A microscopic derivation of the condensate phase operator evolution in a weakly excited gas

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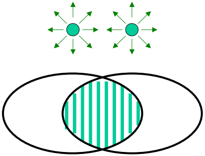
Plan

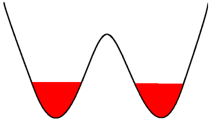
Temporal coherence of a BEC

A fundamental property of BEC, useful for applications

Macroscopic population of a single particle state → macroscopic coherence



Well-defined relative phase at time $t = 0$: How long do the BECs remember their (relative) phase ?



Spatial coherence of a single condensed gas

$$g_1(r) = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(0)\rangle \overset{r\to\infty}{\sim} \phi_0^*(\mathbf{r})\phi_0(0)\langle \hat{a}_0^\dagger \hat{a}_0\rangle;$$

Temporal coherence of a single condensed gas

$$g_1(t,0) = \langle \hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}(\mathbf{r},0)\rangle \overset{t\to\infty}{\sim} |\phi_0(\mathbf{r})|^2 \langle \hat{\mathbf{a}}_0^\dagger(\mathbf{t})\hat{\mathbf{a}}_0(\mathbf{0})\rangle$$

System in equilibrium, isolated, homogeneous, condensed

The condensate phase operator $\hat{\theta}_0$

We introduce the following representation for a bosonic mode ϕ

$$\hat{a}_\phi = \hat{A}_\phi \sqrt{\hat{n}_\phi} \qquad \hat{n}_\phi = \hat{a}_\phi^\dagger \hat{a}_\phi \qquad \hat{A}_\phi = \frac{1}{\sqrt{\hat{n}_\phi + 1}} \hat{a}_\phi \qquad (af(n) = f(n+1)a)$$

$$\hat{A}_\phi |n:\phi\rangle = |n-1:\phi\rangle \quad \text{for} \quad n > 0 \qquad \text{and} \quad \hat{A}_\phi |0:\phi\rangle = 0$$

$$\hat{A}_\phi^\dagger |n:\phi\rangle = |n+1:\phi\rangle \quad \text{for} \quad n \in \mathbb{N}$$

$\hat{A}_\phi$ **is "almost unitary" :** $\hat{A}_\phi \hat{A}_\phi^\dagger = 1 \qquad \hat{A}_\phi^\dagger \hat{A}_\phi = 1 - |0:\phi\rangle\langle 0:\phi|$

For a macroscopically populated mode ϕ_0 , we approximate $\hat{A}_{\phi_0} \simeq e^{i\hat{\theta}_0}$ with $\hat{\theta}_0$ an hermitian operator

Modulus-phase representation of condensate operator $\hat{a}_0$

$$\hat{a}_0 = e^{i\hat{\theta}_0} \sqrt{\hat{N}_0} \quad , \quad [\hat{n}_0, \hat{\theta}_0] = i$$

Hamiltonian on a lattice (spinless bosons)

$$\hat H = b^3 \sum_{\mathbf{r}} \hat\psi^\dagger(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\Delta_{\mathbf{r}}\right)\hat\psi(\mathbf{r}) + g_0 b^3 \sum_{\mathbf{r}} \hat\psi^\dagger(\mathbf{r})\hat\psi^\dagger(\mathbf{r})\hat\psi(\mathbf{r})\hat\psi(\mathbf{r})$$

Space discretisation step b, **consequent cut-off in** $\mathbf{k} \in \mathcal{D} \equiv [-\frac{\pi}{b}, \frac{\pi}{b}[^3$

Commutators $[\hat\psi(\mathbf{r}), \hat\psi^\dagger(\mathbf{r}')] = \frac{\delta_{\mathbf{r},\mathbf{r}'}}{b^3}$; **Kinetic energy** $\Delta_{\mathbf{r}}\langle \mathbf{r}|\mathbf{k}\rangle = -k^2\langle \mathbf{r}|\mathbf{k}\rangle$

Contact Interaction potential $V = g_0 \frac{\delta_{\mathbf{r},\mathbf{0}}}{b^3}$ **with** $g_0 \neq g = \frac{4\pi\hbar^2 a}{m}$

g_0 **adjusted to obtain scattering length** a **on the lattice**

$$\frac{1}{g_0} = \frac{1}{g} - \int_{\mathcal{D}} \frac{d^3k}{(2\pi)^3}\frac{m}{\hbar^2 k^2} \qquad \int_{\mathcal{D}} \frac{d^3k}{(2\pi)^3}\frac{m}{\hbar^2 k^2} = \frac{C}{g}\left(\frac{a}{b}\right)$$

$\xi = \sqrt{\frac{\hbar^2}{m\rho g}}$ **Bogoliubov :** $\frac{a}{\xi} \propto \sqrt{\rho a^3} \to 0$ **Lattice :** $\frac{b}{\xi} \overset{\textbf{fixed}}{=} \eta < 1$ **(Born)**

Ground state energy expansion in powers of a/ξ **at fixed** η **:**

$$\frac{E_0}{N} = \frac{\rho g}{2}\left(1 + C_\eta^{(1)}\frac{a}{\xi} + \dots\right)$$ Then $\eta \to 0$ in the coeff (no divergence)

Bogoliubov theory (homogeneous $\phi_0 = 1/\sqrt{V}$)

Splitting of the field operator

$$\hat{\psi}(r) = \frac{\hat{a}_0}{\sqrt{V}} + \hat{\psi}_\perp(\mathbf{r}) \qquad \hat{N} = \hat{N}_0 + \sum_{\mathbf{r}} |\hat{\psi}_\perp(\mathbf{r})|^2$$

Orthogonal number-conserving field $\hat{\Lambda}$

$$\hat{\Lambda}(\mathbf{r}) = e^{-i\hat{\theta}_0}\hat{\psi}_\perp(\mathbf{r}) \qquad [\hat{\Lambda}, \hat{\theta}_0] = [\hat{\Lambda}^\dagger, \hat{\theta}_0] = 0, \qquad [\hat{N}, \hat{\theta}] = i$$

Elimination of the condensate variables from the Hamiltonian

$$H(\hat{\psi}, \hat{\psi}^\dagger) \to H(\hat{N}, \hat{\Lambda}, \hat{\Lambda}^\dagger) \qquad \hat{N}_0 = \hat{N} - \sum_{\mathbf{r}} |\hat{\Lambda}(\mathbf{r})|^2$$

The Bogoliubov Hamiltonian is quadratic in $\hat{\Lambda}$ and $\hat{\Lambda}^\dagger$

$$H_{\mathrm{Bog}}(\hat{N}) = \frac{g_0 \hat{N}^2}{2V} + \sum_{\mathbf{r}} b^3 \left[\hat{\Lambda}^\dagger \left(h_0 + \frac{g_0 \hat{N}}{V}\right)\hat{\Lambda} + \frac{g_0 \hat{N}}{2V}\left(\hat{\Lambda}^2 + \hat{\Lambda}^{\dagger\,2}\right)\right]$$

Explicit calculation : coarse-grain time average $\overline{\frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t}}^t$

Heisenberg picture : $i\hbar\frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t} = -i\frac{\partial H_{\rm Bog}(\hat{N},\hat{\Lambda},\hat{\Lambda}^\dagger)}{\partial N}|_{\Lambda,\Lambda^\dagger}$

$$\frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t} = -\frac{1}{\hbar}\left\{\frac{g_0\hat{N}}{V} + \frac{g_0}{V}\sum_{\mathbf{r}} b^3\left[\hat{\Lambda}^\dagger\hat{\Lambda} + \frac{1}{2}\left(\hat{\Lambda}^2 + \hat{\Lambda}^{\dagger\,2}\right)\right]\right\}$$

Expansion over eigenmodes of linear equations of motion for $\Lambda, \Lambda^\dagger$

$$\begin{pmatrix}\hat{\Lambda}(\mathbf{r})\\ \hat{\Lambda}^\dagger(\mathbf{r})\end{pmatrix} = \sum_{\mathbf{k}\neq\mathbf{0}}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{V^{1/2}}\left[\begin{pmatrix}U_k\\ V_k\end{pmatrix}\hat{b}_{\mathbf{k}} + \begin{pmatrix}V_k\\ U_k\end{pmatrix}\hat{b}^\dagger_{-\mathbf{k}}\right]$$

Coarse-grain time average of $\mathrm{d}\hat{\theta}_0/\mathrm{d}t$

$$-\hbar\overline{\frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t}}^t = \mu_0(\hat{N}) + \sum_{\mathbf{k}\neq\mathbf{0}}\frac{\partial\epsilon_k}{\partial N}\hat{n}_{\mathbf{k}}$$

with $\mu_0(\hat{N}) = \frac{\mathrm{d}E_0(N)}{\mathrm{d}N}$ **and** $E_0(N) = \frac{g_0N^2}{2V} - \sum_{\mathbf{k}\neq\mathbf{0}}\epsilon_k V_k^2$

$U_k \pm V_k = \left(\frac{E_k}{\epsilon_k}\right)^{\pm 1/2}$; $\epsilon_k = \sqrt{E_k(E_k + 2\rho g_0)}$; $E_k = \frac{\hbar^2k^2}{2m}$; $\rho = \frac{N}{V}$

Physical interpretation of $\overline{\frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t}}^t$: contribution of thermal excitations

Canonical ensemble

$$\hat{\sigma}_{\mathrm{can}} = \frac{e^{-\beta \hat{H}_{\mathrm{Bog}}}}{Z} \qquad \textbf{with} \qquad \hat{H}_{\mathrm{Bog}} = E_0(N) + \sum_{\mathbf{k}\neq\mathbf{0}} \epsilon_k \hat{n}_{\mathbf{k}} \qquad \bar{n}_k = \frac{1}{e^{\beta\epsilon_k} - 1}$$

Free energy of ideal bose gas (Bogoliubov quasi particles)

$$F = E_0(N) + k_B T \sum_{\mathbf{k}} \ln(1 - e^{\beta\epsilon_k})$$

$$\mu_{\mathrm{can}} = \left(\frac{dF}{dN}\right)_{V,T} = \mu_0(\hat{N}) + \sum_{\mathbf{k}\neq\mathbf{0}} \frac{\partial\epsilon_k}{\partial N}\bar{n}_k = -\hbar \left\langle \overline{\frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t}}^t \right\rangle_{\mathrm{can}}$$

PHASE DERIVATIVE ↔ “CHEMICAL POTENTIAL OPERATOR”

$$\overline{\frac{d\hat{\theta}_0}{dt}}^t = -\frac{\hat{\mu}}{\hbar}$$

Fermionic case : Phase operator of the condensate of pairs and time correlation function

2-body density matrix

$$\rho_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}'_1,\mathbf{r}'_2) = \langle \hat\psi^\dagger_\uparrow(\mathbf{r}'_1)\hat\psi^\dagger_\downarrow(\mathbf{r}'_2)\hat\psi_\downarrow(\mathbf{r}_2)\hat\psi_\uparrow(\mathbf{r}_1)\rangle$$

Condensate wavefunction $\phi(\mathbf{r}_1,\mathbf{r}_2)$ ($\bar N_0$ = mean number of cond. pairs)

$$b^6 \sum_{\mathbf{r}'_1,\mathbf{r}'_2} \rho_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}'_1,\mathbf{r}'_2)\phi(\mathbf{r}'_1,\mathbf{r}'_2) = \bar N_0 \phi(\mathbf{r}_1,\mathbf{r}_2)$$

Condensate phase operator $\hat\theta_0$

$$\hat a_0 \equiv b^6 \sum_{\mathbf{r}_1\mathbf{r}_2} \phi^*(\mathbf{r}_1,\mathbf{r}_2)\hat\psi_\downarrow(\mathbf{r}_2)\hat\psi_\uparrow(\mathbf{r}_1) \qquad \hat a_0 = e^{i\hat\theta_0}\hat N_0^{1/2}$$

- **"never empty condensate" approx. : $\mathrm{e}^{\mathrm{i}\hat\theta_0}$ unitary operator**
- **contrarily to the bosonic case, $[\hat N_0, \hat\theta_0] \neq \mathrm{i}$**

Time correlation function of the pairing field :

$$\langle \hat\psi^\dagger_\uparrow(\mathbf{r}_1,t)\hat\psi^\dagger_\downarrow(\mathbf{r}_2,t)\hat\psi_\downarrow(\mathbf{r}_2,0)\hat\psi_\uparrow(\mathbf{r}_1,0)\rangle \overset{t\to\infty}{\sim} |\phi(\mathbf{r}_1,\mathbf{r}_2)|^2 \langle \hat a_0^\dagger(t)\hat a_0(0)\rangle$$

Goal : Phase derivative for a condensate of pairs

Two states ↑ ↓ contact interactions g_0 on a lattice $\hat{H} = \hat{H}_{\rm can} - \mu\hat{N}$

$$\hat{H} = b^3 \sum_{\mathbf{r},\sigma} \hat{\psi}^\dagger_\sigma(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\Delta_{\mathbf{r}} - \mu\right) \hat{\psi}_\sigma(\mathbf{r}) + g_0 b^3 \sum_{\mathbf{r}} \hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}^\dagger_\downarrow(\mathbf{r})\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r})$$

WE EXPECT COARSE GRAIN AVERAGE OF PHASE DERIVATIVE

$$-\frac{\hbar}{2}\overline{\frac{d\hat{\theta}_0}{dt}}^t = \mu_0(\hat{N}) + \sum_{s=F,B}\sum_{\alpha} \frac{d\epsilon_{s,\alpha}}{dN}\hat{n}_{s,\alpha}$$

- $s = F$: fermionic, BCS pair breaking, gapped Δ, $\alpha = \mathbf{k}, \uparrow (\downarrow)$
- $s = B$: bosonic, pair into motion, phononic start, $\alpha = \mathbf{q}$

We shall use :

- **Fermionic excitations** Anderson's Random Phase Approximation
- **Bosonic excitations** Time-dependent BCS including moving pairs.

A few reminders on the BCS theory

Coherent state of pairs $|\psi_{\rm BCS}\rangle = \mathcal{N} \exp\left(\sum_{\mathbf{k}} \Gamma_{\mathbf{k}} a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow}\right)|0\rangle$

In terms of the (canonical) variables $V_{\mathbf{k}}$, $V^*_{\mathbf{k}}$ $V_{\mathbf{k}} = -\frac{\Gamma_k}{\sqrt{1+|\Gamma_{\mathbf{k}}|^2}}$

BCS variational state

$$|\psi_{\rm BCS}\rangle = \prod_{\mathbf{k}} \left(U_{\mathbf{k}} - V_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}\uparrow} \hat{a}^\dagger_{-\mathbf{k}\downarrow}\right)|0\rangle \quad ; \quad U_{\mathbf{k}} = \sqrt{1-|V_{\mathbf{k}}|^2} \in \mathbb{R}$$

$$\Delta \equiv g_0 \langle \hat{\psi}_\downarrow \hat{\psi}_\uparrow \rangle = -\frac{g_0}{L^3}\sum_{\mathbf{k}} U_{\mathbf{k}} V_{\mathbf{k}} \quad ; \quad \rho_\sigma \equiv \langle \hat{\psi}^\dagger_\sigma \hat{\psi}_\sigma \rangle = \frac{1}{L^3}\sum_{\mathbf{k}} |V_{\mathbf{k}}|^2$$

- **Ground BCS $V^0_{\mathbf{k}}$ minimize** $E[V_{\mathbf{k}}, V^*_{\mathbf{k}}] = \langle \psi_{\rm BCS}|\hat{H}|\psi_{\rm BCS}\rangle$

$$V^0_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(1-\frac{\xi_k}{\epsilon_k}\right)} \quad ; \quad \xi_k = \frac{\hbar^2 k^2}{2m} - \mu + g_0 \rho^0_\uparrow \quad ; \quad \epsilon_k = \sqrt{\xi_k^2 + \Delta_0^2}$$

- **For a time-dependent BCS state :** $|\psi(t)\rangle = |\psi_{\rm BCS}(t)\rangle$

$$i\hbar \frac{\mathrm{d}V_{\mathbf{k}}(t)}{\mathrm{d}t} = \frac{\partial E}{\partial V^*_{\mathbf{k}}} \quad ; \quad i\hbar \frac{\mathrm{d}V^*_{\mathbf{k}}(t)}{\mathrm{d}t} = -\frac{\partial E}{\partial V_{\mathbf{k}}}$$

A few reminders on the BCS : elementary excitations

- **BCS Elementary excitations (fermionic)**

Within BCS states, minimize $\hat{H} \leftrightarrow$ minimize $\hat{H}_{\mathrm{BCS}}$ (quadratic, by incomplete Wick contractions)

$$\hat{H}_{\mathrm{BCS}} = E_0 + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} (\hat{\gamma}^{\dagger}_{\mathbf{k}\uparrow}\hat{\gamma}_{\mathbf{k}\uparrow} + \hat{\gamma}^{\dagger}_{-\mathbf{k}\downarrow}\hat{\gamma}_{\mathbf{k}\downarrow})$$

$\hat{\gamma}^{\dagger}_{\mathbf{k}\uparrow} = U^0_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}\uparrow} + V^0_{\mathbf{k}}\hat{a}_{\mathbf{k}\downarrow}$: elementary excitation $\epsilon_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + \Delta^2_0}$

Condensate phase in the BCS approximation

$$\hat{d}_{\mathbf{k}} = \hat{a}^{\dagger}_{\mathbf{k}\uparrow}\hat{a}^{\dagger}_{-\mathbf{k}\downarrow} \qquad \hat{d}_{\mathbf{k}} = \langle \hat{d}_{\mathbf{k}} \rangle_0 + \delta(\hat{d}_{\mathbf{k}})$$

$$\delta\hat{\theta}_0 = \frac{i}{4\bar{N}_0}\sum_{\mathbf{k}} \frac{\Delta_0}{\epsilon_k}\left(\delta\hat{d}_{\mathbf{k}} - \delta\hat{d}^{\dagger}_{\mathbf{k}}\right) \quad ; \quad \bar{N}_0 = \sum_{\mathbf{k}} \frac{\Delta_0^2}{4\epsilon^2_{\mathbf{k}}}$$

Anderson's Random Phase Approximation

- **RPA variables**

$$\hat{n}^{\mathbf{q}}_{\mathbf{k}\uparrow} = \hat{a}^{\dagger}_{\mathbf{k}+\mathbf{q}\uparrow}\hat{a}_{\mathbf{k}\uparrow}, \qquad \hat{\bar{n}}^{\mathbf{q}}_{\mathbf{k}\downarrow} = \hat{a}^{\dagger}_{-\mathbf{k}\downarrow}\hat{a}_{-\mathbf{k}-\mathbf{q}\downarrow}$$

$$\hat{d}^{\mathbf{q}}_{\mathbf{k}} = \hat{a}_{-\mathbf{k}-\mathbf{q}\downarrow}\hat{a}_{\mathbf{k}\uparrow}, \qquad \hat{\bar{d}}^{\mathbf{q}}_{\mathbf{k}} = \hat{a}^{\dagger}_{\mathbf{k}+\mathbf{q}\uparrow}\hat{a}^{\dagger}_{-\mathbf{k}\downarrow}$$

- The Exact Heisenberg equations of motion are quartic
- RPA linearize the equations by incomplete Wick contractions

$$\begin{aligned}\hat{a}\hat{b}\hat{c}\hat{d} \quad \to \quad & \hat{a}\hat{b}\langle\hat{c}\hat{d}\rangle_0 + \langle\hat{a}\hat{b}\rangle_0\hat{c}\hat{d} - \langle\hat{a}\hat{b}\rangle_0\langle\hat{c}\hat{d}\rangle_0 \\ & -\hat{a}\hat{c}\langle\hat{b}\hat{d}\rangle_0 - \langle\hat{a}\hat{c}\rangle_0\hat{b}\hat{d} + \langle\hat{a}\hat{c}\rangle_0\langle\hat{b}\hat{d}\rangle_0 \\ & +\hat{a}\hat{d}\langle\hat{b}\hat{c}\rangle_0 + \langle\hat{a}\hat{d}\rangle_0\hat{b}\hat{c} - \langle\hat{a}\hat{d}\rangle_0\langle\hat{b}\hat{c}\rangle_0\end{aligned}$$

equivalent to say $\hat{a}\hat{b} = \langle\hat{a}\hat{b}\rangle_0 + \delta(\hat{a}\hat{b})$

Equation for the phase (fermionic excitations)

$$-\frac{\hbar}{2}\overline{\frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t}}^{t} \underset{\mathrm{RPA}}{=} \mu(\bar{N}) + \frac{\mathrm{d}\mu}{\mathrm{d}\bar{N}}(\hat{N}-\bar{N}) + \sum_{\mathbf{k}}\frac{\mathrm{d}\epsilon_{\mathbf{k}}}{\mathrm{d}\bar{N}}(\hat{\gamma}^{\dagger}_{\mathbf{k}\uparrow}\hat{\gamma}_{\mathbf{k}\uparrow} + \hat{\gamma}^{\dagger}_{-\mathbf{k}\downarrow}\hat{\gamma}_{-\mathbf{k}\downarrow})$$

Problem to include the bosonic branch within RPA

Within the RPA equations

- Linear RPA eqs for fluctuations of pair operators are closed for a given $\mathbf{q}$ (center-of-mass momentum of the pair)
- Collective excitations $\mathbf{q} \to \epsilon_{\mathbf{q}}$ (moving pairs, phononic start), appear in the linear problem at fixed $\mathbf{q}$.
- $\delta\hat{\theta}_0$ involves only $\mathbf{q} = 0$ pair operators, $\to \frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t}$ does not involve collective excitations of the moving pairs with the RPA.

Push RPA beyond the linear order ?

- Idea : solve the linear RPA problem at fixed $\mathbf{q}$, find the eigenmodes and construct the collective quaisparticle operators $\hat{b}_{\mathbf{q}}$, $\hat{b}_{\mathbf{q}}^\dagger$ (made of combinations of pair operators for the given $\mathbf{q}$)
- In the quartic (beyond RPA) equation of $\frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t}$, expand pair operators over the eigenmodes of the linear $\mathbf{q}-$problem and "recognize"

$$-\frac{\hbar}{2}\overline{\frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t}}^{t} = \sum_{\mathbf{q}} \frac{\mathrm{d}\epsilon_{\mathbf{q}}}{\mathrm{d}\bar{N}} \hat{b}_{\mathbf{q}}^\dagger \hat{b}_{\mathbf{q}} + \dots$$

- **Problem :** RPA operators are linked by non-linear relations e.g. $a^\dagger_{\mathbf{k}+\mathbf{q}\uparrow} a_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}-\mathbf{q}\downarrow} a_{-\mathbf{k}\downarrow} = a^\dagger_{\mathbf{k}+\mathbf{q}\uparrow} a^\dagger_{-\mathbf{k}-\mathbf{q}\downarrow} a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} \to$ **non unicity**

Bosonic excitations : Time-dependent BCS state including moving pairs (Blaizot-Ripka 1985)

$$|\psi\rangle = \mathcal{N}(t)\exp\left(b^6\sum_{\mathbf{r},\mathbf{r}'}\Gamma(\mathbf{r},\mathbf{r}';t)\hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}^\dagger_\downarrow(\mathbf{r}')\right)|0\rangle$$

Key point : introduction of the canonical variables $\Phi(\mathbf{r},\mathbf{r}')$, $\Phi(\mathbf{r},\mathbf{r}')^*$

$$i\hbar\, b^6\,\frac{\partial\Phi(t)}{\partial t} = \frac{\partial\mathcal{H}(\Phi,\Phi^*)}{\partial\Phi^*} \qquad \leftrightarrow \qquad i\hbar\frac{\mathrm{d}V_\mathbf{k}}{\mathrm{d}t} = \frac{\partial E(V_\mathbf{k},V_\mathbf{k}^*)}{\partial V_\mathbf{k}^*}$$

Moving pairs : $\underline{\underline{\Phi}} = b^3\Phi(\mathbf{r},\mathbf{r}')$ **Pairs at rest : $V_\mathbf{k}$**

$$\underline{\underline{\Phi}} = -\underline{\underline{\Gamma}}\left(1+\underline{\underline{\Gamma}}^\dagger\,\underline{\underline{\Gamma}}\right)^{-1/2} \qquad \leftrightarrow \qquad V_\mathbf{k} = -\Gamma_\mathbf{k}\left(1+|\Gamma_\mathbf{k}|^2\right)^{-1/2}$$

$$\underline{\underline{\langle\hat{\psi}_\uparrow\hat{\psi}_\downarrow\rangle}} = \underline{\underline{\Phi}}\left(1-\underline{\underline{\Phi}}^\dagger\,\underline{\underline{\Phi}}\right)^{1/2} \qquad \leftrightarrow \qquad \langle\hat{\psi}_\uparrow\hat{\psi}_\downarrow\rangle = \frac{1}{L^3}\sum_\mathbf{k} V_\mathbf{k}(1-|V_\mathbf{k}|^2)^{1/2}$$

$$\frac{N}{2} = ||\underline{\underline{\Phi}}||^2 \equiv b^6\sum_{\mathbf{r},\mathbf{r}'}|\Phi(\mathbf{r},\mathbf{r}',t)|^2 \qquad \leftrightarrow \qquad \frac{N}{2} = \sum_\mathbf{k}|V_\mathbf{k}|^2$$

Variational approach with moving pairs - II

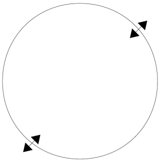
Minimisers of $\mathcal{H}(\Phi, \Phi^*)$

- $\mathcal{H}$, invariant under $\Phi(\mathbf{r}, \mathbf{r}') \to \mathrm{e}^{i\gamma}\Phi(\mathbf{r}, \mathbf{r}')$, is minimized on a circle

$$\gamma \mapsto \mathrm{e}^{i\gamma}\Phi_0(\mathbf{r}, \mathbf{r}')$$

- $\Phi_0(\mathbf{r}, \mathbf{r}') = (N/2)^{1/2}\phi_0(\mathbf{r}, \mathbf{r}')$
- $\phi_0(\mathbf{r}, \mathbf{r}')$ normalized, (Fourier transform of $V^0_{\mathbf{k}}$) depends on N

Field Splitting along and orthogonally to ϕ_0



$$\Phi(\mathbf{r}, \mathbf{r}') = \mathrm{e}^{i\theta}[n^{1/2}\phi_0(\mathbf{r}, \mathbf{r}') + \Lambda(\mathbf{r}, \mathbf{r}')]$$

Expansion of $\mathcal{H}$ for small deviations from the circle of minima

$$\mathcal{H}(\Phi, \Phi^*) = T_0[n, \phi_0, \phi_0^*] + \sum_{j=1,3} T_j[n, \phi_0, \phi_0^*](\Lambda, \Lambda^*) + O(||\Lambda||^4)$$

T_j is of order j in Λ, Λ^*

Variational approach with moving pairs -III

- Equation for θ : $-\hbar \mathrm{d}\theta/\mathrm{d}t = \partial_n \mathcal{H}(\Phi, \Phi^*)$
- Eliminate $n = N/2 - \|\Lambda\|^2$ and limit to the order two in Λ
- Difficulty : Evaluation of time average of a linear term in $\mathrm{d}\theta/\mathrm{d}t$ ($\overline{\Lambda}^t \neq 0$ and quadratic in Λ due to cubic terms in $\mathcal{H}$)
- Including this contribution we reconstruct the total derivative

$$\frac{\hbar}{2}\overline{\frac{\mathrm{d}\theta}{\mathrm{d}t}}^t = \mu_0(N) + \overline{\frac{\mathrm{d}T_2}{\mathrm{d}N}[N](\Lambda, \Lambda^*)}^t + O(||\Lambda||^3)$$

$$T_2 = \frac{1}{2}(\Lambda^*, \Lambda)\sigma_z \mathcal{L}[N]\begin{pmatrix}\Lambda \\ \Lambda^*\end{pmatrix}$$

- Expanding over eigenmodes of the linearized problem $\mathcal{L}\binom{u_\alpha}{v_\alpha} = \epsilon_\alpha \binom{u_\alpha}{v_\alpha} = \epsilon_\alpha$ + using Helmann Feynman

$$\begin{pmatrix}\Lambda(\mathbf{r}, \mathbf{r}'; t) \\ \Lambda^*(\mathbf{r}, \mathbf{r}'; t)\end{pmatrix} = \sum_\alpha b_\alpha(t)\begin{pmatrix}u_\alpha(\mathbf{r}, \mathbf{r}') \\ v_\alpha(\mathbf{r}, \mathbf{r}')\end{pmatrix} + b_\alpha^*(t)\begin{pmatrix}v_\alpha^*(\mathbf{r}, \mathbf{r}') \\ u_\alpha^*(\mathbf{r}, \mathbf{r}')\end{pmatrix}$$

Equation for the phase (bosonic excitations)

$$-\frac{\hbar}{2}\overline{\frac{d\theta}{dt}}^t = \mu_0(N) + \sum_\alpha \frac{d\epsilon_\alpha}{dN}|b_\alpha|^2 + O(\|\Lambda\|^3)$$

Variational approach with moving pairs -IV

Equation for the phase (bosonic excitations)

$$-\frac{\hbar}{2}\overline{\frac{d\theta}{dt}}^{t} = \mu_0(N) + \sum_{\alpha}\frac{d\epsilon_\alpha}{dN}|b_\alpha|^2 + O(\|\Lambda\|^3)$$

Spectrum of the linear problem For s-wave contact interactions, $b \to 0$, and homogeneous system, for a fixed total wavevector $\mathbf{q}$

(*i*) one discrete value $\epsilon_{\mathbf{q}}$ same bosonic spectrum as the RPA

$$\epsilon_{\mathbf{q}} \overset{q\to 0}{\sim} c\mathbf{q} \qquad mc^2 = \rho\frac{\mathrm{d}\mu}{\mathrm{d}\rho}$$

(*ii*) a continuum $(\mathbf{k}_1,\uparrow;\mathbf{k}_2,\downarrow) \mapsto \epsilon_{F,\mathbf{k}_1,\uparrow} + \epsilon_{F,\mathbf{k}_2,\downarrow}$ with $(\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{q})$

Quantum Hydrodynamics
A similar result for the **linear part of the bosonic dispersion relation** (low T) can be obtained with quantum hydrodynamics **L. Landau I. Khalatnikov 1949** with exact (not mean field) equation of state.

References

Bosons

- A. Sinatra, Y. Castin, E. Witkowska, "Nondiffusive phase spreading of a Bose-Einstein condensate at finite temperature", PRA **75** (2007).
- A. Sinatra, Y. Castin, E. Witkowska, "Coherence time of a Bose-Einstein condensate", PRA (2009).
- A. Sinatra, Y. Castin, E. Witkowska, "Limit of spin squeezing in trapped Bose-Einstein condensates", EPL (2013).
- Y. Castin, A. Sinatra, "Temps de cohérence d'un condensat de Bose-Einstein dans un gaz isolé harmoniquement piégé", C. R. Phys. (2018) [free access] (english arXiv:1803.03168).

Fermions

- H. Kurkjian, Y. Castin, A. Sinatra, *"Phase operator and blurring time of a pair-condensed Fermi gas"* PRA (2013).
- H. Kurkjian, Y. Castin, A. Sinatra, "Brouillage thermique d'un gaz cohérent de fermions", C. R. Phys. (2016) [free access] (english arXiv:1502.05644).
- Y. Castin, "Étalement de la phase et cohérence temporelle d'un gaz de fermions condensé par paires à basse température", C. R. Phys. (2019) [free access] (english arXiv:1807.10476).