Mathematical basis for evolution PDEs
La Habana CIMPA school - 2013
Lesson 3 - More about the heat equation
(Poincaré and Log-Sobolev inequalities and applications)

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In this chapter we present some qualitative properties of the heat equation and more particularly we present several results on the self-similar behavior of the solutions in large time. These results are deduced from several functional inequalities, among them the Nash inequality, the Poincaré inequality and the Log-Sobolev inequality.
Let us emphasize that the used methods lie on an interplay between evolution PDEs and functional inequalities and, although we only deal with (simpler) linear situations, these methods are robust enough to be generalized to (some) nonlinear situations.

1 The heat equation (June 28th, July 1st and 2nd, 2013)

1.1 Nash inequality and heat equation

We consider the heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\,\Delta f \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d. \tag{1.1}$$

One can classically prove thanks to the representation formula

$$f(t,.) = \gamma_t * f_0, \quad \gamma_t(x) := \frac{1}{(2\pi t)^{d/2}}\,\exp\Big(-\frac{|x|^2}{2t}\Big)$$

and the Hölder inequality that $f(t,.) \to 0$ as $t\to\infty$, and more precisely, that for any $p \in (1,\infty]$ and a constant $C_{p,d}$ the following rate of decay holds :

$$\|f(t,.)\|_{L^p} \le \frac{C_{p,d}}{t^{\frac{d}{2}(1-\frac{1}{p})}}\,\|f_0\|_{L^1} \qquad \forall\, t>0. \tag{1.2}$$

We aim to give a second proof of (1.2) in the case $p=2$ which is not based on the above representation formula, which is clearly longer and more complicated, but which is also more robust in the sense that it applies to more general equations, even sometimes nonlinear.

Nash inequality. There exists a constant C_d such that for any $f \in L^1(\mathbb{R}^d)\cap H^1(\mathbb{R}^d)$, there holds

$$\|f\|_{L^2}^{1+2/d} \le C_d\,\|f\|_{L^1}\,\|\nabla f\|_{L^2}^{2/d}.$$

Proof of Nash inequality. We write for any $R>0$

$$\begin{aligned} \|f\|^2_{L^2} &= \|\hat f\|^2_{L^2} = \int_{|\xi|\le R} |\hat f|^2 + \int_{|\xi|\ge R} |\hat f|^2 \\ &\le c_d\,R^d\,\|\hat f\|_{L^\infty} + \frac{1}{R^2}\int_{|\xi|\ge R} |\xi|^2\,|\hat f|^2 \\ &\le c_d\,R^d\,\|f\|_{L^1} + \frac{1}{R^2}\,\|\nabla f\|_{L^2}, \end{aligned}$$

and we take the optimal choice for R by setting $R := (\|\nabla f\|^2_{L^2}/\|f\|^2_{L^1})^{\frac{1}{d+2}}$. $\square$

We assume for the sake of simplicity that $f_0 \ge 0$, and then $f(t,.) \ge 0$ thanks to the maximum principle. We then compute

$$\frac{d}{dt}\|f(t,.)\|_{L^1} = \frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)\,dx = \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{div}_x\Big(\nabla_x f(t,x)\Big)\,dx = 0,$$

so that the mass is conserved (by the flow of the heat equation)

$$\|f(t,.)\|_{L^1} = \|f_0\|_{L^1} \quad \forall\, t \ge 0.$$

On the other hand, there holds

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)^2\,dx = \int_{\mathbb{R}^d} f\Delta f\,dx = -\int_{\mathbb{R}^d} |\nabla f|^2\,dx.$$

Putting together that last equation, the Nash inequality and the mass conservation, we obtain the following ordinary differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)^2\,dx \le -K\Big(\int_{\mathbb{R}^d} f(t,x)^2\,dx\Big)^{\frac{d+2}{d}}, \quad K = C_d\,\|f_0\|_{L^1}^{-4/d}.$$

We last observe that for any solution u of the ordinary differential inequality

$$u' \le -K\,u^{1+\alpha}, \quad \alpha = 2/d > 0,$$

some elementary computations lead to the inequality

$$u^{-\alpha}(t) \ge \alpha\,K\,t + u_0^{\alpha} \ge \alpha\,K\,t,$$

from which we conclude that

$$\int_{\mathbb{R}^d} f^2(t,x)\,dx \le C\,\frac{\Big(\|f_0\|^{4/d}\Big)^{d/2}}{t^{d/2}}.$$

That is nothing but the announced estimate.

1.2 Self-similar solutions and the Fokker-Planck equation

It is in fact possible to describe in a more accurate way that the mere estimate (1.2) how the heat equation solution $f(t,.)$ converges to 0 as time goes on. In order to do so, the first step consists in looking for particular solutions to the heat equation that we will discover by identifying some good change of scaling. We thus look for a self-similar solution to (1.2), namely we look for a solution F with particular form

$$F(t,v) = t^{\alpha}\,G(t^{\beta}\,x),$$

for some $\alpha, \beta \in \mathbb{R}$ and a *"self-similar profile"* G. As F must be mass conserving, we have

$$\int_{\mathbb{R}^d} F(t,x)\,dx = \int_{\mathbb{R}^d} F(0,x)\,dx = t^{\alpha}\int_{\mathbb{R}^d} G(t^{\beta}\,x)\,dx,$$

and we get from that the first equation $\alpha = \beta\,d$. On the other hand, we easily compute

$$\partial_t F = \alpha\,t^{\alpha-1}G(t^{\beta}\,x) + \beta\,t^{\alpha-1}\,(t^{\beta}\,v)(\nabla G)(t^{\beta}\,x), \quad \Delta F = t^{\alpha}\,t^{2\beta}\,(\Delta G)(t^{\beta}\,x).$$

In order that (1.1) is satisfied, we have to take $2\,\beta + 1 = 0$. We conclude with

$$F(t,x) = t^{-d/2}\,G(t^{-1/2}\,x), \qquad \frac{1}{2}\Delta G + \frac{1}{2}\mathrm{div}(x\,G) = 0. \tag{1.3}$$

We observe (and that is not a surprise!) that a solution $G \in L^1(\mathbb{R}^d) \cap \mathbf{P}(\mathbb{R}^d)$ to (1.3) will satisfy $\nabla G + x\, G = 0$, it is thus unique and given by

$$G(x) := c_0\, e^{-|x|^2/2}, \quad c_0^{-1} = (2\,\pi)^{d/2} \quad \text{(normalized Gaussian function)}.$$

To sum up, we have proved that F is our favorite solution to the heat equation : that is the fundamental solution to the heat equation.

Changing of point view, we may now consider G as a stationary solution to the harmonic Fokker-Planck equation (sometimes also called the Ornstein-Uhlenbeck equation)

$$\frac{\partial}{\partial t} g = \frac{1}{2}\, L\, g = \frac{1}{2}\, \nabla \cdot (\nabla g + g\, x) \qquad \text{in } (0,\infty) \times \mathbb{R}^d. \tag{1.4}$$

Le link between the heat equation (1.1) and the Fokker-Planck equation (1.4) is as follows. If f is a solution to the Fokker-Planck equation(1.4), some elementary computations permit to show that

$$f(t,x) = (1+t)^{-d/2}\, g(\log(1+t), (1+t)^{-1/2}\, x)$$

is a solution to the heat equation (1.1), with $f(0,x) = g(0,x)$. Reciprocally, if f is a solution to the heat equation (1.1) then

$$g(t,x) := e^{d\,t/2}\, f(e^t - 1, e^{t/2}\, x)$$

solves the Fokker-Planck equation (1.4). The last expression also gives the existence of a solution *in the sense of distributions* to the Fokker-Planck equation (1.4) for any initial datum $f_0 = \varphi \in L^1(\mathbb{R}^d)$ as soon as we know the existence of a solution to the heat equation for the same initial datum (what we get thanks to the usual representation formula for instance).

2 Fokker-Planck equation and Poincaré inequality

2.1 Long time asymptotic behaviour of the solutions to the Fokker-Planck equation

We consider the Fokker-Planck equation

$$\frac{\partial}{\partial t} f = L\, f = \Delta f + \nabla \cdot (f\, \nabla V) \quad \text{in } (0,\infty) \times \mathbb{R}^d \tag{2.1}$$

$$f(0,x) = f_0(x) = \varphi(x), \tag{2.2}$$

and we assume that the *"confinement potential"* V is the harmonic potential

$$V(x) := \frac{|x|^2}{2} + V_0, \quad V_0 := \frac{d}{2}\, \log 2\pi.$$

We start observing that

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t,x)\, dx = \frac{d}{dt} \int_{\mathbb{R}^d} \nabla_x \cdot (\nabla_x f + f\, \nabla_x V)\, dx = 0,$$

so that the mass (of the solution) is conserved. Moreover, the function $G = e^{-V} \in L^1(\mathbb{R}^d) \cap \mathbf{P}(\mathbb{R}^d)$ is nothing but the normalized Gaussian function, and since $\nabla G = -G\, \nabla V$ it is a stationary solution to the Fokker-Planck equation (2.1).

Theorem 2.1 *Let us fix $\varphi \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$.*
(1) There exists a unique global solution $f \in C([0,\infty); L^p(\mathbb{R}^d))$ to the Fokker-Planck equation (2.1)*. This solution is mass conservative*

$$\langle f(t,.)\rangle := \int_{\mathbb{R}^d} f(t,x)\, dx = \int_{\mathbb{R}^d} f_0(x)\, dx =: \langle f_0 \rangle, \quad \textit{if } f_0 \in L^1(\mathbb{R}^d), \tag{2.3}$$

and the following maximum principle holds

$$f_0 \geq 0 \qquad \Rightarrow \qquad f(t,.) \geq 0 \quad \forall\, t \geq 0.$$

(2) Asymptotically in large time the solution converges to the unique stationary solution with same mass, namely

$$\|f(t,.) - \langle f_0 \rangle\, G\|_E \leq e^{-\lambda_P t}\, \|f_0 - \langle f_0 \rangle\, G\|_E \quad \textit{as} \quad t \to \infty, \tag{2.4}$$

where $\|\cdot\|_E$ *stands for the norm of the Hilbert space* $E := L^2(G^{-1/2})$ *defined by*

$$\|f\|_E^2 := \int_{\mathbb{R}^d} f^2\, G^{-1}\, dx$$

and λ_P *is the best (larger) constant in the Poincaré inequality.*

More generally, we will denote by $L^p(F)$ the Lebesgue space associated to the norm $\|f\|_{L^p(F)} := \|f\, F\|_{L^p}$ and we will just write $L^p_k := L^p(\langle x \rangle^k)$.

For the proof of point (1) we refer to the preceding chapters as well as the final remark of section 1. We are going to give the main lines of the proof of point 2. Because the equation is linear, we may assume in the sequel that $\langle f_0 \rangle = 0$.

We start writing the Fokker-Planck equation in the equivalent form

$$\begin{aligned} \frac{\partial}{\partial t} f &= \mathrm{div}_x\big(\nabla_x f + G\, f\, \nabla_x G^{-1}\big) \\ &= \mathrm{div}_x\big(G \nabla_x (f\, G^{-1})\big). \end{aligned}$$

We then compute

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f^2\, G^{-1} &= \int_{\mathbb{R}^d} (\partial_t f)\, f\, G^{-1}\, dx = \int_{\mathbb{R}^d} \mathrm{div}_x\Big(G \nabla_x\big(\frac{f}{G}\big)\Big)\, \frac{f}{G}\, dx \\ &= -\int_{\mathbb{R}^d} G\, \Big|\nabla_x \frac{f}{G}\Big|^2\, dx. \end{aligned}$$

Using the Poincaré inequality established in the next Theorem 2.2 with the choice of function $g := f(t,.)/F$ and observing that $\langle g \rangle_G = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int f^2\, G^{-1} \leq -\lambda_P \int_{\mathbb{R}^d} G\, \Big(\frac{f}{G}\Big)^2\, dx = -\lambda_P \int_{\mathbb{R}^d} f^2\, G^{-1}\, dx,$$

and we conclude using the Gronwall lemma.

Theorem 2.2 (Poincaré inequality) *There exists a constant* $\lambda_P > 0$ *(which only depends on the dimension) such that for any* $g \in L^2(F^{1/2})$*, there holds*

$$\int_{\mathbb{R}^d} |\nabla g|^2\, G\, dx \geq \lambda_P \int_{\mathbb{R}^d} |g - \langle g \rangle_G|^2\, G\, dx, \tag{2.5}$$

where we have defined

$$\langle g \rangle_\mu := \int_{\mathbb{R}^d} g(x)\, \mu(dx)$$

for any given probability measure $\mu \in \mathbf{P}(\mathbb{R}^d)$ *and any function* $g \in L^1(\mu)$*.*

2.2 Proof of the Poincaré inequality

We split the proof into three steps.

2.2.1 Poincaré-Wirtinger inequality (in a open and bounded set Ω)

Lemma 2.3 *Let us denote $\Omega = B_R$ the ball of $\mathbb{R}^d$ with center 0 and radius $R > 0$, and let us consider $\nu \in \mathbf{P}(\Omega)$ a probability measure such that (abusing notations) $\nu, 1/\nu \in L^\infty(\Omega)$. There exists a constant $\kappa \in (0,\infty)$, such that for any (smooth) function f, there holds*

$$\kappa \int_\Omega |f - \langle f\rangle_\nu|^2 \, \nu \le \int_\Omega |\nabla f|^2 \, \nu, \qquad \langle f\rangle_\nu := \int_\Omega f \, \nu,$$

and therefore

$$\int_\Omega f^2 \, \nu \le \langle f\rangle_\nu^2 + \frac{1}{\kappa} \int_\Omega |\nabla f|^2 \, \nu.$$

Proof of Lemma 2.3. We start with

$$f(x) - f(y) = \int_0^1 \nabla f(z_t) \cdot (x-y) \, dt, \quad z_t = (1-t)\, x + t\, y.$$

Multiplying that identity by $\nu(y)$ and integrating in the variable $y \in \Omega$ the resulting equation, we get

$$f(x) - \langle f\rangle_\nu = \int_\Omega \int_0^1 \nabla f(z_t) \cdot (x-y) \, dt \, \nu(y) \, dy.$$

Using the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}
\int_\Omega (f(x) - \langle f\rangle_\nu)^2 \, \nu(x) \, dx &\le \int_\Omega \int_\Omega \int_0^1 |\nabla f(z_t)|^2 \, |x-y|^2 \, dt \, \nu(y) \, \nu(x) dy dx \\
&\le C_1 \int_\Omega \int_\Omega \int_0^{1/2} |\nabla f(z_t)|^2 \, dt dx \, \nu(y) dy + C_1 \int_\Omega \int_\Omega \int_{1/2}^1 |\nabla f(z_t)|^2 \, dt dy \, \nu(x) dx \\
&= C_1 \int_\Omega \int_0^{1/2} \int_{\Omega(t,y)} |\nabla f(z)|^2 \, \frac{dz}{1-t} \, dt \, \nu(y) dy + C_1 \int_\Omega \int_{1/2}^1 \int_{\Omega'(t,x)} |\nabla f(z)|^2 \, \frac{dz}{t} \, dt \, \nu(x) dx \\
&\le 2 C_1 \int_\Omega |\nabla f(z)|^2 \, dz,
\end{aligned}$$

with $C_1 := \|\nu\|_{L^\infty} \, \mathrm{diam}(\Omega)^2$. We immediately deduce the Poincaré-Wirtinger inequality with the constant $\kappa^{-1} := 2\, C_1 \, \|1/\nu\|_{L^\infty}$. ◻

2.2.2 A Liapunov function

There exists a function W such that $W \ge 1$ and there exist some constants $\theta > 0$, $b, R \ge 0$ such that

$$(L^* W)(x) := \Delta W(x) - \nabla V \cdot \nabla W(x) \le -\theta \, W(x) + b \, \mathbf{1}_{B(0,R)}(x) \qquad \forall \, x \in \mathbb{R}^d. \tag{2.6}$$

The proof is elementary. We look for W as $W(x) := e^{\gamma \langle x\rangle}$. We then compute

$$\nabla W = \gamma \, \frac{x}{\langle x\rangle} \, e^{\gamma\langle x\rangle} \quad \text{and} \quad \Delta W = (\gamma^2 + \gamma \, \frac{d-1}{\langle x\rangle}) \, e^{\gamma\langle x\rangle},$$

and then

$$\begin{aligned}
L^* W = \Delta W - x \cdot \nabla W \quad &= \quad \gamma \, \frac{d-1}{\langle x\rangle} \, W + (\gamma^2 - \gamma \, \frac{|x|^2}{\langle x\rangle}) \, W \\
&\le \quad -\theta W + b \, \mathbf{1}_{B_R}
\end{aligned}$$

with the choice $\theta = \gamma = 1$ and then R and b large enough. ◻

Exercise 2.4 *Establish* (2.6) *in the following situations :*
(i) $V(x) := \langle x\rangle^\alpha$ *with* $\alpha \ge 1$ *;*

(ii) there exist $\alpha > 0$ and $R \geq 0$ such that

$$x \cdot \nabla V(x) \geq \alpha \qquad \forall\, x \notin B_R;$$

(iii) there exist $a \in (0,1)$, $c > 0$ and $R \geq 0$ such that

$$a\,|\nabla V(x)|^2 - \Delta V(x) \geq c \qquad \forall\, x \notin B_R;$$

(iv) V is convex (or it is a compact supported perturbation of a convex function) and satisfies $e^{-V} \in L^1(\mathbb{R}^d)$.

2.2.3 End of the proof of the Poincaré inequality

We write (2.6) as

$$1 \leq -\frac{L^* W(x)}{\theta\, W(x)} + \frac{b}{\theta\, W(x)}\, \mathbf{1}_{B(0,R)}(x) \qquad \forall x \in \mathbb{R}^d.$$

For any $g \in \mathcal{D}(\mathbb{R}^d)$, we deduce

$$\int g^2\, G \quad \leq \quad -\int g^2\, \frac{L^* W(x)}{\theta\, W(x)}\, G + \frac{b}{\theta} \int_{B(0,R)} g^2\, \frac{1}{W}\, G \quad =: \; T_1 + T_2.$$

On the one hand, we have

$$\begin{aligned}
\theta\, T_1 \quad &= \quad \int \nabla W \cdot \Big\{ \nabla \left(\frac{g^2}{W} \right) G + \frac{g^2}{W}\, \nabla G \Big\} + \int \frac{g^2}{W}\, \nabla V \cdot \nabla W\, G \\
&= \quad \int \nabla W \cdot \nabla \left(\frac{g^2}{W} \right) G \\
&= \quad \int 2\, \frac{g}{W}\, \nabla W \cdot \nabla g\, G - \int \frac{g^2}{W^2}\, |\nabla W|^2\, G \\
&= \quad \int |\nabla g|^2\, G - \int \Big| \frac{g}{W}\, \nabla W - \nabla g \Big|^2\, G \\
&\leq \quad \int |\nabla g|^2\, G.
\end{aligned}$$

On the other hand, using the Poincaré-Wirtinger inequality in $B(0,R)$, we have

$$\begin{aligned}
\frac{\theta}{b}\, T_2 \quad &= \quad \int_{B(0,R)} g^2\, \frac{1}{W}\, G \\
&\leq \quad F(B(0,R)) \int_{B(0,R)} g^2\, \nu_R, \qquad \nu_R := G(B(0,R))^{-1}\, G|_{B(0,R)} \\
&\leq \quad G(B(0,R)) \Big(\langle g \rangle_R^2 + C_R \int_{B(0,R)} |\nabla g|^2\, \nu_R \Big), \qquad \langle g \rangle_R = \int_{B(0,R)} g\, \nu_R.
\end{aligned}$$

Gathering the two above estimates, we have shown

$$\int g^2\, G \leq C \Big(\langle g \rangle_R^2 + \int_{\mathbb{R}^d} |\nabla g|^2\, G \Big). \tag{2.7}$$

Consider now $h \in C^2 \cap L^\infty$. We know that for any $c \in \mathbb{R}$, there holds

$$\int_{\mathbb{R}^d} (h - \langle h \rangle_G)^2 G \leq \phi(c) := \int_{\mathbb{R}^d} (h - c)^2\, G \tag{2.8}$$

because ϕ is a polynomial function of second degree which reaches is minimum value in $c_h := \langle h \rangle_G$. We last define $g := h - \langle h \rangle_R$, so that $\langle g \rangle_R = 0$, $\nabla g = \nabla h$. Using first (2.8) and next (2.7), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}(h-\langle h\rangle_G)^2\,G &\le \int_{\mathbb{R}^d} g^2\,G\\ &\le C\Big(\langle g\rangle_R^2+\int_{\mathbb{R}^d}|\nabla g|^2\,G\Big)\\ &= C\int_{\mathbb{R}^d}|\nabla h|^2\,G.\end{aligned}$$

That ends the proof of the Poincaré inequality (2.5). $\square$

Exercise 2.5 *Generalize the above Poincaré inequality to a general superlinear potential $V(x) = \langle x\rangle^\alpha/\alpha + V_0$, $\alpha \ge 1$, in the following strong (weighted) formulation*

$$\int |\nabla g|^2\,\mathcal{G} \ge \kappa \int |g-\langle g\rangle_{\mathcal{G}}|^2\,(1+|\nabla V|^2)\,\mathcal{G} \qquad \forall\, g \in \mathcal{D}(\mathbb{R}^d),$$

where we have defined $\mathcal{G} := e^{-V} \in \mathbf{P}(\mathbb{R}^d)$ (for an appropriate choice of $V_0 \in \mathbb{R}$).

3 Fokker-Planck equation and Log Sobolev inequality.

The estimate (2.4) gives a satisfactory (optimal) answer to the convergence to the equilibrium issue for the Fokker-Planck equation (2.1). However, we may formulate two criticisms. The proof is "completely linear" (in the sense that it can not be generalized to a nonlinear equation) and the considered initial data are very confined/localized (in the sense that they belong to the strong weighted space E, and again that it is not always compatible with the well posedness theory for nonlinear equations).

We present now a series of results which apply to more general initial data but, above all, which can be adapted to nonlinear equations. On the way, we will establish several functional inequalities of their own interest, among them the famous Log-Sobolev (or logarithmic Sobolev) inequality.

3.1 Fisher information.

We are still interested in the harmonic Fokker-Planck equation (2.1)-(2.2). We define

$$D := \{f \in L^1(\mathbb{R}^d);\quad f \ge 0,\quad \int f = 1,\quad \int f\,x = 0,\quad \int f\,|x|^2 = d\}$$

and

$$D_\le := \{f \in L^1(\mathbb{R}^d);\quad f \ge 0,\quad \int f = 1,\quad \int f\,x = 0,\quad \int f\,|x|^2 \le d\}.$$

We observe that D (and $D_\le$) are invariant set for the flow of Fokker-Planck equation (2.1). We also observe that G is the unique stationary solution which belongs to D. Indeed, the equations for the first moments are

$$\partial_t\langle f\rangle = 0,\quad \partial_t\langle fx\rangle = -d\,\langle fx\rangle,\quad \partial_t\langle f|x|^2\rangle = 2d\langle f\rangle - 2\langle f|x|^2\rangle.$$

It is therefore quite natural to think that any solution to the Fokker-Planck equation (2.1)-(2.2) with initial datum $\varphi \in D$ converges to G. It is what we will establish in the next paragraphs.

We define the Fisher information (or Linnik functional) $I(f)$ and the relative Fisher information by

$$I(f) = \int \frac{|\nabla f|^2}{f} = 4\int |\nabla\sqrt{f}|^2, \qquad I(f|G) = I(f) - I(G) = I(f) - d.$$

Lemma 3.1 *For any $f \in D_\le$, there holds*

$$I(f|G) \ge 0, \tag{3.1}$$

with equality if, and only if, $f = G$.

Proof of Lemma 3.1. We define $V := \{f \in D_{\le} \text{ and } \nabla\sqrt{f} \in L^2\}$. We start with the proof of (3.1). For any $f \in V$, we have

$$\begin{aligned} 0 \le J(f) &:= \int \left|2\nabla\sqrt{f} + x\,\sqrt{f}\right|^2 \,dx \\ &= \int \Big(4\,|\nabla\sqrt{f}|^2 + 2\,x\cdot\nabla f + |x|^2\,f\Big)\,dx = I(f) + \langle f|x|^2\rangle - 2\,d \\ &\le I(f) - d = I(f) - I(G) =: I(f|G). \end{aligned}$$

We consider now the case of equality. If $I(f|G) = 0$ then $J(f) = 0$ and $2\nabla\sqrt{f} + x\,\sqrt{f} = 0$ a.e.. By a bootstrap argument (Sobolev inequality, Morrey inequality, and then classical differential calculus) we deduce that $f \in C^\infty$. Consider $x_0 \in \mathbb{R}^d$ such that $f(x_0) > 0$ (which exists because $f \in V$) and then $\mathcal{O}$ the open and connected to x_0 component of the set $\{f > 0\}$. We deduce from the preceding identity that $\nabla(\log\sqrt{f} + |x|^2/4) = 0$ in $\mathcal{O}$ and then $f(x) = e^{C-|x|^2/2}$ on $\mathcal{O}$ for some constant $C \in \mathbb{R}$. By continuity of f, we deduce that $\mathcal{O} = \mathbb{R}^d$, and then $C = -\log(2\,\pi)^{d/2}$ (because of the normalized condition imposed by the fact that $f \in V$).

Lemma 3.2 *For any (smooth) function f, we have*

$$\frac{1}{2}I'(f)\cdot\Delta f = -\sum_{ij}\int\Big(\frac{1}{f^2}\,\partial_i f\,\partial_j f - \frac{1}{f}\partial_{ij}f\Big)^2 f, \tag{3.2}$$

$$\frac{1}{2}I'(f)\cdot(\nabla\cdot(f\,x)) = I(f), \tag{3.3}$$

$$\frac{1}{2}I'(f)\cdot L\,f = -\sum_{ij}\int\Big(\frac{1}{f^2}\,\partial_i f\,\partial_j f - \frac{1}{f}\partial_{ij}f + \delta_{ij}\Big)^2 f - \Big(I(f) - I(G)\Big). \tag{3.4}$$

As a consequence, there holds

$$\frac{1}{2}I'(f)\cdot L(f) \le -I(f|G) \le 0.$$

Proof of Lemma 3.2. Proof of (3.2). First, we have

$$I'(f)\cdot h = 2\int\frac{\nabla f}{f}\nabla h - \frac{|\nabla f|^2}{f^2}\,h.$$

Integrating by part with respect to the x_i variable, we get

$$\begin{aligned} \frac{1}{2}I'(f)\cdot\Delta f &= \int\frac{1}{f}\partial_j f\,\partial_{iij}f - \int\frac{1}{2\,f^2}\partial_{ii}f\,(\partial_j f)^2 \\ &= \int\frac{\partial_i f}{f^2}\partial_j f\,\partial_{ij}f - \frac{1}{f}\partial_{ij}f\,\partial_{ij}f + \int\frac{1}{f^2}\partial_i f\,\partial_j f\,\partial_{ij}f - \frac{\partial_i f}{f^3}\partial_i f\,(\partial_j f)^2 \\ &= -\sum_{ij}\int\Big(\frac{1}{f^2}\,\partial_i f\,\partial_j f - \frac{1}{f}\partial_{ij}f\Big)^2 f. \end{aligned}$$

Proof of (3.3). We write

$$\frac{1}{2}I'(f)\cdot(\nabla\cdot(f\,x)) = \int\frac{\partial_j f}{f}\partial_{ij}(f\,x_i) - \frac{(\partial_j f)^2}{2\,f^2}\,\partial_i(f\,x_i).$$

We observe that

$$\begin{aligned} \partial_{ij}(f\,x_i) - \frac{(\partial_j f)}{2\,f}\,\partial_i(f\,x_i) &= \partial_{ij}f\,x_i + d\,\partial_j f + \delta_{ij}\,\partial_j f - \partial_i f\,\partial_j f\,\frac{x_i}{2\,f} - \frac{d}{2}\partial_j f \\ &= \partial_{ij}f\,x_i + (\frac{d}{2}+1)\,\partial_j f - \partial_i f\,\partial_j f\,\frac{x_i}{2\,f}. \end{aligned}$$

Gathering the two preceding equalities, we obtain

$$\frac{1}{2}I'(f)\cdot(\nabla\cdot(f\,x)) = (\frac{d}{2}+1)\,I(f) + \int \frac{\partial_j f}{f}\,\partial_{ij} f\,x_i - \int \frac{\partial_j f}{f}\,\partial_i f\,\partial_j f\,\frac{x_i}{2\,f}.$$

Last, we remark that

$$-\frac{d}{2}\,I(f) = \frac{1}{2}\int \partial_i\left(\frac{(\partial_j f)^2}{f}\right)\,x_i = \int \frac{\partial_j f\,\partial_{ij} f}{f}\,x_i - \frac{1}{2}\frac{(\partial_j f)^2}{f^2}\,\partial_i f\,x_i,$$

and we then conclude

$$\frac{1}{2}I'(f)\cdot(\nabla\cdot(f\,x)) = I(f).$$

Proof of (3.4). Developing the expression below and using (3.2), we have

$$\begin{aligned} 0 \quad &\le \quad \sum_{ij}\int\Big(\frac{1}{f^2}\,\partial_i f\,\partial_j f - \frac{1}{f}\partial_{ij} f + \delta_{ij}\Big)^2 f \\ &= \quad -\frac{1}{2}I'(f)\cdot\Delta f - 2\sum_i\int\Big(\partial_{ii} f - \frac{1}{f}\,(\partial_i f)^2\Big) + d\int f. \end{aligned}$$

From $\int f = 1$, $\int \partial_{ii} f = 0$ and (3.3), we then deduce

$$0 \quad \le \quad -\frac{1}{2}I'(f)\cdot\Delta f - 2I(f) + d = -\frac{1}{2}I'(f)\cdot Lf + d - I(f),$$

which ends the proof of (3.4). □

Theorem 3.3 *The Fisher information I is decreasing along the flow of the Fokker-Planck equation, i.e. I is a Liapunov functional, and more precisely*

$$I(f(t,.)|G) \le e^{-2\,t}\,I(\varphi|G). \tag{3.5}$$

That implies the convergence in large time to G of any solution to the Fokker-Planck equation associated to any initial condition $\varphi \in D\cap V$. More precisely,

$$\forall\,\varphi\in D\cap V \qquad f(t,.)\to G \quad \textit{in} \quad L^q\cap L^1_2 \quad \textit{as} \quad t\to\infty, \tag{3.6}$$

for any $q\in[1,2^*/2)$.

Proof of Theorem 3.3. On the one hand, thanks to (3.4), we have

$$\frac{d}{dt}I(f|G) \le -2\,I(f|G), \tag{3.7}$$

and we conclude to (3.5) thanks to the Gronwall lemma. On the other hand, thanks to the Sobolev inequality, we have

$$\|f\|_{L^{2^*/2}} = \|\sqrt{f}\|^2_{L^{2^*}} \le C\,\|\nabla\sqrt{f}\|_{L^2} = C\,I(f)^2 \le C\,I(\varphi)^2.$$

Consider now an increasing sequence (t_n) which converges to $+\infty$. Thanks to estimate (3.5) and the Rellich Theorem, we may extract a subsequence $\sqrt{f(t_{n_k})}$ which converges a.e. and strongly in $L^{2\,q}$ and weakly in $\dot H^1$ to a limit denoted by $\sqrt{g}$. That implies that $f(t_{n_k})$ converges to g strongly in $L^q\cap L^1_k$ for any $q\in[1,2^*/2)$, $k\in[0,2)$, and that $I(g)\le\limsup I(f(t_{n_k}))<\infty$, so that $g\in V$. Finally, since $2\nabla\sqrt{f(t_{n_k})} - x\sqrt{f(t_{n_k})} \to 2\nabla\sqrt{g} - x\sqrt{g}$ weakly in L^2_{loc} (for instance) we have

$$0 \le J(g) \le \liminf_{k\to\infty} J(f(t_{n_k},.) = \liminf_{k\to\infty} I(f(t_{n_k},.)|G) = 0.$$

From $J(g)=0$ and $g\in V\cap D_\le$ we get $g=G$ as a consequence of Lemma 3.1, and it is then the all family $(f(t))_{t\ge0}$ which converges to G as $t\to\infty$. The L^1_2 convergence is a consequence of the fact that the sequence $(f(t_n)\,|v|^2)_n$ is tight because $\langle f(t)\,|v|^2\rangle = \langle G\,|v|^2\rangle$ for any time $t\ge0$. □

Exercise 3.4 *Prove that* $0 \le f_n \to f$ *in* $L^q \cap L^1_k$, $q > 1$, $k > 0$, *implies that* $H(f_n) \to H(f)$. *(Hint. Use the splitting*

$$s\,|\log s| \le \sqrt{s}\,\mathbf{1}_{0 \le s \le e^{-|x|^k}} + s\,|x|^k\,\mathbf{1}_{e^{-|x|^k} \le s \le 1} + s(\log s)_+\,\mathbf{1}_{s \ge 1} \quad \forall\, s \ge 0$$

and the dominated convergence theorem).

Exercise 3.5 *Prove the convergence* (3.5) *for any* $\varphi \in \mathbf{P}(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ *such that* $I(\varphi) < \infty$. *(Hint. Compute the equations for the moments of order* 1 *and* 2*).*

3.2 Entropy and Log-Sobolev inequality.

For a function $f \in \mathbf{P}(\mathbb{R}^d) \cap L^1_k(\mathbb{R}^d)$, $k > 0$, we define the entropy $H(f) \in \mathbb{R} \cup \{+\infty\}$ and the relative entropy $H(f|G) \in \mathbb{R} \cup \{+\infty\}$ by

$$H(f) = \int_{\mathbb{R}^d} f\,\log f\,dx, \quad H(f|G) = H(f) - H(G) = \int_{\mathbb{R}^d} j(f/G)\,G\,dx,$$

where $j(s) = s\,\log s - s + 1$.
We start observing that for $f \in \mathbf{P}(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d)$, there holds

$$\begin{aligned} H'(f) \cdot L(f) &:= \int_{\mathbb{R}^d} (1 + \log f)\,[\Delta f + \nabla(x\,f)] \\ &= -\int_{\mathbb{R}^d} \nabla f \cdot \nabla \log f - \int_{\mathbb{R}^d} x\,f \cdot \nabla \log f \\ &= -I(f) + d\,\langle f \rangle = -I(f|G). \end{aligned}$$

As a consequence, the entropy is a Liapunov functional for the Fokker-Planck equation and more precisely

$$\frac{d}{dt} H(f) = -I(f|G) \le 0. \tag{3.8}$$

Theorem 3.6 (Logarithmic Sobolev inequality). *For any* $\varphi \in D$, $\sqrt{\varphi} \in \dot{H}^1$, *the following Log-Sobolev inequality holds*

$$H(\varphi|G) \le \frac{1}{2} I(\varphi|G). \tag{3.9}$$

That one also writes equivalently as

$$\int_{\mathbb{R}^d} f/G \ln(f/G)\,G\,dx = \int_{\mathbb{R}^d} f \ln f - \int_{\mathbb{R}^d} G \ln G \le \frac{1}{2}\Big(\int_{\mathbb{R}^d} \frac{\nabla f\,\nabla f}{f} - d\Big)$$

or also as

$$\int_{\mathbb{R}^d} u^2\,\log(u^2)\,G(dx) \le 2\int_{\mathbb{R}^d} |\nabla u|^2\,G(dx).$$

For some applications, it is worth noticing that the constant in the Log-Sobolev inequality does not depend on the dimension, what it is not true for the Poincaré inequality.

Proof of Theorem 3.6. On the one hand, from (3.6) (and more precisely the result of Exercise 3.4) and (3.8), we get

$$\begin{aligned} H(\varphi) - H(G) &= \lim_{T\to\infty} [H(\varphi) - H(f_T)] = \lim_{T\to\infty} \int_0^T \Big[-\frac{d}{dt} H(f)\Big]\,dt \\ &= \lim_{T\to\infty} \int_0^T [I(f|G)]\,dt. \end{aligned}$$

From that identity and (3.7), we deduce

$$\begin{aligned} H(\varphi) - H(G) &\le \lim_{T\to\infty} \int_0^T \Big[-\frac{1}{2}\frac{d}{dt} I(f|G)\Big]\, dt \\ &= \lim_{T\to\infty} \frac{1}{2}\,[I(\varphi|G) - I(f_T|G)] = \frac{1}{2} I(\varphi|G), \end{aligned}$$

thanks to (3.5). ◻

Lemma 3.7 (Csiszár-Kullback inequality). *Consider μ and ν two probability measures such that $\nu = g\,\mu$ for a given nonnegative measurable function g. Then*

$$\|\mu - \nu\|_{VT}^2 := \|g-1\|^2_{L^1(d\mu)} \le 2 \int g \log g \, d\mu. \tag{3.10}$$

Proof of Lemma 3.7. First proof. One easily checks (by differentiating three times both functions) that

$$\forall u \ge 0 \qquad 3\,(u-1)^2 \le (2\,u+4)\,(u\log u - u + 1).$$

Thanks to the Cauchy-Schwarz inequality one deduces

$$\int |g-1|\, d\mu \le \sqrt{\frac{1}{3}\int (2\,g+4)\, d\mu}\,\sqrt{\int (g\log g - g + 1)\, d\mu} = \sqrt{2\int g\log g\, d\mu}.$$

Second proof. Thanks to the Taylor-Laplace formula, there holds

$$\begin{aligned} h(g) &:= g\log g - g + 1 = h(1) + (g-1)\,h'(1) + (g-1)^2 \int_0^1 h''(1+s\,(g-1))\,(1-s)\, ds \\ &= (g-1)^2 \int_0^1 \frac{1-s}{1+s\,(g-1)}\, ds. \end{aligned}$$

Using Fubini theorem, we get

$$H(g) := \int (g\log g - g + 1)\, d\mu = \int_0^1 (1-s) \int \frac{(g-1)^2}{1+s\,(g-1)} d\mu\, ds.$$

For any $s \in [0,1]$, we use the Cauchy-Schwarz inequality and the fact that both ν and $g\,\nu$ are probability measures in order to deduce

$$\Big(\int |g-1|\, d\mu\Big)^2 \le \Big(\int \frac{(g-1)^2}{1+s\,(g-1)} d\mu\Big)\Big(\int [1+s\,(g-1)] d\mu\Big) = \int \frac{(g-1)^2}{1+s\,(g-1)} d\mu.$$

As a conclusion, we obtain

$$H(g) \ge \int_0^1 \Big(\int |g-1|\, d\mu\Big)^2 (1-s)\, ds = \frac{1}{2}\Big(\int |g-1|\, d\mu\Big)^2,$$

which ends the proof of the Csiszár-Kullback inequality. ◻

Putting together (3.8), (3.9) and (3.10), we immediately obtain the following convergence result.

Theorem 3.8 *For any $\varphi \in D$ such that $H(\varphi) < \infty$ the associated solution f to the Fokker-Planck equation* (2.1)-(2.2) *satisfies*

$$H(f|G) \le e^{-2t}\, H(\varphi|G),$$

and then

$$\|f - G\|_{L^1} \le \sqrt{2}\, e^{-t}\, H(\varphi|G)^{1/2}.$$

Exercise 3.9 *Generalize Theorem 3.6 and Theorem 3.8 to the case of a super-harmonic potential $V(x) = \langle x\rangle^\alpha/\alpha$, $\alpha \ge 2$, and to an initial datum $\varphi \in \mathbf{P}(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ such that $H(\varphi) < \infty$.*

3.3 From log-Sobolev to Poincaré.

Lemma 3.10 *If the log-Sobolev inequality*

$$\lambda\, H(f|G) \le I(f|G) \quad \forall\, f \in D$$

holds for some constant $\lambda > 0$, then the Poincaré inequality

$$(\lambda + d)\, \|h - G\|^2_{L^2(G^{-1/2})} \le \int |\nabla h|^2\, G^{-1} \quad \forall\, h \in \mathcal{D}(\mathbb{R}^d),\ \langle h[1, x, |x|^2]\,\rangle = 0,$$

also holds (for the same constant $\lambda > 0$).

That lemma gives an alternative proof of the Poincaré inequality. Of course that proof is not very "cheap" in the sense that one needs to prove first the log-Sobolev inequality which in somewhat more difficult to prove that the Poincaré inequality. Moreover, the log-Sobolev inequality is known to be true under more restrict assumption on the confinement potential than the Poincaré inequality. However, that allows to compare the constants involved in the two inequalities and the proof is robust enough so that it can be adapted to nonlinear situations.

Proof of Lemma 3.10. Consider $h \in \mathcal{D}(\mathbb{R}^d)$ such that $\int h(v)\,[1, v, |v|^2]\, dv = [0, 0, 0]$. Applying the Log-Sobolev inequality to the function $f = G + \varepsilon\, h \in D$ for $\varepsilon > 0$ small enough, we have

$$\lambda\, \frac{H(G + \varepsilon\, h) - H(G)}{\varepsilon^2} = \frac{\lambda}{\varepsilon^2}\, H(f|G) \le \frac{1}{2\,\varepsilon^2} I(f|G) = \frac{I(G + \varepsilon\, h) - I(G)}{2\,\varepsilon^2}.$$

Expending up to order 2 the two functionals, we have

$$f \log f = G \log G + \varepsilon\, h\,(1 + \log G) + \frac{\varepsilon^2}{2}\, \frac{h^2}{G} + \mathcal{O}(\varepsilon^3),$$

$$\frac{|\nabla f|^2}{f} = \frac{|\nabla G|^2}{G} + \varepsilon \Big\{ 2\, \frac{\nabla G}{G} \cdot \nabla h - \frac{|\nabla G|^2}{G^2}\, h \Big\} + \frac{\varepsilon^2}{2} \Big\{ \frac{|\nabla h|^2}{G} - 2\, h\, \frac{\nabla G}{G^2} \cdot \nabla h + \frac{|\nabla G|^2}{G^3}\, h^2 \Big\} + \mathcal{O}(\varepsilon^3).$$

Passing now to the limit $\varepsilon \to 0$ in the first inequality and using that the zero and first order terms vanish because (performing one integration by parts)

$$H'(G) \cdot h = \int_{\mathbb{R}^d} (\log G + 1)\, h = 0,$$

$$I'(G) \cdot h = \int_{\mathbb{R}^d} \Big\{ \frac{|\nabla G|^2}{G^2} - 2\, \frac{\Delta G}{G} \Big\}\, h = 0,$$

we get

$$\lambda\, H''(G) \cdot (h, h) \le I''(G) \cdot (h, h).$$

More explicitly, we have

$$\lambda \int \frac{h^2}{G} \le \int \Big\{ \frac{|\nabla h|^2}{G} + \nabla \Big(\frac{\nabla G}{G^2} \Big)\, h^2 + \frac{|\nabla G|^2}{G^3}\, h^2 \Big\},$$

and then

$$(\lambda + d) \int \frac{h^2}{G} = \int \frac{h^2}{G} \Big\{ \lambda - \frac{\Delta G}{G} + \frac{|\nabla G|^2}{G^2} \Big\} \le \int \frac{|\nabla h|^2}{G},$$

which is nothing but the Poincaré inequality. ◻

4 Weighted L^1 semigroup spectral gap

In that last section, we establish that as a consequence of the Poincaré inequality, the following weighted L^1 semigroup spectral gap estimate holds.

Theorem 4.11 *For any $a \in (-\lambda_P, 0)$ and for any $k > k^* := \lambda_P + d/2$ there exists $C_{k,a}$ such that for any $\varphi \in L^1_k$ the associated solution f to the Fokker-Planck equation* (2.1)-(2.2) *satisfies*

$$\|f - \langle\varphi\rangle\, G\|_{L^1_k} \le C_{k,a}\, e^{a\,t}\, \|\varphi - \langle\varphi\rangle\, G\|_{L^1_k}.$$

A refined version of the proof below shows that the same estimate holds with $a := -\lambda_P$ and for any $k > k^{**} := \lambda_P$.

Proof of Theorem 4.11. We introduce the splitting $L = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{B}f := \Delta f + \nabla\cdot(f\,x) - M\,f\,\chi_R, \quad \mathcal{A}f := M\,f\,\chi_R,$$

where $\chi_R(x) = \chi(x/R)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $0 \le \chi \le 1$, $\chi \equiv 1$ on B_1, and where $R, M > 0$ are two real constants to be chosen later. We splits the proof into several steps.

Step 1. The operator $\mathcal{A}$ is clearly bounded in any Lebesgue space and more precisely

$$\forall\, f \in L^p \quad \|\mathcal{A}\,f\|_{L^p(G^{1/p})} \le C_{p,R,M}\,\|f\|_{L^p}.$$

Step 2. For any $k, \varepsilon > 0$ and for any $M, R > 0$ large enough (which may depend on k and ε) the operator $\mathcal{B}$ is dissipative in L^1_k in the sense that

$$\forall\, f \in \mathcal{D}(\mathbb{R}^d) \quad \int_{\mathbb{R}^d} (\mathcal{B}f)\,(\mathrm{sign} f)\,\langle x\rangle^k \le (\varepsilon - k)\,\|f\|_{L^1_k}. \tag{4.11}$$

We set $\beta(s) = |s|$ (and more rigorously we must take a smooth version of that function) and $m = \langle x\rangle^k$, and we compute

$$\begin{aligned}
\int (L\,f)\,\beta'(f)\,m &= \int (\Delta f + d\,f + x\cdot\nabla f)\,\beta'(f)\,m \\
&= \int \{-\nabla f\,\nabla(\beta'(f)m) + d\,|f|\,m + m\,x\cdot\nabla|f|\} \\
&= -\int |\nabla f|^2\,\beta''(f)\,m + \int |f|\,\{\Delta m + d - \nabla(x\,m)\} \\
&\le \int |f|\,\{\Delta m - x\cdot\nabla m\},
\end{aligned}$$

where we have used that β is a convex function. Defining

$$\begin{aligned}
\psi &:= \Delta m - x\cdot\nabla m - M\chi_R\,m \\
&= (k^2\,|x|^2\,\langle x\rangle^{-4} - k\,|x|^2\,\langle x\rangle^{-2} - M\,\chi_R)\,m
\end{aligned}$$

we easily see that we can choose $M, R > 0$ large enough such that $\psi \le (\varepsilon - k)\,m$ and then (4.11) follows.

Step 3. Fix now $k > k^*$. For any $a \in (-\lambda_P, 0)$, there holds

$$\forall\, \varphi \in \mathcal{D}(\mathbb{R}^d) \quad \|e^{\mathcal{B}t}\varphi\|_{L^2_k} \le \frac{C_{a,k}}{t^{d/2}}\, e^{a\,t}\,\|\varphi\|_{L^1_k} \tag{4.12}$$

A similar computation as in step 2 shows

$$\begin{aligned}
\int (\mathcal{B}\,f)\,f\,m^2 &= -\int |\nabla(f\,m)|^2 + \int |f|^2\,\Big\{\frac{|\nabla m|^2}{m^2} + \frac{d}{2} - x\cdot\nabla m - M\,\chi_R\Big\} m^2 \\
&= -\int |\nabla(f\,m)|^2 + (\frac{d}{2} + \varepsilon - k)\int |f|^2\,m^2,
\end{aligned}$$

for $M, R > 0$ chosen large enough. Denoting by $f(t) = S_\mathcal{B}(t)\varphi = e^{\mathcal{B}t}\varphi$ the solution to the evolution PDE

$$\partial_t f = \mathcal{B} f, \qquad f(0) = \varphi,$$

we (formally) have

$$\frac{1}{2}\frac{d}{dt}\int f^2\, m^2 = \int (\mathcal{B}\, f)\, f\, m^2 \le -\int |\nabla (f\, m)|^2 + a \int |f|^2\, m^2,$$

from which (4.13) follows by using the Nash inequality similarly as in the proof of estimate (1.2) in section 1.1.

Step 4. For any $k > k^*$ and $a \in (-\lambda_P, 0)$, there holds

$$\|(\mathcal{A}S_\mathcal{B})^{(*n)}\varphi\|_{L^2(G^{-1/2})} \le C_{k,n,a}\, e^{at}\, \|\varphi\|_{L^1_k} \quad \forall\, t \ge 0, \tag{4.13}$$

for $n = d+1$ for instance. We just establish (4.13) when $d = 1$. We denote $\mathcal{E} := L^1_k$, $E := L^2(G^{-1/2})$. Observing that

$$\|\mathcal{A}S_\mathcal{B}(t)\|_{\mathcal{E}\to E} \le \frac{C_1}{t^{1/2}}\, e^{a' t} \quad \text{and} \quad \|\mathcal{A}S_\mathcal{B}(t)\|_{\mathcal{E}\to\mathcal{E}} \le C_2\, e^{a' t},$$

we compute

$$\begin{aligned}
\|(\mathcal{A}S_\mathcal{B})^{(*2)}\|_{\mathcal{E}\to E} &\le \int_0^t \|\mathcal{A}S_\mathcal{B}(t-s)\|_{\mathcal{E}\to E}\, \|\mathcal{A}S_\mathcal{B}(s)\|_{\mathcal{E}\to\mathcal{E}}\, ds \\
&\le e^{a' t} \int_0^t \frac{C_1}{(t-s)^{1/2}}\, C_2\, ds \\
&= e^{a' t}\, C_1\, C_2\, t^{1/2} \int_0^1 \frac{du}{u^{1/2}},
\end{aligned}$$

from which we immediately conclude by taking $a' \in (-\lambda_P, a)$.

Step 5. We define in both spaces E and $\mathcal{E}$ the projection operator

$$\Pi f := \langle f \rangle\, G.$$

We denote by $\mathcal{L}$ the differential Fokker-Planck operator in $\mathcal{E}$ and still by L the same operator in E. We also denote by $S_\mathcal{L}$ and S_L the associated semigroups. Since $G \in E \subset \mathcal{E}$ is a stationary solution to the Fokker-Planck equation and the mass is preserved by the associated flow, we have $S_L(I - \Pi) = (I - \Pi)\, S_L$ as well as

$$\|S_L(t)(I - \Pi)\|_{E\to E} = \|(I - \Pi)\, S_L(t)\|_{E\to E} \le e^{-\lambda_P t} \quad \forall\, t \ge 0, \tag{4.14}$$

which is nothing but (2.4). Now, we decompose the semigroup on invariant spaces

$$S_\mathcal{L} = \Pi\, S_\mathcal{L} + (I - \Pi)\, S_\mathcal{L}\, (I - \Pi)$$

and by iterating once the Duhamel formula

$$\begin{aligned}
S_\mathcal{L}(t) &= S_\mathcal{B}(t) + \int_0^t S_\mathcal{L}(t-s)\, \mathcal{A}S_\mathcal{B}(s)\, ds \\
&= S_\mathcal{B}(t) + S_\mathcal{L} * \mathcal{A}S_\mathcal{B}(t),
\end{aligned}$$

we have

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{B} * (\mathcal{A}S_\mathcal{B}) + \, S_\mathcal{L}\, (\mathcal{A}S_\mathcal{B})^{(*2)}.$$

These two identities together, we have

$$S_\mathcal{L} \;=\; \Pi\, S_\mathcal{L} + (I - \Pi)\, \{S_\mathcal{B} + S_\mathcal{B} * (\mathcal{A}S_\mathcal{B}) + \, S_\mathcal{L}\, (\mathcal{A}S_\mathcal{B})^{(*2)}\}\, (I - \Pi)$$

or in other words

$$S_\mathcal{L} - \Pi \;=\; (I - \Pi)\, \{S_\mathcal{B} + S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})\}(I - \Pi) + \{(I - \Pi)\, S_L\} * (\mathcal{A}S_\mathcal{B})^{(*2)}(I - \Pi).$$

We conclude by observing that the RHS in the above expression is $\mathcal{O}(e^{at})$ thanks to estimate (4.14) and thanks to steps 2 and 4 above. ◻