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An introduction to evolution PDEs

Chapter 4 - Relative entropy

This short chapter is an introduction to entropy (or Liapunov) methods for the scattering (or linear Boltzmann) equation first and a general class of evolution PDEs next.

1 Weighted L^2 inequality for the scattering equation

The linear Boltzmann (or scattering) equation of the density function $f = f(t,v) \ge 0$, $t \ge 0$, $v \in \mathcal{V} \subset \mathbb{R}^d$, writes

$$\partial_t f = \mathcal{L}\, f := \int_{\mathcal{V}} (b_* f_* - b\, f)\, dv_*, \tag{1.1}$$

where $b = b(v, v_*)$ and $b_* = b(v_*, v)$, $b \ge 0$ is a given function (the rate of collisions), or more generally

$$\partial_t f = \mathcal{L}\, f := \int_{\mathcal{V}} b_* f_*\, dv_* - B(v)\, f, \tag{1.2}$$

and we assume that there exists a function $\phi > 0$ such that

$$\mathcal{L}^* \phi := \int_{\mathcal{V}} b\, \phi_*\, dv_* - B\, \phi = 0, \quad \text{in other words} \quad B(v) := \int_{\mathcal{V}} \frac{\phi_*}{\phi}\, b\, dv_*,$$

with gain $\phi = \phi(v)$ and $\phi_* = \phi(v_*)$. The first equation (1.1) corresponds to the choice

$$B(v) = \int_{\mathcal{V}} b\, dv_*, \quad \phi \equiv 1,$$

in the second equation (1.2).

Example 1. We assume $\mathcal{V} \subset \mathbb{R}^d$, $b_* = k(v, v_*)\, F(v)$, for a symmetric function $k(v, v_*) = k(v_*, v) > 0$ and a given function $0 < F \in L^1(\mathcal{V}) \cap \mathbf{P}(\mathcal{V})$. The equation (1.1) becomes

$$\partial_t f = \mathcal{L}\, f := \int_{\mathcal{V}} k\, (F\, f_* - F_*\, f)\, dv_*. \tag{1.3}$$

It is worth noticing that $F = F(v)$ is a stationary solution to the equation (1.5) since

$$\partial_t F = 0 = \mathcal{L}\, F. \tag{1.4}$$

Example 2. We assume $\mathcal{V} = (0, \infty)$, $b_* = b_*\, \mathbf{1}_{v_* > v}$, $\phi(v) = v$, and then the equation (1.2) becomes the *fragmentation equation*

$$\partial_t f = \mathcal{L}\, f := \int_0^\infty b_*\, f_*\, dv_* - B(v)\, f(v), \quad B(v) := \int_0^v \frac{v_*}{v}\, b\, dv_*. \tag{1.5}$$

Conservation law. Without any additional assumption, we immediately deduce that the equation (1.2) has one law of conservation : any solution satisfies (at least formally)

$$\int_{\mathcal{V}} f(t,v)\,\phi(v)\,dv = \int_{\mathcal{V}} f(0,v)\,\phi(v)\,dv,$$

because

$$\frac{d}{dt}\int_{\mathcal{V}} f\,\phi\,dv = \int_{\mathcal{V}} (\mathcal{L}f)\,\phi\,dv = \int_{\mathcal{V}} f\,(\mathcal{L}^*\phi)\,dv = 0.$$

Liapunov/entropy functional. We assume that there exists a function $0 < F \in L^1(\mathcal{V}) \cap \mathbf{P}(\mathcal{V})$ which is a stationary solution

$$\mathcal{L}F = \int_{\mathcal{V}} b_* F_*\,dv_* - \int_{\mathcal{V}} \frac{\phi_*}{\phi}\,b\,dv_*\,F = 0,$$

what it is the situation in Example 1. Then any solution f to the equation (1.2) satisfies (at least formally)

$$\frac{d}{dt}\int_{\mathcal{V}} f^2\,\frac{\phi}{F}\,dv = 2\int_{\mathcal{V}} (\mathcal{L}\,f)\,\frac{f\,\phi}{F}\,dv = -D_2(f) \tag{1.6}$$

with

$$D_2(f) := \int_{\mathcal{V}}\int_{\mathcal{V}} b_*\,F\,\phi\left(\frac{f_*}{F_*} - \frac{f}{F}\right)^2 dv dv_*. \tag{1.7}$$

We then say that

$$\mathcal{H}_2(f) := \int_{\mathcal{V}} f^2\,\frac{\phi}{F}\,dv$$

is a Liapunov (or generalized relative entropy) for the equation (1.2).

To prove (1.6) in the case $\phi = 1$, we perform the following computations

$$\begin{aligned}
(\mathcal{L}f, f/F) &= \iint b_* F_*\,\frac{f_*}{F_*}\,\frac{f}{F} - \frac{1}{2}\iint b\,F\,\frac{f^2}{F^2} - \frac{1}{2}\iint b\,F\,\frac{f^2}{F^2} \\
&= \iint b_* F_*\,\frac{f_*}{F_*}\,\frac{f}{F} - \frac{1}{2}\iint b_*\,F_*\,\frac{(f_*)^2}{(F_*)^2} - \frac{1}{2}\iint b_*\,F_*\,\frac{f^2}{F^2} \\
&= -\frac{1}{2}\iint b_* F_*\left(\frac{f_*}{F_*} - \frac{f}{F}\right)^2,
\end{aligned}$$

where in order to pass from the first to the second line we have just changed the name of the variables in the second term

$$\iint b\,F\,\frac{f^2}{F^2} = \iint b_*\,F_*\,\frac{(f_*)^2}{(F_*)^2}$$

and we have used the fact that F is a stationary solution in the third term

$$\int b\,F\,dv_* = \int b_*\,F_*\,dv_*.$$

For a general law of conservation ϕ, the computation is almost the same

$$\begin{aligned}
(\mathcal{L}f, \phi\,f/F) &= \iint b_*\,\phi\,F_*\,\frac{f_*}{F_*}\,\frac{f}{F} - \frac{1}{2}\iint b\,\phi_*\,F\,\frac{f^2}{F^2} - \frac{1}{2}\iint b\,\phi_*\,F\,\frac{f^2}{F^2} \\
&= \iint b_*\,\phi\,F_*\,\frac{f_*}{F_*}\,\frac{f}{F} - \frac{1}{2}\iint b_*\,\phi\,F_*\,\frac{(f_*)^2}{(F_*)^2} - \frac{1}{2}\iint b_*\,\phi\,F_*\,\frac{f^2}{F^2} \\
&= -\frac{1}{2}\iint b_*\,\phi\,F_*\left(\frac{f_*}{F_*} - \frac{f}{F}\right)^2.
\end{aligned}$$

A theorem. We now consider the same situation as in example 1, and we assume furthermore that there exist some constants $0 < k_0 \le k_1 < \infty$ such that

$$\forall\, v, v_* \in \mathcal{V}, \qquad k_0 \le k(v, v_*) \le k_1.$$

We consider the scattering equation (1.1) in that case, that we complement with an initial condition

$$f(0, v) = f_0(v) \quad \forall\, v \in \mathcal{V}.$$

Theorem 1.1 *Assume* $f_0 \in L^1(\mathcal{V})$, $\mathcal{V} = \mathbb{R}^d$.
(1) There exists a unique global solution $f \in C([0,\infty); L^1(\mathcal{V}))$ *to the scattering equation* (1.1). *That solution is mass conserving*

$$\int_{\mathcal{V}} f(t,v)\, dv = \int_{\mathcal{V}} f_0(v)\, dv =: \langle f_0 \rangle$$

and satisfies the maximum principle

$$f_0 \ge 0 \qquad \Rightarrow \qquad f(t,.) \ge 0 \quad \forall\, t \ge 0.$$

(2) In the large time asymptotic, the solution converges to the unique stationary solution with same mass

$$\|f(t,.) - \langle f_0 \rangle\, F\|_E \le e^{-k_0 t/2}\, \|f_0 - \langle f_0 \rangle\, F\|_E,$$

where $\| \cdot \|_E$ *is the Hilbert norm defined by*

$$\|f\|_E^2 := \int_{\mathcal{V}} f^2\, F^{-1}\, dv.$$

For the proof of point (1) we refer to the precedent chapters where the needed arguments have been introduced. We are going to give now the (formal) proof of point (2).

Functional inequality and long time behaviour. The following functional inequality holds true : for any function $f \in E$, we have

$$D_2(f) \ge k_0 \|f - \langle f \rangle\, F\|_E^2. \tag{1.8}$$

It is worth observing that the Cauchy-Schwarz inequality implies

$$|\langle f \rangle| \le \int_{\mathcal{V}} (|f| F^{-1/2})\, F^{1/2} \le \Big(\int_{\mathcal{V}} f^2 F^{-1}\Big)^{1/2} \Big(\int_{\mathcal{V}} F\Big)^{1/2} = \|f\|_E,$$

so that the mass $\langle f \rangle$ is well defined if $f \in E$. Let us accept for a while the inequality (1.8) and let us prove then the convergence result (2) in Theorem 1.1. Thanks to (1.6), the fact that F is a stationary solution, the fact that f is mass conserving and (1.8), we have

$$\frac{d}{dt} \|f - \langle f \rangle F\|_E^2 = -D_2(f) \le -k_0 \|f - \langle f \rangle\, F\|_E^2,$$

and we conclude by applying the Gronwall lemma.
Let us prove now the functional inequality (1.8). From the lower bound assumption made on k, the following first inequality holds

$$D_2(f) := \iint b_*\, F_* \left(\frac{f_*}{F_*} - \frac{f}{F} \right)^2 \ge k_0 \iint F\, F_* \left(\frac{f_*}{F_*} - \frac{f}{F} \right)^2.$$

On the other hand, by integrating (in the v_* variable) the identity

$$f\, F_* - f_*\, F = \left(\frac{f}{F} - \frac{f_*}{F_*} \right) F\, F_*,$$

we get

$$g = F \int_{\mathcal{V}} \left(\frac{f}{F} - \frac{f_*}{F_*} \right) F_*\, dv_*$$

with $g = f - \langle f \rangle F$. Thanks to the Cauchy-Schwarz inequality, we deduce

$$g^2 \le \int_{\mathcal{V}} \left(\frac{f}{F} - \frac{f_*}{F_*} \right)^2 F\, F_*\, dv_* \times \int_{\mathcal{V}} F\, F_*\, dv_*,$$

so that we get the second inequality

$$\int_{\mathcal{V}} \frac{g^2}{F}\, dv \quad \le \quad \int_{\mathcal{V}} \int_{\mathcal{V}} \left(\frac{f}{F} - \frac{f_*}{F_*} \right)^2 F\, F_*\, dv_*\, dv.$$

We conclude by gathering these two estimates.

2 Relative entropy for linear and positive PDE

We consider the general PDE evolution equation

$$\partial_t f = \Delta f - a \cdot \nabla f + cf + \int b\, f_*, \quad \int b\, f_* := \int b(x, x_*) f(x_*)\, dx_*, \quad b \ge 0,$$

and we establish that if $g > 0$ is another solution

$$\partial_t g = \Delta g - a \cdot \nabla g + cg + \int b\, g_*$$

and if $\phi \ge 0$ is a solution to the dual evolution problem

$$-\partial_t \phi = \Delta \phi + \mathrm{div}(a\, \phi) + c\, \phi + \int b_*\, \phi_*, \quad \int b_*\, \phi_* := \int b(x_*, x)\, \phi(x_*)\, dx_*,$$

then we can exhibit a family of entropies on the form

$$\mathcal{H}(f) := \int_{\mathbb{R}^d} H(f/g)\, g\, \phi$$

for any convex function H.

Step 1. First order PDE. We assume that

$$\begin{aligned} \partial_t f &= -a \cdot \nabla f + cf \\ \partial_t g &= -a \cdot \nabla g + cg \\ -\partial_t \phi &= \mathrm{div}(a\, \phi) + c\, \phi, \end{aligned}$$

and we show that

$$\partial_t (H(X) g\phi) + \mathrm{div}(aH(X) g\phi) = 0, \quad X = f/g.$$

We compute

$$\begin{aligned} &\partial_t (H(X) g\phi) + \mathrm{div}(aH(X) g\phi) \\ &= H'(X) g\phi\, [\partial_t X + a \nabla X] + H(x)\, [\partial_t (g\phi) + \mathrm{div}(ag\phi)] \end{aligned}$$

The first term vanishes because

$$\partial_t X + a \nabla X = \frac{1}{g}\, (\partial_t f + a \nabla f) - \frac{f}{g^2}\, (\partial_t g + a \nabla g) = \frac{1}{g}\, (cf) - \frac{f}{g^2}\, (cg) = 0.$$

The second term also vanishes because

$$\partial_t (g\phi) + \mathrm{div}(ag\phi) = \phi\, [\partial_t g + a \nabla g] + g\, [\partial_t \phi + \mathrm{div}(a\phi)] = \phi\, [\, - cg] + g\, [\, + c\phi] = 0.$$

Step 2. Second order PDE. We assume that

$$\begin{aligned} \partial_t f &= \Delta f + cf \\ \partial_t g &= \Delta g + cg \\ -\partial_t \phi &= \Delta \phi + c\,\phi, \end{aligned}$$

and we show

$$\partial_t(H(X)g\phi) - \operatorname{div}(\phi\nabla(H(X)g)) + \operatorname{div}(gH(X)\nabla\phi) = -H''(X)g\phi|\nabla X|^2.$$

We first observe that

$$\begin{aligned} \Delta X &= \operatorname{div}\Big(\frac{\nabla f}{g} - f\,\frac{1}{g^2}\,\nabla g\Big) \\ &= \frac{\Delta f}{g} - 2\nabla f\,\frac{\nabla g}{g^2} + 2\,f\,\frac{|\nabla g|^2}{g^3} - \frac{f}{g^2}\,\Delta g \\ &= \frac{\Delta f}{g} - \frac{f\,\Delta g}{g^2} - 2\,\frac{\nabla g}{g}\cdot\nabla X, \end{aligned}$$

which in turn implies

$$\partial_t X - \Delta X = 2\,\frac{\nabla g}{g}\cdot\nabla X.$$

We then compute

$$\begin{aligned} &\partial_t(H(X)g\phi) - \operatorname{div}(\phi\nabla(H(X)g)) + \operatorname{div}(gH(X)\nabla\phi) = \\ &= (\partial_t H(X))\,g\phi + H(X)\,\partial_t(g\phi) - \phi\operatorname{div}[gH'(X)\nabla X + H(X)\nabla g] + gH(X)\Delta\phi \\ &= H'(X)g\phi\left\{\partial_t X - \Delta X - 2\frac{\nabla g}{g}\cdot\nabla X\right\} - g\phi\,H''(X)\,|\nabla X|^2 + H(X)\,[\partial_t(g\phi) - \phi\Delta g + g\Delta\phi] \\ &= -g\phi\,H''(X)\,|\nabla X|^2, \end{aligned}$$

since the first term and the last term independently vanish.

Step 3. Integral equation. We assume that

$$\begin{aligned} \partial_t f &= cf + \int bf_* \\ \partial_t g &= cg + \int bg_* \\ -\partial_t \phi &= c\,\phi + \int b_*\phi_*, \end{aligned}$$

with the notations

$$\int b\psi_* := \int b(x,x_*)\,\psi(x_*)\,dx_*, \qquad \int b_*\psi_* := \int b(x_*,x)\,\psi(x_*)\,dx_*,$$

and we show

$$\partial_t(H(X)g\phi) + \int H(X)gb_*\phi_* - \int bH(X_*)g_*\phi = -\int bg_*\phi\Big\{H(X_*) - H(X) - H'(X)(X*-X)\Big\}$$

We compute indeed

$$\begin{aligned} \partial_t(g\phi H(X)) &= H(X)g\partial_t\phi + H(X)\phi\partial_t g + H'(X)\phi(\partial_t f - X\partial_t g) \\ &= -\int H(X)gb_*\phi_* + \int bH(X_*)g_*\phi \\ &\quad + \int bg_*\phi\Big\{ - H(X_*) + H(X) + H'(X)X_* - H'(X)X\Big\} \end{aligned}$$

Step 4. Conclusion. For any solutions (f, g, ϕ) to the system of (full) equations, we have summing up the three computations

$$\begin{aligned}
&\partial_t(g\phi H(X)) + \\
&+\mathrm{div}(aH(X)g\phi) - \mathrm{div}(\phi\nabla(H(X)g)) + \mathrm{div}(gH(X)\nabla\phi) + \int bH(X_*)g_*\phi - \int H(X)gb_*\phi_* \\
&= -g\phi\, H''(X)\, |\nabla X|^2 - \int bg_*\phi \Big\{ H(X_*) - H(X) - H'(X)(X * -X) \Big\}.
\end{aligned}$$

Since when we integrate in the x variable the term on the second line vanishes, we find out

$$\frac{d}{dt}\mathcal{H}(f) = -D_{\mathcal{H}}(f)$$

with

$$D_{\mathcal{H}}(f) := \int g\phi\, H''(X)\, |\nabla X|^2 + \int\int bg_*\phi \Big\{ H(X_*) - H(X) - H'(X)(X * -X) \Big\} \geq 0.$$