Chapter 2 - Transport equation : characteristics method and DiPerna-Lions renormalization theory

1 Introduction (October 2nd, 2013)

We consider the PDE (transport equation)

$$\partial_t f = \Lambda\, f = -a(x) \cdot \nabla f(x) \quad \text{in} \quad (0,\infty) \times \mathbb{R}^d, \tag{1.1}$$

that we complement by an initial condition

$$f(0,x) = f_0(x) \quad \text{in} \quad \mathbb{R}^d.$$

We assume that the drift force field satisfies

$$a \in C^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$$

(a globally Lipschitz would be suitable) and that the initial datum satisfies

$$f_0 \in L^p(\mathbb{R}^d),\ 1 \le p \le \infty. \tag{1.2}$$

We prove that there exists a unique solution in the renormalization sense to the transport equation (1.1) associated to the initial datum f_0.

2 Characteristics method and existence of solutions

2.1 Smooth initial datum.

As a first step we consider $f_0 \in C^1_c(\mathbb{R}^d;\mathbb{R})$.
Thanks to the Cauchy-Lipschitz theorem on ODE, we know that for any $x \in \mathbb{R}^d$ the equation

$$\dot x(t) = a(x(t)), \quad x(0) = x, \tag{2.1}$$

admits a unique solution $t \mapsto x(t) = \Phi_t(x) \in C^1(\mathbb{R};\mathbb{R}^d)$. Moreover, for any $t \ge 0$, the vectors valued function $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism and the application $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $(t,x) \mapsto \Phi_t(x)$ is globally Lipschitz.
The characteristics method makes possible to build a solution to the transport equation (1.1) thanks to the solutions (characteristics) of the above ODE problem.
We start with a simple case. Assuming $f_0 \in C^1(\mathbb{R}^d;\mathbb{R})$, we define the function $f \in C^1(\mathbb{R} \times \mathbb{R}^d;\mathbb{R})$

$$\forall\, t \ge 0,\ \forall\, x \in \mathbb{R}^d \quad f(t,x) := f_0(\Phi_t^{-1}(x)). \tag{2.2}$$

From the associated implicit equation $f(t,\Phi_t(x)) = f_0(x)$, we deduce

$$\begin{aligned} 0 &= \frac{d}{dt}[f(t,\Phi_t(x)] = (\partial_t f)(t,\Phi_t(x)) + \dot\Phi_t(x) \cdot (\nabla_x f)(t,\Phi_t(x)) \\ &= (\partial_t f + a(x) \cdot \nabla_x f)(t,\Phi_t(x)). \end{aligned}$$

The above equation holding true for any $t > 0$ and $x \in \mathbb{R}^d$ and the function Φ_t mapping $\mathbb{R}^d$ onto $\mathbb{R}^d$, we deduce that $f \in C^1((0,T) \times \mathbb{R}^d)$ satisfies the transport equation (1.1) in the sense of the classical differential calculus.

If furtheremore $f_0 \in C^1_c(\mathbb{R}^d)$, by using that $|\Phi_t(x) - x| \le L\,t$ for any $x \in \mathbb{R}^d$, $t \ge 0$ we deduce that $f(t) \in C^1_c(\mathbb{R}^d)$ for any $t \ge 0$, with $\operatorname{supp} f(t) \subset \operatorname{supp} f_0 + B(0, L\,t)$. In other words, transport occurs with finite speed : that makes a great difference with the instantaneous positivity of solution (related of a "infinite speed" of propagation of particles) known for the heat equation and more generally for parabolic equations.

Exercise 2.1 *Make explicit the construction and formulas in the three following cases :*
(1) $a(x) = a \in \mathbb{R}^d$ *is a constant vector.*
(2) $a(x) = x$
(3) $a(x) = v$, $f_0 = f_0(x,v) \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ *and we look for a solution* $f = f(t,x,v) \in C^1((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$.
(4) Prove that (S_t) *is a group on* $C(\mathbb{R}^d)$*, where*

$$\forall\, f_0 \in C(\mathbb{R}^d),\ \forall\, t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d \quad (S_t f_0)(x) = f(t,x) := f_0(\Phi_t^{-1}(x)). \tag{2.3}$$

(5) Repeat a similar construction in the case of a time depending drift force field $a = a(t,x) \in C^1([0,T] \times \mathbb{R}^d)$ *and for the transport equation with a gain source term added at the RHS of* (1.1).

2.2 L^p initial datum.

As a second step we want to generalize the construction of solutions to more general initial data as announced in (1.2). We observe that at least formally the following computation holds for a given positive solution f of the transport equation (1.1) :

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} f^p\,dx &= \int_{\mathbb{R}^d} \partial_t f^p\,dx = \int_{\mathbb{R}^d} p f^{p-1}\,\partial_t f\,dx \\
&= \int_{\mathbb{R}^d} p f^{p-1}\,a \cdot \nabla_x\,dx = \int_{\mathbb{R}^d} a(x) \cdot \nabla_x f^p\,dx \\
&= \int_{\mathbb{R}^d} (-\mathrm{div}_x a) f^p\,dx \le \|\mathrm{div}_x a\|_{L^\infty} \int_{\mathbb{R}^d} f^p\,dx.
\end{aligned}$$

With the help of the Gronwall lemma, we learn from that differential inequality that the following (still formal) estimate holds

$$\|f(t)\|_{L^p} \le e^{bt/p}\,\|f_0\|_{L^p} \quad \forall\, t \ge 0, \tag{2.4}$$

with $b := \|\mathrm{div} a\|_{L^\infty}$. As a consequence, we may propose the following natural definition of solution.

Definition 2.2 *We say that* $f = f(t,x)$ *is a weak solution to the transport equation* (1.1) *associated to the initial datum* $f_0 \in L^p(\mathbb{R}^d)$ *if it satisfies the bound*

$$f \in L^\infty(0,T; L^p(\mathbb{R}^d))$$

and it satisfies the equation in the following weak sense :

$$\int_0^T \int_{\mathbb{R}^d} f\,L^*\varphi\,dxdt = \int_{\mathbb{R}^d} f_0\,\varphi(0,.)\,dx$$

for any $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$*, where we define de primal operator* L *by*

$$Lg := \partial_t g + a \cdot \nabla_x g$$

and the (formal) dual operator L^* *by*

$$L^*\varphi := -\partial_t \varphi - div_x(a\varphi).$$

Exercise 2.3 *1. Prove that a classical solution is a weak solution.*

2. Prove that a weak solution f is weakly continuous (after modification of $f(t)$ on a time set of measure zero) in the following sense :

(i) $f \in C([0,T]; \mathcal{D}'(\mathbb{R}^d)$ in general (and even $f \in Lip([0,T]; w*-(C^1_c(\mathbb{R}^d))'))$;

(ii) $f \in C([0,T]; w*-(C_c(\mathbb{R}^d))')$ when $p = 1$ (for the weak topology $*\sigma(M^1, C_c)$);

(ii) $f \in C([0,T]; w-L^p(\mathbb{R}^d))$ when $p \in (1,\infty)$ (for the weak topology $\sigma(L^p, L^{p'})$);

(ii) $f \in C([0,T]; w-L^p_{loc}(\mathbb{R}^d))$ for any $p \in [1,\infty)$ when $p = \infty$.

Theorem 2.4 (Existence) *For any $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, there exists a global (defined for any $T > 0$) weak solution to the transport equation* (1.1) *which furthermore satisfies*

$$f \in C([0,\infty); L^p(\mathbb{R}^d)) \text{ when } p \in [1,\infty); \quad f \in C([0,\infty); L^1_{loc}(\mathbb{R}^d)) \text{ when } p = \infty.$$

If moreover $f_0 \ge 0$ then $f(t,.) \ge 0$ for any $t \ge 0$.

Step 1. Rigorous a priori bounds. Take $f_0 \in C^1_c(\mathbb{R}^d)$. For any smooth (renormalizing) function $\beta : \mathbb{R} \to \mathbb{R}_+$, $\beta(0) = 0$, which are C^1 and globally Lipschitz we clearly have that $\beta(f(t,x))$ is a solution to the same equation associated to the initial datum $\beta(f_0)$ and $\beta(f(t,.)) \in C^1_c(\mathbb{R}^d)$ for any $t \ge 0$. The function

$$(0,T) \to \mathbb{R}_+, \quad t \mapsto \int_{\mathbb{R}^d} \beta(f(t,x))\,dx$$

is clearly C^1 (that is an exercise using the Lebesgue's dominated convergence Theorem) and

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} \beta(f(t,x))\,dx &= \int_{\mathbb{R}^d} \partial_t \beta(f(t,x))\,dx = \int_{\mathbb{R}^d} \beta'(f(t,x))\partial_t f(t,x)\,dx \\
&= \int_{\mathbb{R}^d} \beta'(f(t,x))\,a(x)\cdot\nabla_x f(t,x)\,dx = \int_{\mathbb{R}^d} a(x)\cdot\nabla_x \beta(f(t,x))\,dx \\
&= \int_{\mathbb{R}^d} (-\mathrm{div}_x a)(x)\beta(f(t,x))\,dx.
\end{aligned}$$

We deduce from that identity the differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d} \beta(f(t,x))\,dx \le b\int_{\mathbb{R}^d} \beta(f(t,x))\,dx,$$

with $b := \|\mathrm{div}_x a\|_{L^\infty}$, and then thanks to the Gronwall lemma

$$\int_{\mathbb{R}^d} \beta(f(t,x))\,dx \le e^{bt}\int_{\mathbb{R}^d} \beta(f_0(x))\,dx.$$

Since $f_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ by assumption, for any $1 \le p < \infty$, we can define a sequence of renormalized functions (β_n) such that $0 \le \beta_n(s) \nearrow |s|^p$ for any $s \in \mathbb{R}$ and we can pass to the limit in the preceding inequality using the monotonous Lebesgue Theorem at the RHS and the Fatou Lemma at the LHS in order to get

$$\int_{\mathbb{R}^d} |f(t,x)|^p\,dx \le e^{bt}\int_{\mathbb{R}^d} |f_0(x)|^p\,dx,$$

or in other words

$$\|f(t,.)\|_{L^p} \le e^{bt/p}\,\|f_0\|_{L^p} \quad \forall\, t \ge 0.$$

Passing to the limit $p \to \infty$ in the above equation we obtain (maximum principle)

$$\|f(t,.)\|_{L^\infty} \le \|f_0\|_{L^\infty} \quad \forall\, t \ge 0.$$

Moreover, $f \in C([0,T]; L^p(\mathbb{R}^d)$ for any $p \in [1,\infty)$.

Step 2. Existence in the case $p \in [1, \infty)$**.** For any function $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$, we may define a sequence of functions $f_{0,n} \in C^1_c(\mathbb{R}^d)$ (take for instance $f_{0,n} := (\chi_n f_0) * \rho_n$ such that $f_{0,n} \to f_0$ in $L^p(\mathbb{R}^d)$: here comes the restriction $p < \infty$). Because of the first step we may define $f_n(t)$ a solution to the transport equation. Moreover, thanks to the first step and because the equation is linear we have

$$\sup_{t \in [0,T]} \|f_n(t,.) - f_m(t,.)\|_{L^p} \le e^{bt/p} \, \|f_{0,n} - f_{0,m}\|_{L^p} \to 0 \quad \forall T \ge 0.$$

The sequence (f_n) being a Cauchy sequence, there exists $f \in C([0,T]; L^p(\mathbb{R}^d))$ such that $f_n \to f$ in $C([0,T]; L^p(\mathbb{R}^d))$ as $n \to \infty$. Now, writing

$$\begin{aligned} 0 &= -\int_0^T \int_{\mathbb{R}^d} \varphi \left\{ \partial_t f_n + a \cdot \nabla f_n \right\} dxdt \\ &= \int_0^T \int_{\mathbb{R}^d} f_n \left\{ \partial_t \varphi + \mathrm{div}_x (a\varphi) \right\} dxdt + \int_{\mathbb{R}^d} f_{0,n} \, \varphi(0,.) \, dx, \end{aligned}$$

we may pass to the limit in the above equation and we get that f is a solution in the convenient sense.
If moreover $f_0 \ge 0$ then the same holds for $f_{0,n}$, then f_n and finally for f. ◻

Exercise 2.5 *(1) Show that for any charactericties solution f to the transport equation associated to an initial datum $f_0 \in C^1_c(\mathbb{R}^d)$, for any times $T > 0$ and radius R there exists some constants $C_T, R_T \in (0, \infty)$ such that*

$$\sup_{t \in [0,T]} \int_{B_R} |f(t,x)| \, dx \le C_T \int_{B_{R_T}} |f_0(x)| \, dx.$$

(Hint. Use the property of finite speed propagation of the transport equation).
(2) Adapt the proof of existence to the case $f_0 \in L^\infty$.

3 Weak solutions are renormalized solutions

We start with a remark. For any $g \in C^2$ classical solution of (1.1) and $\beta \in C^2(\mathbb{R}; \mathbb{R})$, there holds

$$\partial_t \beta(g) + a \cdot \nabla_x (\beta(g)) = \beta'(g) \, \partial_t g + \beta'(g) \, a \cdot \nabla_x g = \beta'(g) \, G.$$

Definition 3.1 *We say that $g \in L^1_{loc}([0,T] \times \mathbb{R}^d)$ is a renormalized solution to the transport equation* (1.1) *with $G \in L^1_{loc}([0,T] \times \mathbb{R}^d)$, $g_0 \in L^1_{loc}(\mathbb{R}^d)$ if g satisfies the equations*

$$\int_0^T \int_{\mathbb{R}^d} \beta(g) \, L^* \varphi = \int_{\mathbb{R}^d} \beta(g_0) \varphi(0,.) + \int_0^T \int_{\mathbb{R}^d} \varphi \, \beta'(g) \, G \tag{3.1}$$

for any test function $\varphi \in C^2_c([0,T) \times \mathbb{R}^d)$ and any renormalizing function $\beta \in C^2(\mathbb{R})$ such that $\beta'' \in C_c(\mathbb{R})$.

Theorem 3.2 *With the above notations and assumptions, any weak solution $g \in C([0,T]; L^1_{loc}(\mathbb{R}^d))$ to the transport equation* (1.1) *is a renormalized solution.*

We start with two elementary but fundamental lemmas.

Lemma 3.3 *Given $G \in L^1_{loc}([0,T] \times \mathbb{R}^d)$, let $g \in L^1_{loc}([0,T] \times \mathbb{R}^d)$ be a weak solution to the PDE*

$$\Lambda g = G \quad \text{on } (0,T) \times \mathbb{R}^d.$$

For a mollifer sequence

$$\rho_\varepsilon(t,x) := \frac{1}{\varepsilon^{d+1}} \rho(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}), \quad 0 \le \rho \in \mathcal{D}(\mathbb{R}^{d+1}), \; supp \rho \subset (-1,0) \times B(0,1), \; \int_{\mathbb{R}^{d+1}} \rho = 1,$$

and for $\tau \in (0,T)$, $\varepsilon \in (0,\tau)$, we define the function

$$g_\varepsilon := (\rho_\varepsilon *_{t,x} g)(t,x) := \int_0^T \int_{\mathbb{R}^d} g(s,y)\, \rho(t-s, x-y)\, ds dy.$$

Then $g_\varepsilon \in C^\infty([0,T-\tau) \times \mathbb{R}^d)$ and it satisfies the equation

$$L g_\varepsilon = G_\varepsilon + r_\varepsilon$$

in the classical differential calculus sense on $[0,T-\tau) \times \mathbb{R}^d$, with

$$G_\varepsilon := \rho_\varepsilon *_{t,x} G, \quad r_\varepsilon := a \cdot \nabla_x g_\varepsilon - (a \cdot \nabla g) * \rho_\varepsilon.$$

It is worth emphasizing that in the above formula the "commutator" r_ε is defined in a weak sense, namely

$$r_\varepsilon(t,x) := \int_{\mathbb{R}^{d+1}} g(s,y) \Big\{ a(x) \cdot \nabla_x \rho_\varepsilon(t-s,x-y) + div_y \big[a(y)\, \rho_\varepsilon(t-s,x-y) \big] \Big\}\, dyds.$$

Proof of Lemma 3.3. Define $\mathcal{O} := [0,T-\tau) \times \mathbb{R}^d$. For any $(t,x) \in \mathcal{O}$ fixed and any $\varepsilon \in (0,\tau)$, we define

$$(s,y) \mapsto \varphi(s,y) = \varphi_\varepsilon^{t,x}(s,y) := \rho_\varepsilon(t-s,x-y) \in \mathcal{D}((0,T) \times \mathbb{R}^d).$$

We then just write the weak formulation of equation (1.1) for that test function, and we get

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^d} g\, \Lambda^* \varphi - \int_0^T \int_{\mathbb{R}^d} G\, \varphi \\ &= \int_0^T \int_{\mathbb{R}^d} g(s,y)\, \{ -\partial_s \varphi^{t,x}(s,y) - \nabla_y (a(y)\, \varphi^{t,x}(s,y)) \} - \int_0^T \int_{\mathbb{R}^d} G(s,y)\, \varphi^{t,x}(s,y) \\ &= \int_0^T \int_{\mathbb{R}^d} g(s,y)\, \{ \partial_t \varphi^{t,x}(s,y) + a(x) \cdot \nabla_x \varphi^{t,x}(s,y) \} \\ &\quad + \int_0^T \int_{\mathbb{R}^d} g(s,y)\, \{ a(y) \cdot \nabla_y \varphi^{t,x}(s,y) - a(x) \cdot \nabla_x \varphi^{t,x}(s,y) \} - \int_0^T \int_{\mathbb{R}^d} G(s,y)\, \varphi^{t,x}(s,y) \\ &= \partial_t g_\varepsilon(t,x) + a \cdot \nabla_x g_\varepsilon(t,x) - r_\varepsilon(t,x) - G_\varepsilon(t,x), \end{aligned}$$

which is the anounced equation. ◻

Lemma 3.4 *Under the assumptions $B \in W^{1,q}_{loc}(\mathbb{R}^d)$ and $g \in L^p_{loc}(\mathbb{R}^d)$ with $1/r = 1/p + 1/q \le 1$, then*

$$R_\varepsilon := (B \cdot \nabla g) * \rho_\varepsilon - B \cdot \nabla (g * \rho_\varepsilon) \to 0 \quad L^r_{loc},$$

for any mollifer sequence (ρ_ε).

Remark 3.5 *For a time dependent function $g = g(t,x)$ satisfying the boundedness conditions of Theorem 3.2 the same result (with the same proof) holds, so that the commutator r_ε defined in Lemma 3.3 satisfies $r_\varepsilon \to 0$ in $L^1_{loc}([0,T) \times \mathbb{R}^d)$.*

Proof of Lemma 3.4. We only consider the case $p = 1$, $q = \infty$ and $r = 1$. We start writting

$$\begin{aligned} R_\varepsilon(x) &= -\int g(y) \Big\{ \mathrm{div}_y \Big(B(y)\, \rho_\varepsilon(x-y) \Big) + B(x) \cdot \nabla_x \big(\rho_\varepsilon(x-y) \big) \Big\}\, dy \\ &= \int g(y) \Big\{ \Big(B(y) - B(x) \Big) \cdot \nabla_x (\rho_\varepsilon(x-y)) \Big\}\, dy - ((g\, \mathrm{div}\, B) * \rho_\varepsilon)(x) \\ &=: R^1_\varepsilon(x) + R^2_\varepsilon(x). \end{aligned}$$

For the first term, we remark that

$$\begin{aligned} |R^1_\varepsilon(x)| &\le \int |g(y)| \Big| \frac{B(y) - B(x)}{\varepsilon} \Big|\, |(\nabla \rho)_\varepsilon(x-y)|\, dy \\ &\le \|\nabla B\|_{L^\infty} \int_{|x-y| \le 1} |g(y)|\, |(\nabla \rho)_\varepsilon(x-y)|\, dy, \end{aligned}$$

so that

$$\int_{B_R} |R^1_\varepsilon(x)|\, dx \le \|\nabla B\|_{L^\infty}\, \|\nabla \rho\|_{L^1}\, \|g\|_{L^1(B_{R+1})}. \tag{3.2}$$

On the other hand, if g is a smooth (say C^1) function

$$\begin{aligned} R^1_\varepsilon(x) &= \nabla_x((gB)*\rho_\varepsilon) - B\cdot\nabla_x(g*\rho_\varepsilon) \\ &\longrightarrow \nabla_x(gB) - B\cdot\nabla_x g = (\mathrm{div}B)\, g. \end{aligned}$$

Since every things make sense at the limit with the sole assumption $\mathrm{div}B \in L^\infty$ and $g \in L^1$, with the help of (3.2) we can use a density argument in order to get the same result without the additional smoothness hypothesis on g. More precisely, for a sequence g_α in C^1 such that $g_\alpha \to g$ in L^1_{loc}, we have

$$R^1_\varepsilon[g_\alpha] \to (\mathrm{div}B)\, g \text{ in } L^1_{loc}, \quad \|R^1_\varepsilon[h]\|_{L^1} \le C\, \|h\|_{L^1}\ \forall\, h,$$

so that

$$R^1_\varepsilon[g] - (\mathrm{div}B)\, g = \{R^1_\varepsilon[g] - R^1_\varepsilon[g_\alpha]\} + \{R^1_\varepsilon[g_\alpha] - (\mathrm{div}B)\, g_\alpha)\} + \{(\mathrm{div}B)\, g_\alpha - (\mathrm{div}B)\, g)\} \to 0$$

in L^1_{loc} as $\varepsilon \to 0$. For the second term, we clearly have

$$R^2_\varepsilon = (g\,\mathrm{div}\, B) * \rho_\varepsilon \to g\,\mathrm{div}\, B$$

and we conclude by putting all the terms together. ◻

Proof of Theorem 3.2. Step 1. We consider a weak solution $g \in L^1_{loc}$ to the PDE

$$Lg = G \quad \text{in} \quad [0,T)\times\mathbb{R}^d.$$

By mollifying the functions with the sequence (ρ_ε) defined in Lemma 3.3 and using Lemma 3.3, we get

$$Lg_\varepsilon = G_\varepsilon + r_\varepsilon \quad \text{in} \quad [0,T)\times\mathbb{R}^d, \quad r_\varepsilon \to 0 \text{ in } L^1_{loc}.$$

Because g_ε is a smooth function, we may perform the following computation (in the sense of the classical differential calculus)

$$L\beta(g_\varepsilon) = \beta'(g_\varepsilon)\, G_\varepsilon + \beta'(g_\varepsilon)\, r_\varepsilon,$$

so that

$$\int \beta(g_\varepsilon) L^*\varphi = \int_{\mathbb{R}^d} \beta(g_\varepsilon(0,.))\varphi(0,.) + \int \beta'(g_\varepsilon)\, G_\varepsilon\, \varphi + \int \beta'(g_\varepsilon)\, r_\varepsilon \tag{3.3}$$

for any $\varphi \in C^2_c([0,T)\times\mathbb{R}^d$. Using that

$$g_\varepsilon \to g, \quad G_\varepsilon \to G, \quad r_\varepsilon \to 0 \quad \text{in } L^1_{loc} \text{ as } \varepsilon \to 0,$$

we may pass to the limit $\varepsilon \to 0$ in the last identity and we obtain (3.1) for any test function $\varphi \in C^2_c((0,T)\times\mathbb{R}^d)$.
In the case we consider a solution g built thanks to Theorem 2.4 above or Theorem 3.2 in chapter 1, we know that additionally $g \in C([0,T); L^1_{loc}(\mathbb{R}^d))$ and therefore

$$g_\varepsilon \to g \quad \text{in } C([0,T); L^1_{loc}(\mathbb{R}^d)).$$

In particular $g_\varepsilon(0,.) \to g(0,.)$ in $L^1_{loc}(\mathbb{R}^d)$ and we may pass to the limit in equation (3.3) for any test function $\varphi \in C^2_c([0,T)\times\mathbb{R}^d)$, which ends the proof of (3.1). ◻

4 Consequence of the renormalization result

In this section we present several immediate consequences of the renormalization formula established in Theorem 3.2.

4.1 Uniqueness and C_0-semigroup in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$

Corollary 4.1 *Assume $p \in [1,\infty)$. For any initial datum $g_0 \in L^p(\mathbb{R}^d)$, the transport equation admits a unique weak solution $g \in C([0,T];L^p(\mathbb{R}^d))$.*

Proof of Corollary 4.1. Consider two weak solutions g_1 and g_2 to the transport equation (1.1) associated to the same initial datum g_0. The function $g := g_2 - g_1 \in C([0,T];L^p(\mathbb{R}^d))$ is then a weak solution to the transport equation (1.1) associated to the initial datum $g(0) = 0$. Thanks to Theorem 3.2 it is also a renormalized solution, which means

$$\int_{\mathbb{R}^d} \beta(g(t,.))\,\varphi\,dx = \int_0^t\!\!\int_{\mathbb{R}^d} \beta(g)\,\mathrm{div}_x(a\,\varphi)\,dxds$$

for any renormalizing function $\beta \in W^{1,\infty}(\mathbb{R})$, $\beta(0) = 0$, and any test function $\varphi = \varphi(x) \in C^1_c(\mathbb{R}^d)$. We fix β such that furthermore $0 < \beta(s) \le |s|^p$ for any $s \ne 0$, $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \le \chi \le 1$, $\chi \equiv 1$ on $B(0,1)$ and we take $\psi(x) = \chi_R(x) = \chi(x/R)$, so that

$$\int_{\mathbb{R}^d} \beta(g(t,.))\,\chi_R\,dx = \int_0^t\!\!\int_{\mathbb{R}^d} \beta(g)\,(\mathrm{div}_x a)\,\chi_R\,dxds + \frac{1}{R}\int_0^t\!\!\int_{\mathbb{R}^d} \beta(g)\,a(x)\cdot\nabla\chi(x/R)\,dxds$$

Observing that $\beta(g) \in C([0,T];L^1(\mathbb{R}^d))$ and $\chi_R \to 1$, we easilly pass to the limit $R \to \infty$ in the above expression, and we get

$$\int_{\mathbb{R}^d} \beta(g(t,.))\,dx = \int_0^t\!\!\int_{\mathbb{R}^d} \beta(g)\,(\mathrm{div}_x a)\,dxds \tag{4.1}$$

By the Gronwall lemma we conclude that $\beta(g(t,.)) = 0$ and then $g(t,.) = 0$ for any $t \in [0,T]$. ◻

Exercise 4.2 *Prove the same result assuming only that a is globally Lipschitz. (Hint. Use that $|a(x)| \le C\,(1+|x|)$ for any $x \in \mathbb{R}^d$).*

In the same way as in chapter 1, we can deduce from the above existence and uniqueness result on the linear transport equation (1.1) that the formula

$$(S_t g_0)(x) := g(t,x)$$

defines a C_0-semigroup on $L^p(\mathbb{R}^d)$, $1 \le p < \infty$, where g is the solution to the transport equation (1.1) associated to the initial datum g_0.

4.2 Positivity

We can recover in a quite elegant way the positivity as an aposteriori property that we deduce from the renormalization formula.

Corollary 4.3 *Consider a solution $g \in C([0,T];L^p(\mathbb{R}^d))$, $1 \le p < \infty$, to the transport equation (1.1). If $g_0 \ge 0$ then $g(t,.) \ge 0$ for any $t \ge 0$.*

Proof of Corollary 4.4. We argue similarily as in the proof of Corollary 4.1 but fixing a renormalizing function $\beta \in W^{1,\infty}(\mathbb{R})$ such that $\beta(s) = 0$ for any $s \ge 0$, $\beta(s) > 0$ for any $s < 0$. Since then $\beta(g_0) = 0$, we deduce that (4.1) holds again with that choice of function β and then, thanks to Gronwall lemma, $\beta(g(t,.)) = 0$ for any $t \ge 0$. That means $g(t,.) \ge 0$ for any $t \ge 0$. ◻

4.3 A posteriori estimate

Corollary 4.4 *Consider a solution $g \in C([0,T];L^p(\mathbb{R}^d))$, $1 \le p < \infty$, to the transport equation (1.1). If $g_0 \in L^q(\mathbb{R}^d)$, $1 \le q \le \infty$, then $g \in L^\infty(0,T;L^q(\mathbb{R}^d))$ for any $T > 0$.*

Proof of Corollary 4.4. We argue similarily as in the proof of Corollary 4.1 but fixing an arbitray renormalizing function $\beta \in W^{1,\infty}(\mathbb{R})$, $\beta(s) = 0$ on a small neighbourhood of $s = 0$, so that $\beta(g) \in C([0,T]; L^1(\mathbb{R}^d))$. For such a choice, we obtain the time integrale inequality

$$\int_{\mathbb{R}^d} \beta(g(t,.))\, dx = \int_{\mathbb{R}^d} \beta(g_0)\, dx + \int_0^t\!\!\int_{\mathbb{R}^d} \beta(g)\,(\mathrm{div}_x a)\, dxds \qquad \forall\, t \ge 0.$$

From the Gronwall lemma, we obtain with $b = \|\mathrm{div}\, a\|_{L^\infty}$, the estimate

$$\int_{\mathbb{R}^d} \beta(g(t,.))\, dx \le e^{b\,t} \int_{\mathbb{R}^d} \beta(g_0)\, dx \qquad \forall\, t \ge 0. \tag{4.2}$$

Since estimate (4.2) is uniform with respect to β, we may choose a sequence of renormalizing functions (β_n) such that $\beta_n(s) \nearrow |s|^q$ in the case $1 \le q < \infty$ and we get

$$\|g(t,.)\|_{L^q} \le e^{bt/q}\, \|g(t,.)\|_{L^q} \qquad \forall\, t \ge 0.$$

In the case $q = \infty$, we obtain the same conclusion by fixing $\beta \in W^{1,\infty}$ such that $\beta(s) = 0$ for any $|s| \le \|g_0\|_{L^\infty}$, $\beta(s) > 0$ for any $|s| > \|g_0\|_{L^\infty}$ or by passing to the limit $q \to \infty$ in the above inequality. ◻

4.4 Continuity

We can recover the strong L^p continuity property from the renormalization formula for a given solution. We do not present that rather technical issue here.

5 Complementary results

In this section we state and give a sketch of the proof of several complementary results of existence and uniqueness.

- equations by adding a given source term / a nonlinear RHS term ;

- domains by considering the equation set on $\Omega \subset \mathbb{R}^d$ (and we possibly have to add boundary conditions). We do not consider that problem in the present notes.

Regarding the uniqueness issue, we will explain how to obtain it in a L^∞ framework.

5.1 Duhamel formula and existence for transport equation with an additional term

We consider the transport equation with an additional term

$$\partial_t g = \Lambda\, g + G \quad \text{in} \quad (0,\infty) \times \mathbb{R}^d, \tag{5.1}$$

with

$$(\Lambda g)(x) := -a(x) \cdot \nabla g(x) + c(x)\, g(x) + \int_{\mathbb{R}^d} b(x,y)\, g(y)\, dy$$

We interpret that equation as a perturbation equation

$$\partial_t g = \mathcal{B} g + \tilde{G}, \quad \tilde{G} = \mathcal{A} g + G,$$

and we claim that the fiunction

$$g(t) = S_{\mathcal{B}}(t) g_0 + \int_0^t S_{\mathcal{B}}(t-s)\, \tilde{G}(s)\, ds \tag{5.2}$$

is a solution to equation (5.1). Indeed, the semigroup $S_{\mathcal{B}}$ satisfies by definition (and at least formally)

$$\frac{d}{dt} S_{\mathcal{B}}(t) h = \mathcal{B} S_{\mathcal{B}}(t) h,$$

so that (again formally)

$$\begin{aligned}\frac{d}{dt}g(t) &= \frac{d}{dt}S_{\mathcal{B}}(t)g_0 + \int_0^t \frac{d}{dt}S_{\mathcal{B}}(t-s)\,\tilde G(s)\,ds + S_{\mathcal{B}}(0)\,\tilde G(t) \\ &= \mathcal{B}\Big\{S_{\mathcal{B}}(t)g_0 + \int_0^t S_{\mathcal{B}}(t-s)\,\tilde G(s)\,ds\Big\} + \tilde G(t) \\ &= \mathcal{B}\,g(t) + \tilde G(t).\end{aligned}$$

All that computations can be justified when written in a weak sense. The method used here is nothing but the wellknown variation of the constant method in ODE, the expression (5.2) is called the *"Duhamel formula"* and a function $g(t)$ which satisfies (5.2) (in an appropriate and meaningful functional sense) is called a *"mild solution"* to the equation (5.1).

Theorem 5.1 *Assume $a \in W^{1,\infty}$, $c \in L^\infty$, $b \in L^\infty_x(L^{p'}_y) \cap L^\infty_y(L^{p'}_x)$, $1 \le p < \infty$. For any $g_0 \in L^p$ and $G \in L^1(0,T;L^p)$ there exists a unique mild (weak, renormalized) solution to equation* (5.1).

Elements of proof of Theorem 5.1. For any $h \in C([0,T];L^p)$, we define the mapping

$$(\mathcal{U}h)(t) := S_{\mathcal{B}}(t)g_0 + \int_0^t S_{\mathcal{B}}(t-s)\,\Big\{\mathcal{A}h(s) + G(s)\Big\}\,ds$$

and we observe that

$$\mathcal{U} : C([0,T];L^p) \to C([0,T];L^p)$$

with Lipschitz constant bounded by $b\,T$. We just point out that thanks to Young inequality (when $1 < p < \infty$)

$$\begin{aligned}\int\!\!\int b(x,y)h(y)\,g(x)^{p-1}\,dxdy &\le \int\!\!\int b(x,y)[h(y)^p + g(x)^p\,dxdy \\ &\le \|b\|_{L^\infty_x(L^{p'}_y)}\,\|h\|^p_{L^p} + \|b\|_{L^\infty_y(L^{p'}_x)}\,\|g\|^p_{L^p}.\end{aligned}$$

Choosing T small enough, we can apply the Banach-Picard contraction theorem and we get the existence of a fixed point $g \in C([0,T];L^p)$, $g = \mathcal{U}g$. Proceding by induction, we obtain in that way a global mild solution to equation (5.1). ◻

5.2 Duality and uniqueness in the case $p = \infty$

Theorem 5.2 *Assume $a \in W^{1,\infty}$. For any $g_0 \in L^\infty$ there exists at most one weak solution $g \in L^\infty((0,T)\times\mathbb{R}^d)$ to the transport equation* (1.1).

Elements of proof of Theorem 5.2. Since the equation is linear we only have to prove that the unique weak solution $g \in L^\infty((0,T)\times\mathbb{R}^d)$ associated to the initial datum $g_0 = 0$ is $g = 0$.
By definition, for any $\psi \in C^1_c([0,T]\times\mathbb{R}^d)$, there holds

$$\int_0^T\!\!\int_{\mathbb{R}^d} g\,L^*\psi\,dxdt = -\int_{\mathbb{R}^d} g(T)\,\psi(T)\,dx$$

with $L^*\psi := -\partial_t\psi - \mathrm{div}(a\,\psi)$. We claim that for any $\Psi \in C^1_c((0,T)\times\mathbb{R}^d)$ there exists a function $\psi \in C^1_c([0,T]\times\mathbb{R}^d)$ such that

$$\Lambda^*\psi = \Psi, \quad \psi(T) = 0. \tag{5.3}$$

If we accept that fact, we obtain

$$\int_0^T\!\!\int_{\mathbb{R}^d} g\,\Psi\,dxdt = 0 \quad \forall\,\Psi \in C^1_c((0,T)\times\mathbb{R}^d),$$

which in turns implies $g = 0$ and that ends the proof.

Here we can solve easily the backward equation (5.3) thanks to the characterics method which leads to an explicit representation formula. In order to make the discussion simpler we exhibit that formula for the associated forward problem (we do not want to bother with backward time, but one can pass from a formula to another just by changing time $t \to T - t$). We then consider the equation

$$\partial_t \psi + a \cdot \nabla \psi + c\, \psi = \Psi, \quad \psi(0) = 0,$$

with $c := \operatorname{div} a$. Introduction the flow $\Phi_t(x)$ associated to the ODE $\dot{x} = a(x)$, if such a solution exists, we have

$$\frac{d}{dt}\Big[\psi(t, \Phi_t(x))\, e^{\int_0^t c(\Phi_s(x))\, ds}\Big] = \Psi(t, \Phi_t(x))\; e^{\int_0^t c(\Phi_s(x))\, ds},$$

from which we deduce

$$\psi(t, \Phi_t(x)) = e^{-\int_0^t c(\Phi_\tau(x))\, d\tau} \int_0^t \Psi(s, \Phi_s(x))\; e^{\int_0^s c(\Phi_\tau(x))\, d\tau},$$

or equivalently, observing that $\Phi_t^{-1} = \Phi_{-t}$ because the ODE is time autonomous, we have

$$\psi(t, x) := \int_0^t \Psi(s, \Phi_{s-t}(x))\, e^{-\int_s^t c(\Phi_{\tau - t}(x))\, d\tau}\, ds.$$

It is clear that ψ defined by the above formula is the solution to our dual problem from which we get (reversing time) the solution to (5.3) we were trying to find. ◻

6 Transport equation in conservative form

In this section we consider a time depend vectors field $a = a(t, y) : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ of class $C^1 \cap \mathrm{Lip}$ and we note L the Lipschitz constant of a in the second variable :

$$\forall\, t \in [0, T],\ \forall\, x, y \in \mathbb{R}^d \qquad |a(t, x) - a(t, y)| \le L\, |x - y|.$$

We are interested in the transport equation in conservative form

$$\frac{\partial f}{\partial t} + \nabla\,(a\, f) = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{R}^d), \tag{6.4}$$

where $f = f(t, dx) = df_t(x)$ is a mapping from $(0, T)$ into the space of bounded Radon measures $M^1(\mathbb{R}^d)$ or the space of probability measures $\mathbf{P}(\mathbb{R}^d)$. We recall that

$$M^1(\mathbb{R}^d) := \{f \in (C_c(\mathbb{R}^d))';\ \|f\|_{TV} := \sup_{\|\varphi\|_\infty \le 1} |\langle f, \varphi\rangle| < \infty\}$$

and

$$\mathbf{P}(\mathbb{R}^d) := \{f \in (C_c(\mathbb{R}^d))';\ f \ge 0,\ \langle f, 1\rangle = 1\} \subset M^1(\mathbb{R}^d).$$

We also define

$$\mathbf{P}_1(\mathbb{R}^d) := \{f \in \mathbf{P}(\mathbb{R}^d);\ \langle f, |x|\rangle < 1\}$$

and the Monge-Kantorovich-Wasserstein distance on $\mathbf{P}_1(\mathbb{R}^d)$ by

$$\forall\, f, g \in \mathbf{P}_1(\mathbb{R}^d), \quad W_1(f, g) = \|f - g\|_{(Lip)'} := \sup_{\varphi \in C^1; \|\nabla \varphi\|_\infty \le 1} \langle f - g, \varphi\rangle.$$

We just point out that $W_1(f, g)$ is well-defined and finite for any $f, g \in \mathbf{P}_1(\mathbb{R}^d)$ because

$$|\langle f - g, \varphi\rangle| = |\langle f - g, \varphi - \varphi(0)\rangle| \le \langle |f| + |g|, |x|\rangle < \infty,$$

for any $\varphi \in C^1(\mathbb{R}^d)$, $\|\nabla \varphi\|_\infty \le 1$, and that $W_1(f, g) = 0$ implies $f = g$ because $C^1_c(\mathbb{R}^d) \subset C_c(\mathbb{R}^d)$ with continuous and dense embedding.

Definition 6.3 (Image Measure). *Let $(E,\mathcal{E},\mu)$ be a measure space, F be a set and $\Phi : E \to F$ a mapping. We define the σ-algebra $\mathcal{F}$ on F by $\mathcal{F} := \{A \subset F;\ \Phi^{-1}(A) \in \mathcal{E}\}$ (it is the smallest σ-algebra on F for which Φ is a measurable) and we define the measure ν on $\mathcal{F}$ by $\forall\, A \in \mathcal{F}$ $\nu[A] := \mu[\Phi^{-1}(A)]$. We denote $\nu = \Phi \sharp \mu$ and we say that ν is the image measure of μ by Φ. By definition, for any measurable function $\varphi : (F,\mathcal{F}) \to \mathbb{R}_+$, we have*

$$\int_F \varphi\, d(\Phi \sharp \mu) = \int_E \varphi \circ \Phi\, d\mu.$$

Theorem 6.4 (Characterictics). *For any $f_0 \in M^1(\mathbb{R}^d)$, the unique solution $f \in C([0,T]; M^1(\mathbb{R}^d)-w)$ to the transport equation (6.4) associated to the initial datum f_0 is given by*

$$f(t,.) = \Phi_t \sharp f_0 \quad \forall\, t \in [0,T], \tag{6.5}$$

where Φ_t denotes the flow associated to the ODE of characterictics defined in section 2.1. Moreover, given two initial data f_0, $g_0 \in \mathscr{P}_1(\mathbb{R}^d)$, the corresponding solutins $f,g \in C([0,\infty); \mathscr{P}(\mathbb{R}^d) - w)$ to the transport equation (6.4) *satisfy*

$$\forall\, t \in [0,T] \qquad W_1(f_t, g_t) \le e^{L\,t} W_1(f_0, g_0). \tag{6.6}$$

Remark 6.5 *For a deterministic system associated to a vectors field a, we say that* (2.1) *is a Lagragian description of the dynamics while* (6.4) *is an Eulerian description. The formula* (6.5) *shows the equivalence between these two points of view.*

Proof of Theorem 6.6. Step 1. We prove thar $f(t) := \Phi_t \sharp f_0$ is a solution to (6.4). Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and just compute

$$\begin{aligned}
\Big\langle \frac{\partial f}{\partial t}, \varphi \Big\rangle = \frac{d}{dt} \int_{\mathbb{R}^d} \varphi\, f(t) \quad &= \quad \frac{d}{dt} \int_{\mathbb{R}^d} \varphi(\Phi_t(y_0))\, f_0(dy_0) \\
&= \int_{\mathbb{R}^d} (\nabla\varphi)(\Phi_t(y_0)) \cdot \frac{d}{dt}(\Phi_t(y_0))\, f_0(dy_0) \\
&= \int_{\mathbb{R}^d} (\nabla\varphi)(\Phi_t(y_0)) \cdot a(t, \Phi_t(y_0))\, f_0(dy_0) \\
&= \int_{\mathbb{R}^d} (\nabla\varphi)(y) \cdot a(t,y)\, f(t,dy) \\
&= -\Big\langle \nabla(a\, f), \varphi \Big\rangle,
\end{aligned}$$

in the sense of duality in $\mathcal{D}'((0,T))$. That means that (6.4) holds in the sense of duality in $(\mathcal{D}(0,T) \otimes \mathcal{D}(\mathbb{R}^d))'$, and thanks to a denisty argument, in the sense of duality in $\mathcal{D}'((0,T) \times \mathbb{R}^d)$.

Step 2. We establish the uniqueness of the solution. Because the equation is linear, we just have to prove that $f_T = 0$ is $f_0 = 0$. We argue by duality. We defined the backward flow Ψ_t by setting $\Psi_t(z) = z(t)$, where $z(t)$ is the solution to the ODE

$$z'(t) = a(t, z(t)), \quad z(T) = z.$$

For a given function $\varphi_T \in C^1_c(\mathbb{R}^d)$, we define $\varphi(t,y) := \varphi_T(\Psi_t^{-1}(y)) \in C^1_b([0,T] \times \mathbb{R}^d)$. From the implicit equation $\varphi(t, z(t)) = \varphi_T(z)$, we obtain

$$\begin{aligned}
0 \quad &= \quad \frac{d}{dt}\,[\varphi(t,z(t))] = (\partial_t \varphi)(t, z(t)) + (\nabla\varphi)(t,z(t))\, z'(t) \\
&= \quad [\partial_t \varphi + a \cdot \nabla\varphi]\,(t, z(t)),
\end{aligned}$$

and the following transport equation holds (in the sense of classical differential calculus)

$$\partial_t \varphi + a \cdot \nabla \varphi = 0 \qquad [0,T] \times \mathbb{R}^d.$$

We then compute

$$\begin{aligned}
\frac{d}{dt}\langle f_t,\varphi_t\rangle &= \int_{\mathbb{R}^d}[\partial_t\varphi(x)]\,f_t(dx)+\langle\partial_t f_t,\varphi_t\rangle\\
&= \int_{\mathbb{R}^d}[a\cdot\nabla\varphi_t(x)]\;f_t(dx)+\int_{\mathbb{R}^d}[-a\cdot\nabla\varphi_t(x)]\;f_t(dx)=0.
\end{aligned}$$

It implies

$$\int_{\mathbb{R}^d}\varphi_T(x)\,f_T(dx)=\int_{\mathbb{R}^d}\varphi_0(x)\,f_0(dx)=0,$$

for any $\varphi_T\in C^1_b(\mathbb{R}^d)$, which means $f_T\equiv 0$.

Step 3. We start recalling thta the flow Φ_t satisfies

$$\forall\, t\in[0,T]\quad \|\nabla_y\Phi_t\|_\infty\le e^{tL}. \tag{6.7}$$

Indeed, for $x_0,y_0\in\mathbb{R}^d$, the two solutions solutions $x_t=\Phi_t(x_0)$, $y_t=\Phi_t(y_0)$ satisfy

$$\frac{d}{dt}|x_t-y_t|\le|\dot x_t-\dot y_t|\le|a(t,x_t)-a(t,y_t)|\le L\,|x_t-y_t|,$$

and we conclude thanks to the Gronwall lemma.

Now, thanks to Theorem 6.6 and by definition of W_1 and $\sharp$, we have

$$\begin{aligned}
W_1(f_t,g_t) &= W_1(\Phi_t\sharp f_0,\Phi_t\sharp g_0)\\
&= \sup_{\|\nabla\varphi\|\le 1}\int_{\mathbb{R}^d}\varphi\,d(\Phi_t\sharp f_0-\Phi_t\sharp g_0)\\
&= \sup_{\|\nabla\varphi\|\le 1}\int_{\mathbb{R}^d}\varphi\circ\Phi_t\,d(f_0-g_0)\\
&\le \sup_{\|\nabla\varphi\|\le 1}\|\nabla\varphi\circ\Phi_t\|\sup_{\|\nabla\psi\|\le 1}\int_{\mathbb{R}^d}\psi\,d(f_0-g_0)\\
&\le \|\nabla\Phi_t\|\,W_1(f_0,g_0)
\end{aligned}$$

and we conclude thanks to (6.7). ◻

Remark 6.6 *1. When the solution has a density with respect to the Lebesgue measure $f_t(dy)=g_t(y)\,dy$ with $g_t\in L^1(\mathbb{R}^d)$, the theorem of changement of variables in the definition of the image measure implies*

$$g_0((x)=g_t(\Phi_t(x))\,det(D\Phi_t(x)).$$

In particular $g_0(y)$ is not aqual to $g_t(\Phi_t(y))$ in general.

2. However, one can classically show that $J(t,y):=det(D\Phi_t(y))$ satisfies the Liouville equation

$$\frac{d}{dt}J(t,y)=[(div\,a)(t,\Phi_t(y))]\,J(t,y),\quad J(0,y)=det(Id)=1.$$

In the case of a free-diveregnce verctors field a, namely $div\,a=0$, we deduce of it the incompressibility of the flow $J(t,y)\equiv 1$. In that case, $g_t(\Phi_t(x))=g_0(x)$.

3. When $div\,a=0$ we can obtain $g_t(\Phi_t(y))=g_0(y)$ (and thus recover the incompressibility of the flow) in a maybe much simpler way. We come back to the uniqueness argument in the proof of Theorem 6.6. We define $h(t,z):=g_0(\Phi_t^{-1}(z))$ for $g_0\in C^1_b(\mathbb{R}^d)$, and we compute

$$0=\frac{d}{dt}\left[h(t,y(t))\right]=\left[\partial_t h+a\cdot\nabla h\right](t,y(t)).$$

We deduce

$$0=\partial_t h+a\cdot\nabla h=\partial_t h+\nabla(a\,h),\quad h(0,.)=g_0.$$

From the uniqueness of the solution, there holds $h = g$, and then

$$g(t, \Phi_t(x)) = h(t, \Phi_t(x)) = g_0(x) = g(t, \Phi_t(x))\, J(t, x).$$

Choosing $g_0 \to 1$, we get $J \equiv 1$.

4. For $f_0 = \delta_{x_0}$, we have

$$\Phi_t \sharp \delta_{x_0} = \delta_{\Phi_t(x_0)}.$$

Indeed, for any test function $\varphi \in C_b(\mathbb{R}^d)$, we write

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x)(\Phi_t \sharp \delta_{x_0})(dx) &= \int_{\mathbb{R}^d} \varphi(\Phi_t(x))\, \delta_{x_0}(dx) \\ &= \varphi(\Phi_t(x_0)) = \int_{\mathbb{R}^d} \varphi(x)\, \delta_{\Phi_t(x_0)}(dx).\end{aligned}$$

Lemma 6.7 *For an initial $f_0 \in L^1(\mathbb{R}^d)$, the solution $f \in C([0,T]; M^1(\mathbb{R}^d) - w)$ to the conservative transport equation* (6.4) *satisfies*
(1) the mass conservation property :

$$\int_{\mathbb{R}^d} f(t,x)\, dx = \int_{\mathbb{R}^d} f_0\, dx \quad \forall\, t \in [0,T];$$

(2) the L^1 stability property :

$$\int_{\mathbb{R}^d} |f(t,x)|\, dx = \int_{\mathbb{R}^d} |f_0|\, dx \quad \forall\, t \in [0,T].$$

Proof of Lemma 6.7. We only prove (2), point (1) can be proved similarly. We write

$$\partial_t f = -a \cdot \nabla f - (\mathrm{div} a)\, f,$$

which has an unique solution $f \in C([0,\infty); L^1_{loc}(\mathbb{R}^d))$ thanks to Theorem 2.4 or its variant Theorem 5.1. For any renormalizing function $\beta \in C^1 \cap W^{1,\infty}$, we have

$$\partial_t \beta(f) = -a \cdot \nabla \beta(f) - (\mathrm{div} a)\, f\, \beta'(f) \quad \text{in } \mathcal{D}'((0,T))$$

and then

$$\frac{d}{dt} \int_{\mathbb{R}^d} \beta(f)\, \chi = \int_{\mathbb{R}^d} \big\{ \chi (\mathrm{div} a)\, [\beta(f) - f\, \beta'(f)]\, dx + (a \cdot \nabla \chi)\, \beta(f) \big\}$$

for any $\chi \in \mathcal{D}(\mathbb{R}^d)$. We first take $\beta_\varepsilon(s) = s^2/2$ for $|s| \le \varepsilon$, $\beta_\varepsilon(s) = |s| - \varepsilon/2$ for $|s| \ge \varepsilon$, and observing that $|\beta_\varepsilon(s) - s\, \beta'_\varepsilon(s)| \le \varepsilon\ \forall\, s \in \mathbb{R}$, $\forall\, \varepsilon \in (0,1)$ as well as $\beta_\varepsilon(s) \to |s|$ as $\varepsilon \to 0$, we may pass to the limit $\varepsilon \to 0$ and we get

$$\begin{aligned}&\Big| \int_{\mathbb{R}^d} (|f(t)| - |f_0|)\, \chi - \int_0^t \int_{\mathbb{R}^d} (a \cdot \nabla \chi)\, |f(s)|\, dx ds \Big| \\ &\qquad = \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^d} \big\{ \chi (\mathrm{div} a)\, [\beta_\varepsilon(f) - f\, \beta'_\varepsilon(f)]\, dx ds \big\} = 0.\end{aligned}$$

Taking $\chi(x) = \psi(x/R)$ with $\psi \in \mathcal{D}(\mathbb{R}^d)$, $\psi \equiv 1$ on $B(0,1)$, $0 \le \psi \le 1$, $\mathrm{supp}\, \psi \subset B(0,2)$, we may pass to the limit $R \to \infty$, and we get

$$\int_{\mathbb{R}^d} |f(t)|\, dx = \int_{\mathbb{R}^d} |f_0| + \lim_{R \to \infty} \frac{1}{R} \int_0^t \int_{\mathbb{R}^d} a(s,x) \cdot \nabla \psi(x/R))\, |f(s)|\, dx ds = \int_{\mathbb{R}^d} |f_0|,$$

so that the L^1 stability property is proved. $\square$

Exercise 6.8 *We define*

$$J(t,x) := \exp\Big(\int_0^t (div\, a)(s,\Phi_s(x))\, ds\Big).$$

(1) Show that for $f_0 \in C_c^1(\mathbb{R}^d)$, *the function* f *defined implicitely by*

$$f(t,\Phi_t(x))J(t,x) = f_0(x) \quad \forall\, t \in [0,T],\ \forall\, x \in \mathbb{R}^d,$$

is the (unique) solution to the transport equation in divergenece form (6.4) *associated to the inital datum* f_0.
(2) Show or use Liouville Theorem $J(t,.) = det\, D\Phi_t$, *in order to get an alternative proof of Lemma 6.7.*

7 References

The main result of the chapter, namely Theorem 3.2, is due to R. DiPerna and P.-L. Lions and has been established in [1].

– [1] DiPerna, R. J., and Lions, P.-L. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math. 98*, 3 (1989), 511–547.