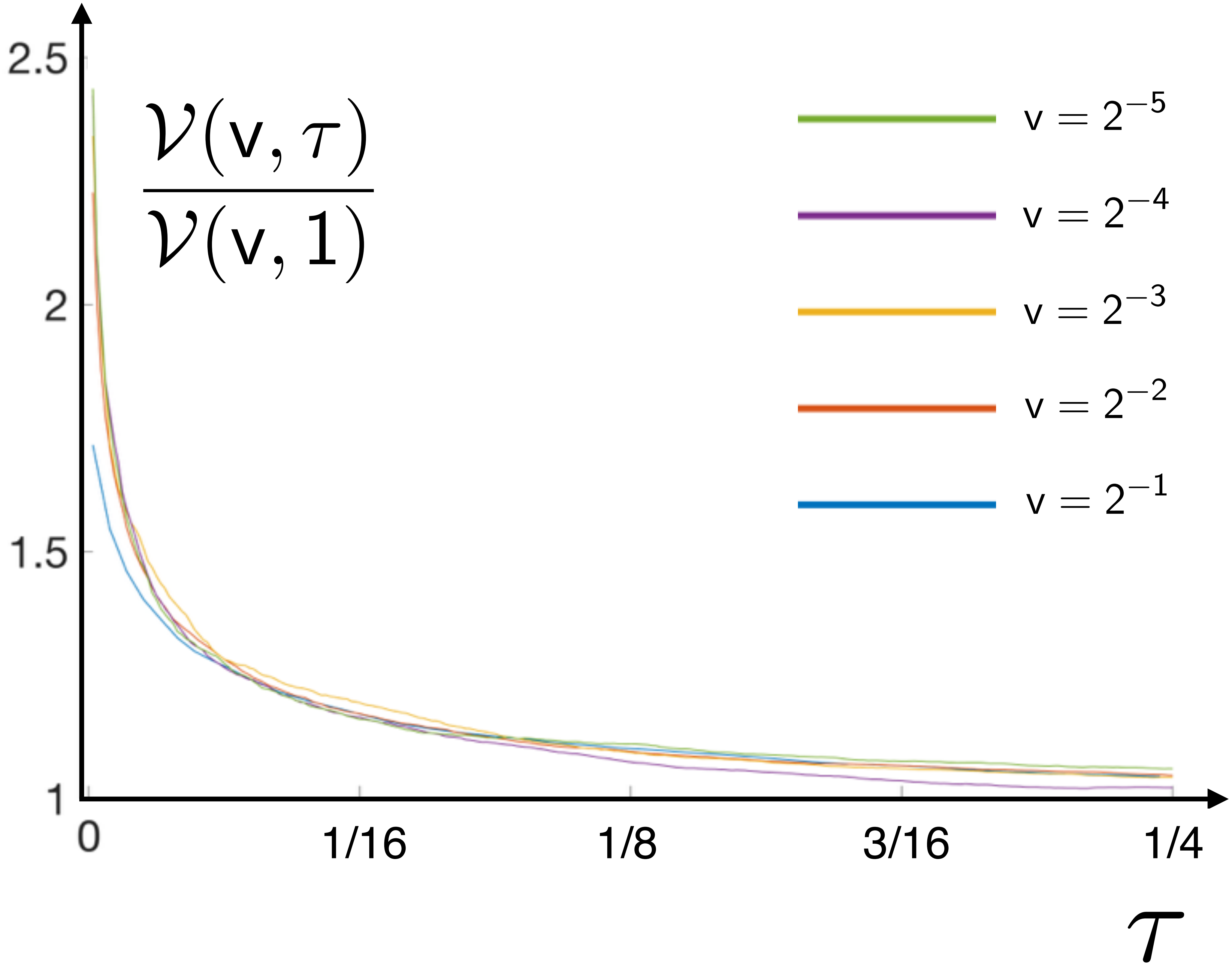
$$t_* = K/Dv^4 \quad (K \gg 1)$$

$$\tau = t/t_*$$

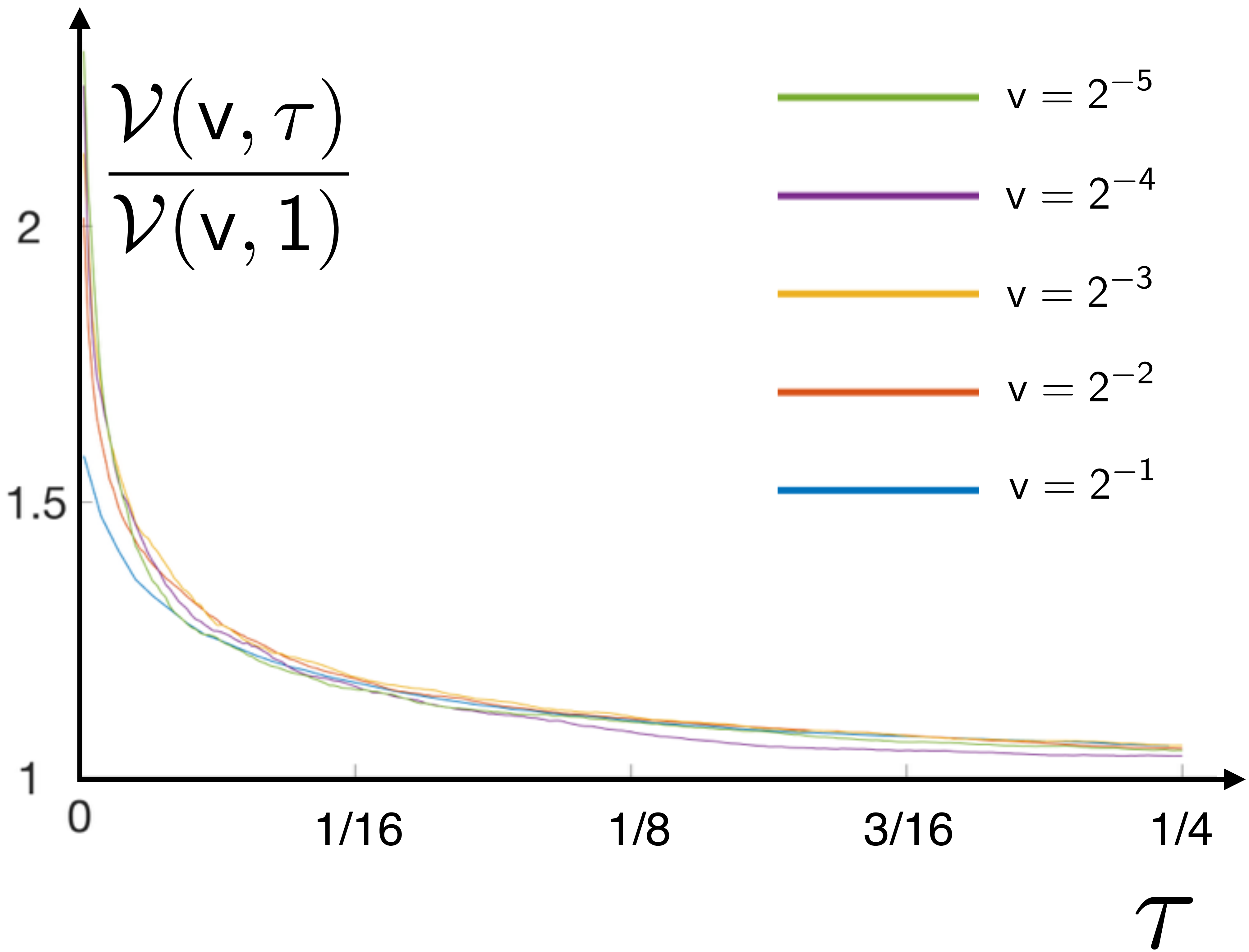
Define V as a function of rescaled time:

$$\mathcal{V}(v,\tau) = \frac{\langle X_{t_*\tau} \rangle}{t_*\tau}$$

$\lambda = 1$ (D = 1 for numerics)

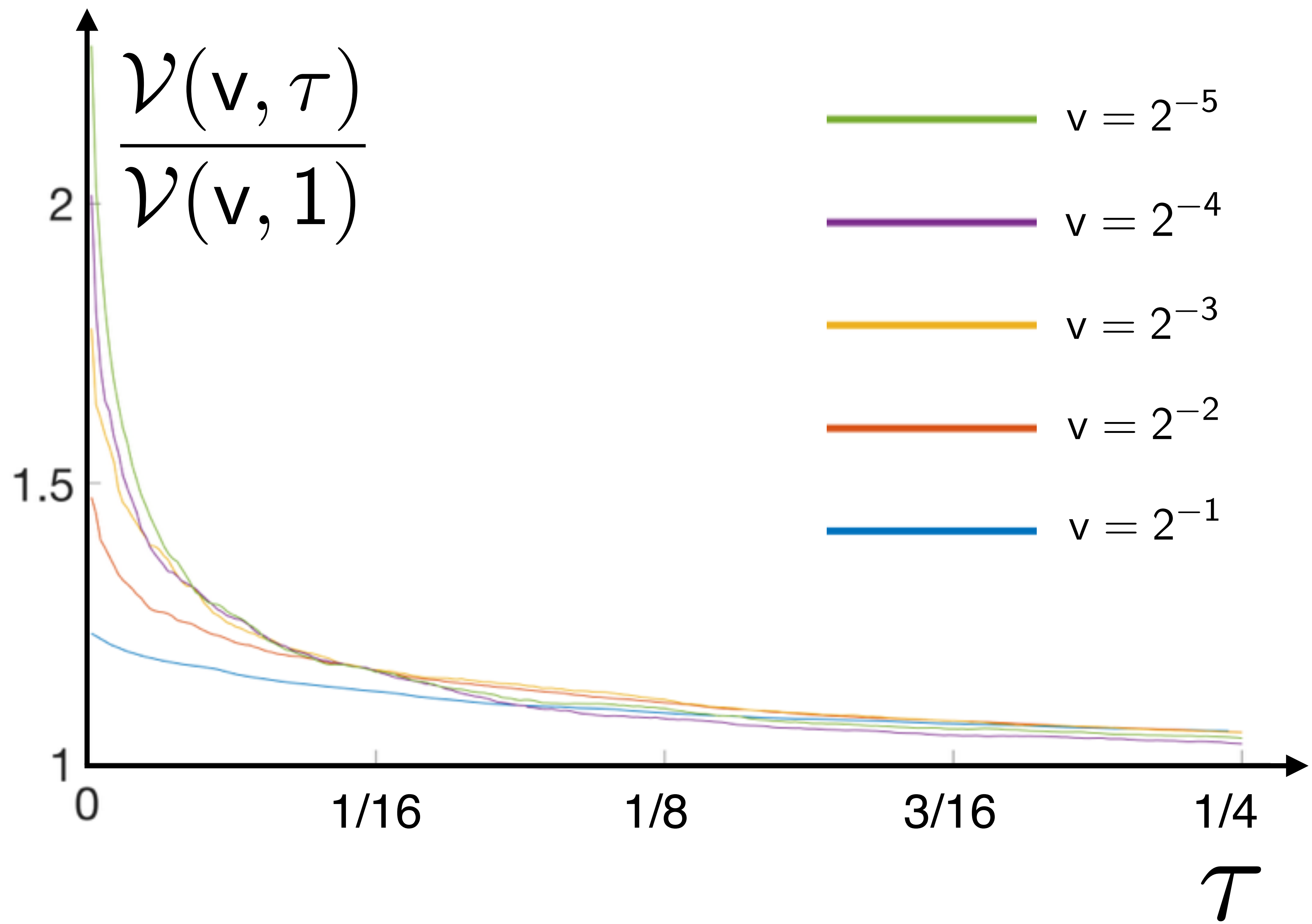


$$\lambda = 1/2$$



$\frac{\mathcal{V}(\mathsf{v},\tau)}{\mathcal{V}(\mathsf{v},1)}$
2
1.5
1
0
1/16
1/8
3/16
1/4
τ
$\mathsf{v} = 2^{-5}$
$\mathsf{v} = 2^{-4}$
$\mathsf{v} = 2^{-3}$
$\mathsf{v} = 2^{-2}$
$\mathsf{v} = 2^{-1}$

$$\lambda = 1/8$$



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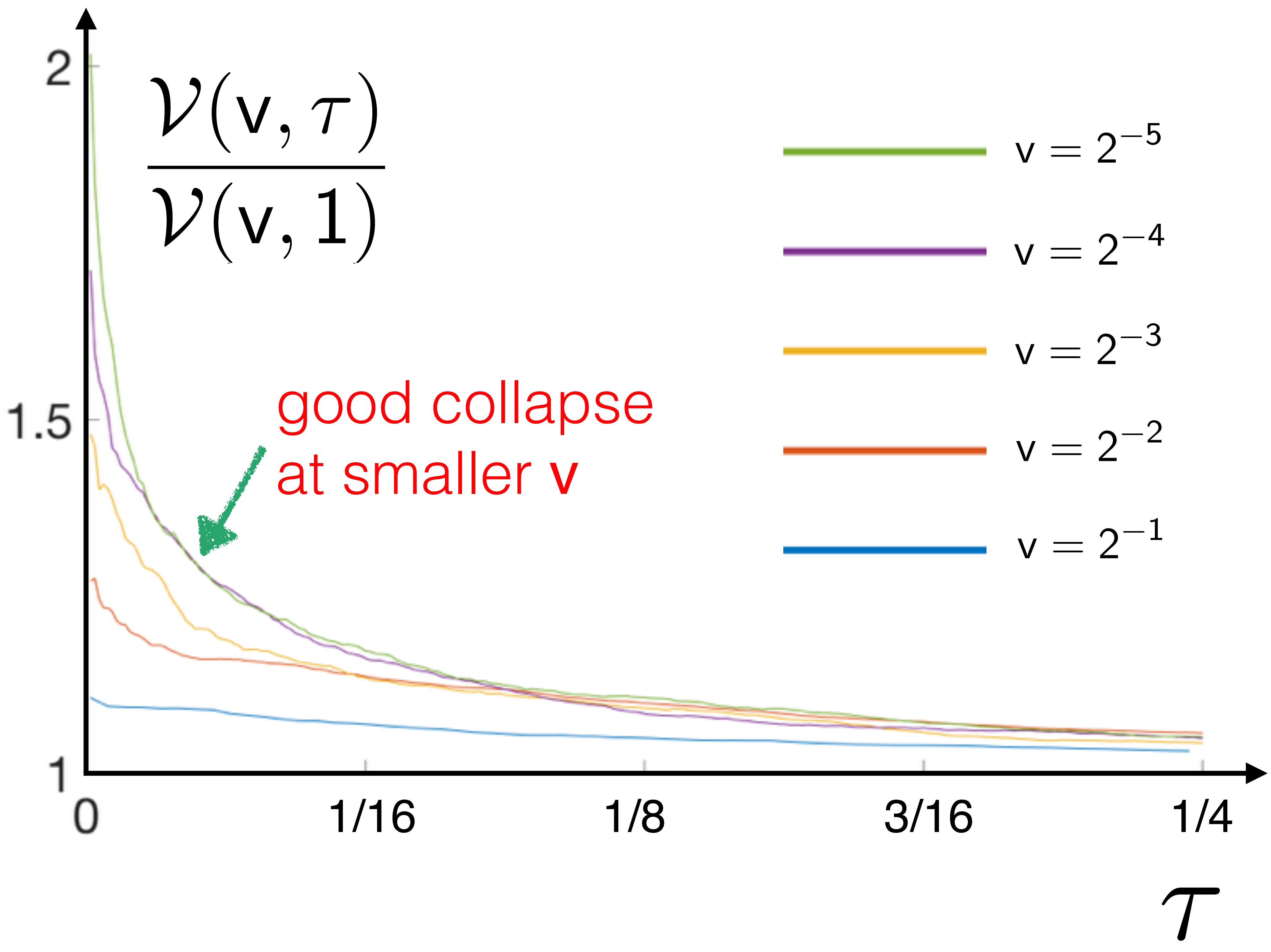
$$\lambda = 1/16$$



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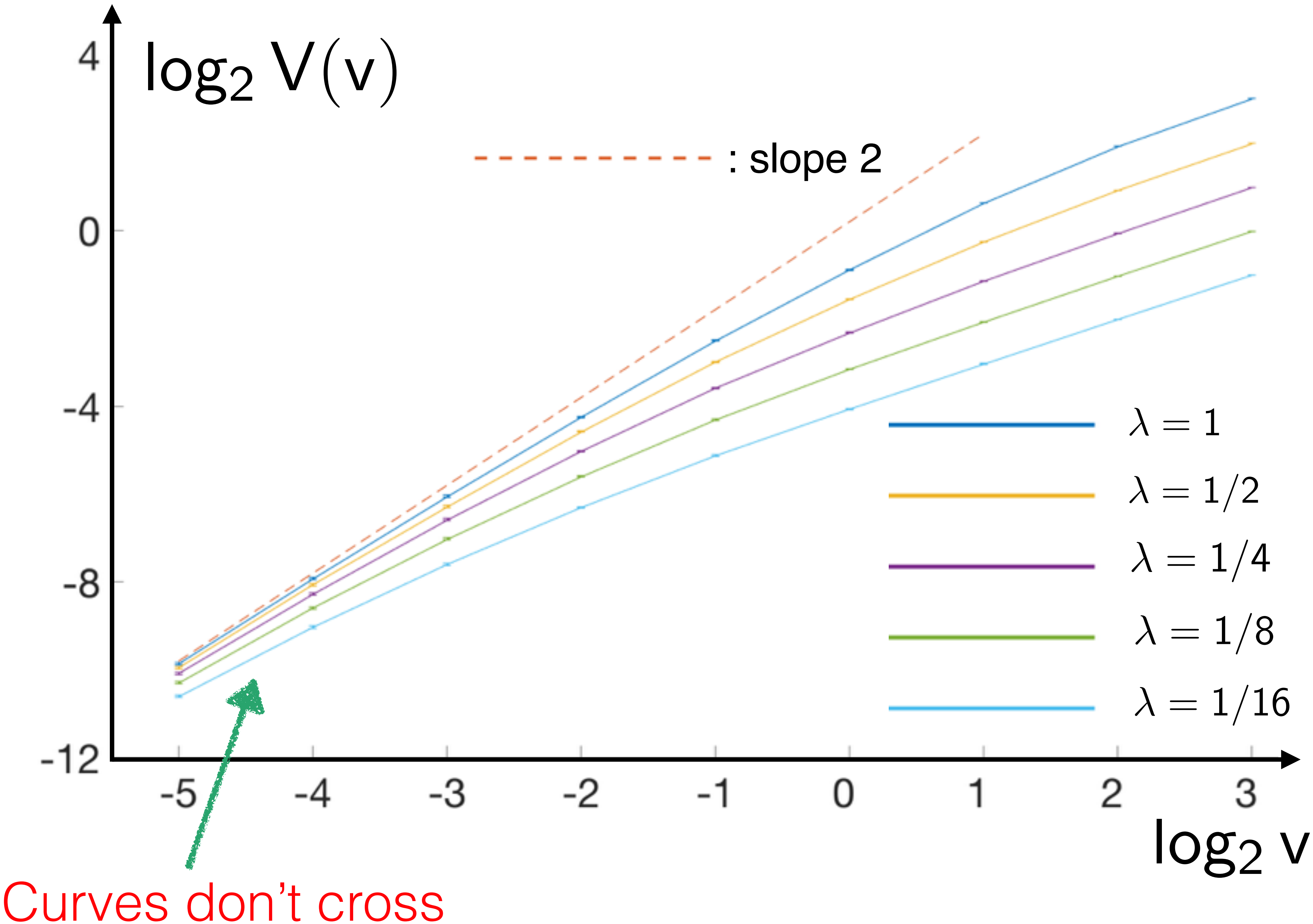
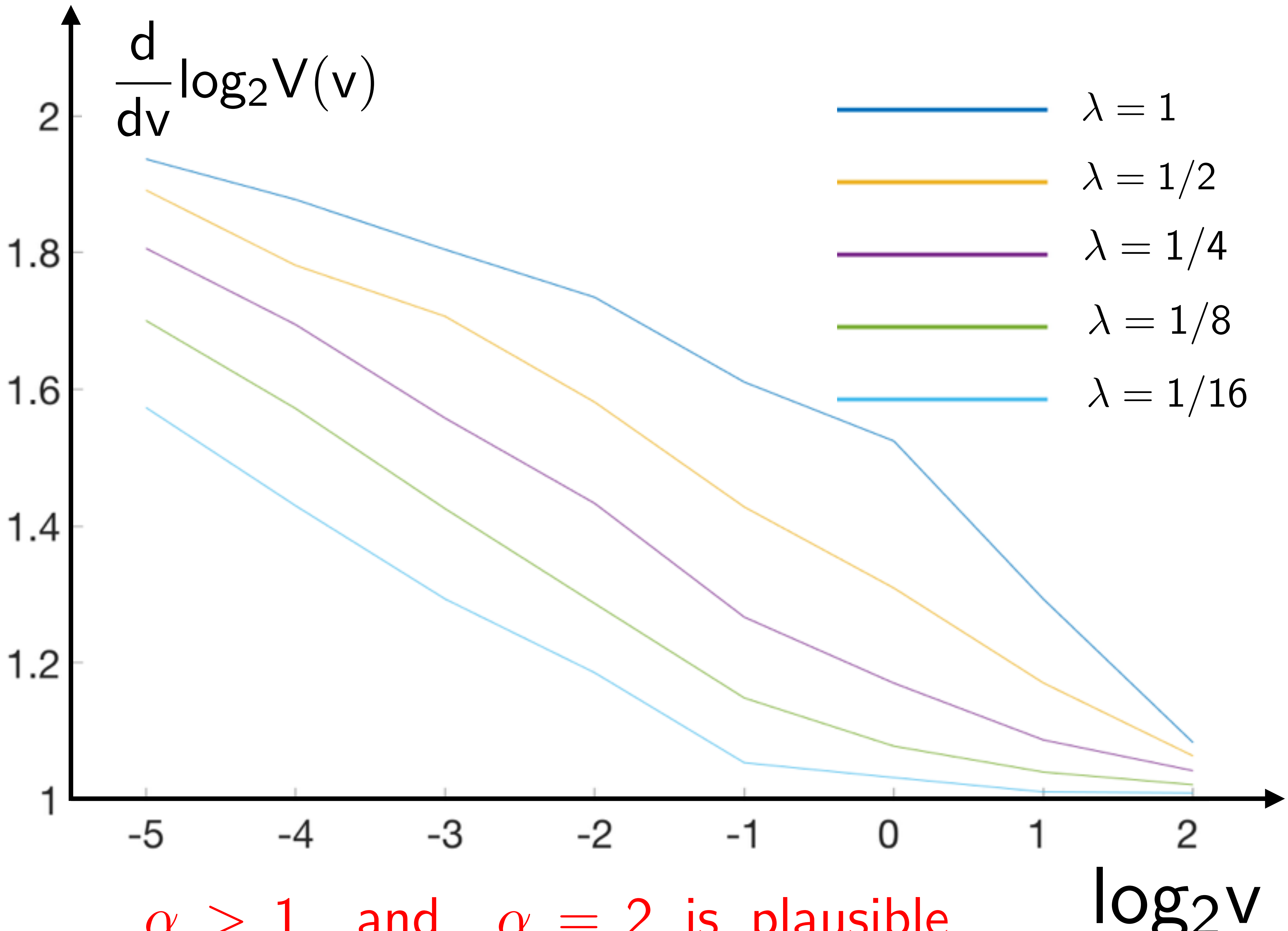
$\mathrm{v} = 2^{-3}$
$\mathrm{v} = 2^{-2}$
$\mathrm{v} = 2^{-1}$
good collapse
at smaller **v**

Exponent α : $V(v) \sim v^{\alpha}$ as $v \to 0$

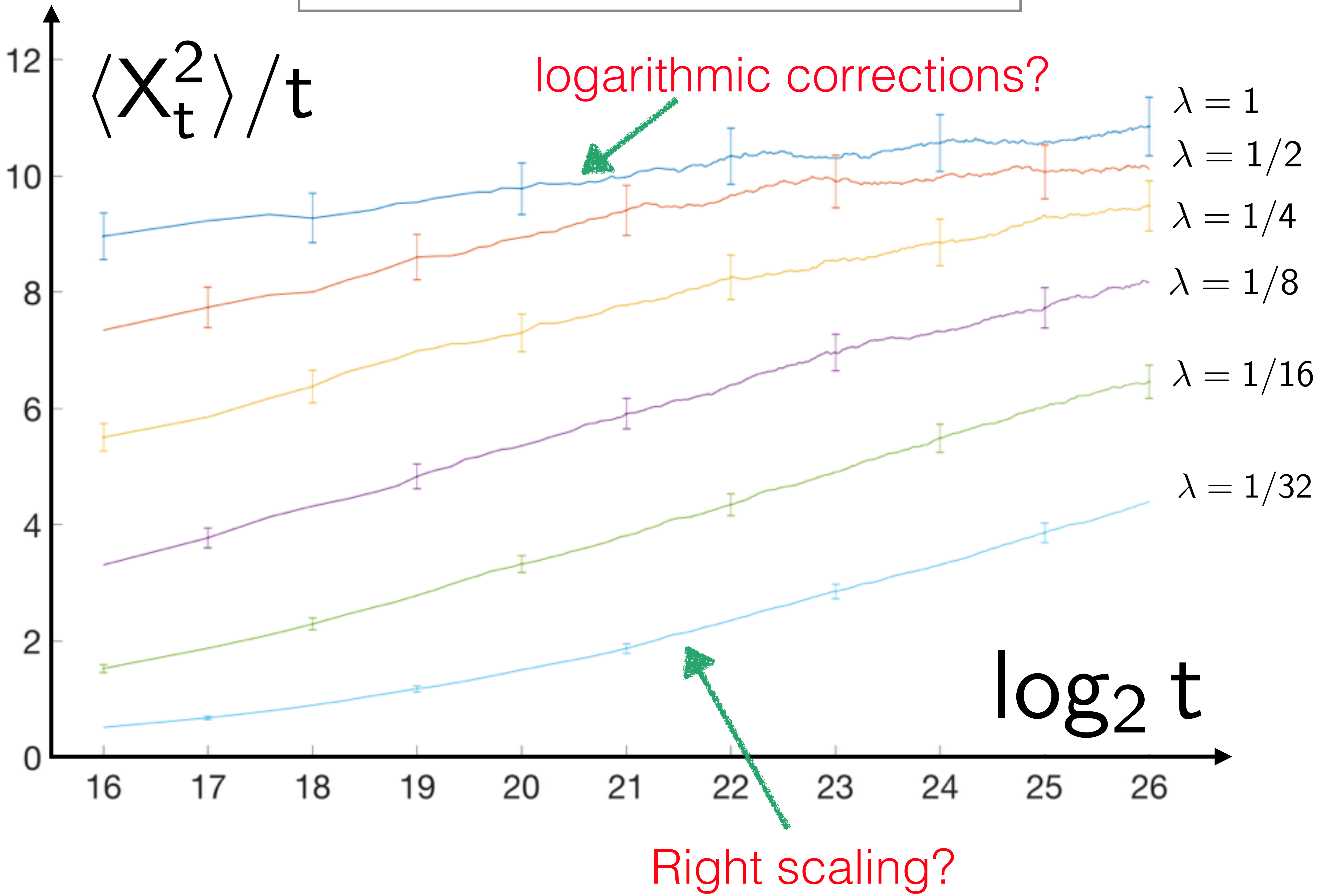


Corresponding slopes

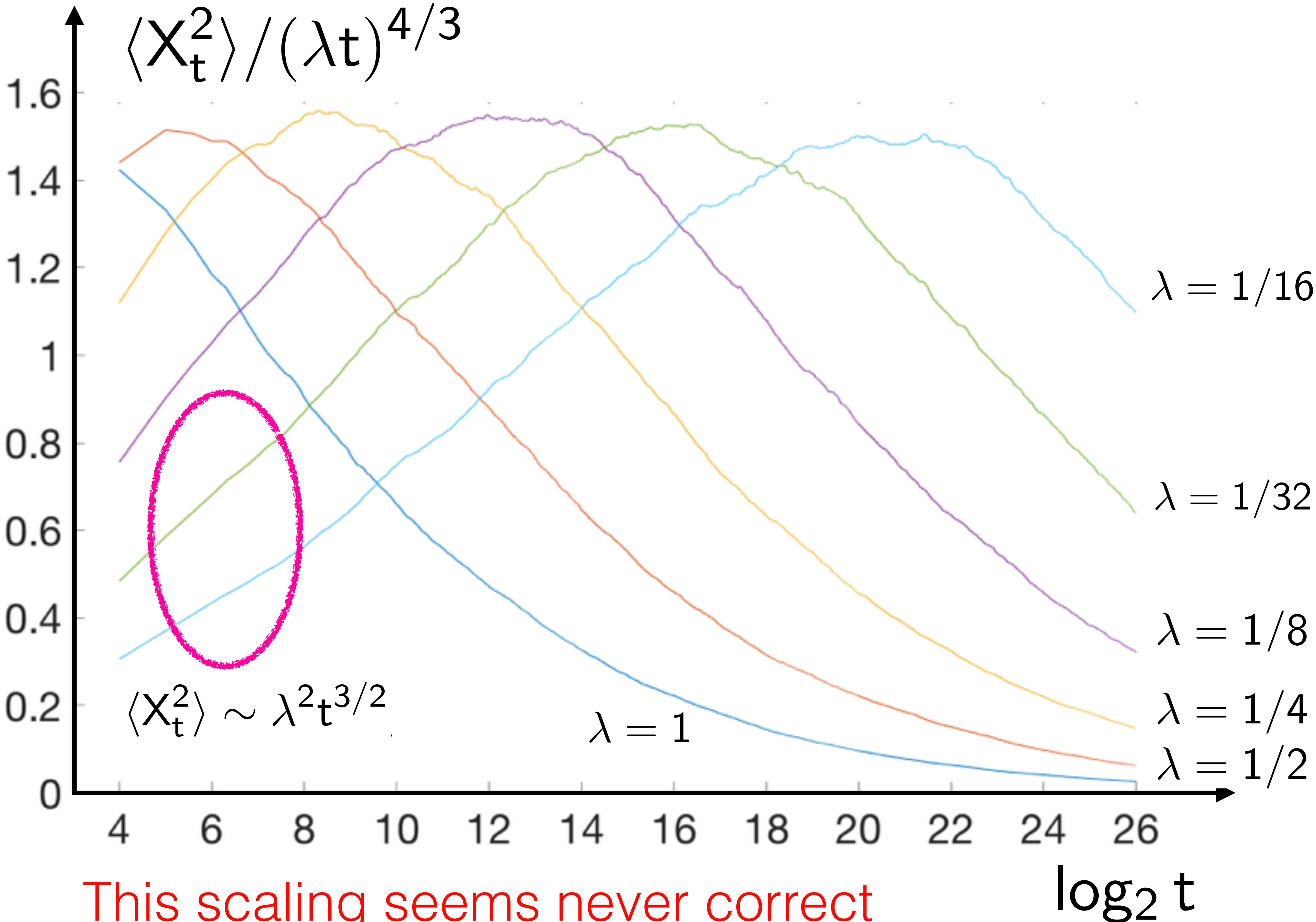


$\alpha > 1$ and $\alpha = 2$ is plausible

Fluctuations at **v = 0**



Scaling from the self-consistent approximation



This scaling seems never correct

Tentative theory

Try to explain:

1) Why is the walker eventually always trapped?
2) Why are some small corrections (log like) wrt the naive explanation plausible?

Proposal:

1) Express a relation between the two exponent α (drift) and β (fluctuations)
2) Observe that the walker is always trapped by the static modes as in Sinai random walk. This leads to logarithmic corrections.

"The walker wants to be super-diffusive but does not succeed", F. Völlering

Relating α and β

1) Assume $V(v) \sim v^{\alpha}$ $(\alpha \to 0)$ for some $1 \leq \alpha \leq 2$

SCA (→ 1)

Strong trapping (→ 2)

2) Let **v = 0** and replace

$$-\partial_x F = \int_{\mathbb{R}} dk \left(A_k(t) e^{ikx} + cc\right)$$

↓

$$\Psi = \int_{\mathbb{R}} dk\, |k|^{(\alpha-1)/2} \left(A_k(t) e^{ikx} + cc\right)$$

Effect: $\int_{|k| \leq v^2} dk(\dots) \sim v$ → $\int_{|k| \leq v^2} dk(\dots) \sim v^{\alpha}$

Relating α and β

3) Compute β through the self-consistent approximation for Ψ instead of $-\partial_x F$

This yields

$$V(v) \sim v^{\alpha} \quad \Rightarrow \quad \langle X_t^2 \rangle \sim t^{\frac{4}{2+\alpha}}, \quad 1 \leq \alpha < 2$$

$$\langle X_t^2 \rangle \sim t\,(\log t)^{1/2}, \quad \alpha = 2$$

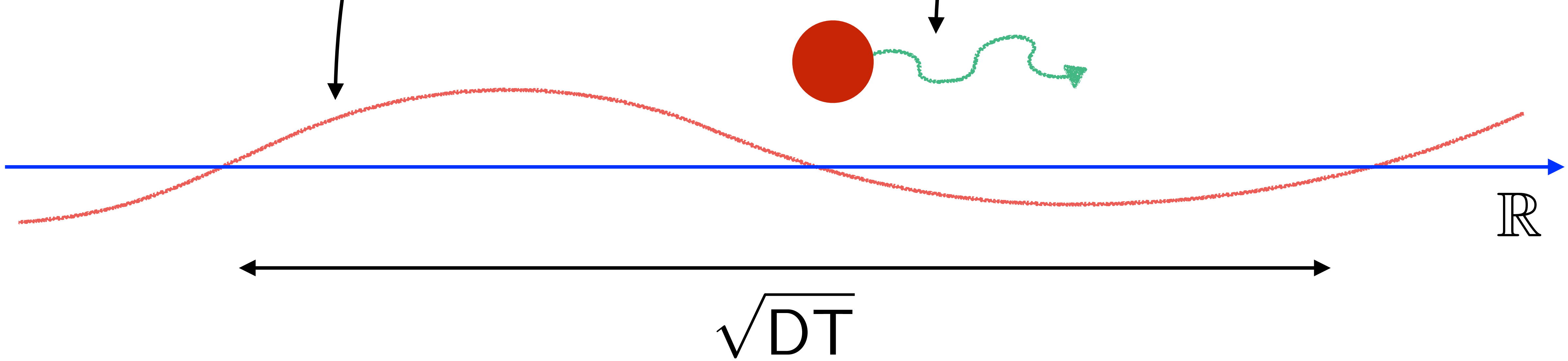
Determining α and β

Fix α tentatively. Let T be some large time. Split

$$\Psi = \int_{|\mathsf{k}| \leq (\mathsf{DT})^{1/2}} \mathsf{dk}(\dots) + \int_{|\mathsf{k}| > (\mathsf{DT})^{1/2}} \mathsf{dk}(\dots)$$

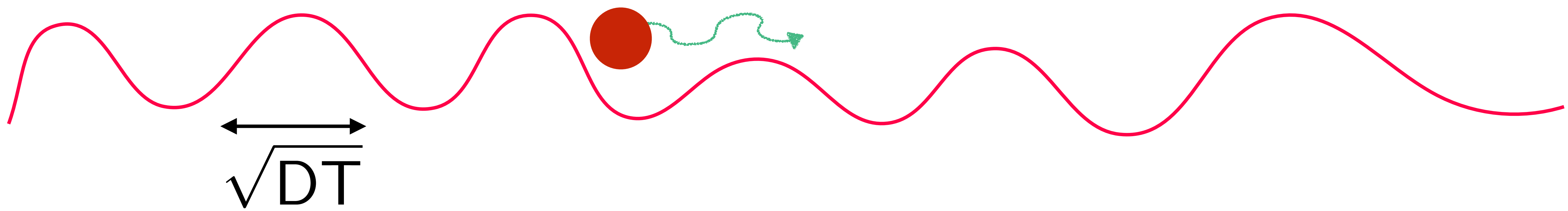
Static part

Fluctuating part,
Self-consistent approx.



Determining α and β

We get thus an effective Sinai random walk:



Computations yields (for any $1 \leq \alpha \leq 2$)

$$\langle X_T^2 \rangle \sim DT\,(\log T)^4$$

Hence:

- $\beta = 1$
- $\alpha = 2$
- possible logarithmic corrections

Conclusions

Approach: response of a random walk to
a small external perturbation

1) The time to reach the NESS diverges
2) No linear response. The system is far from equilibrium
3) Trustable way to detect trapping effects:
   response to a drift and fluctuations
4) Tentative non-linear response theory