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An introduction to evolution PDEs

Chapter 6 - Elliptic-parabolic Keller-Segel equation (for chemotaxis and astrophysics)

1 A compactness argument

We consider a sequence of functions (f_n) such that

$$0 \le f_n \in C([0,\infty); \mathcal{D}'(\mathbb{R}^2))$$

and f_n is a solution to the KS equation

$$\partial_t f_n \quad = \quad \Delta f_n - \nabla (f_n \, \bar{\mathcal{K}}_n) \quad \text{in} \quad (0,\infty) \times \mathbb{R}^2, \tag{1.1}$$

where

$$\bar{\mathcal{K}}_n := \mathcal{K} * f_n, \quad \mathcal{K} := \frac{1}{2\pi} \frac{z}{|z|^2}.$$

We also assume that (f_n) satisfies (uniformly in n) the natural bounds

$$\sup_{[0,T]} \int_{\mathbb{R}^2} f_n \, (1 + |x|^2 + (\log f_n)_+) \, dx + \int_0^T \int_{\mathbb{R}^2} \frac{|\nabla f_n|^2}{f_n} \, dxdt \le C_T.$$

We recall that we have (up to the extraction of a subsequence)

$$f_n \rightharpoonup f \quad \text{weakly in} \quad L^1((0,T) \times \mathbb{R}^2)$$

as a consequence of the Dunford-Pettis lemma, and better

$$f_n \rightharpoonup f \quad \text{weakly in} \quad L^2((0,T) \times \mathbb{R}^2)$$

because thanks to the bound on the Fisher information, the Cauchy-Schwarz inequality and the Sobolev inequality there holds

$$\int_{\mathbb{R}^2} f_n^2 dx \le C \Big(\int_{\mathbb{R}^2} |\nabla f_n| dx \Big)^2 \le C \int_{\mathbb{R}^2} f_n dx \times \int_{\mathbb{R}^2} \frac{|\nabla f_n|^2}{f_n} \, dx,$$

so that (f_n) is bounded in $L^2((0,T) \times \mathbb{R}^2)$.

We aim to explain now why the following strong convergence result holds true, this one allows then to pass to the limit in the weak formulation of (1.1).

Lemma 1.1 *Under the above assumptions, there holds*

$$\bar{\mathcal{K}}_n \to \bar{\mathcal{K}} := \mathcal{K} * f \quad \textit{strongly in} \quad L^2((0,T) \times B_R) \ \forall \, R > 0. \tag{1.2}$$

Proof of the Lemma. Step 1. We recall that we also have proved that

$$(f_n) \text{ is bounded in } L^{p'}(0,T;L^p(\mathbb{R}^2)) \quad \forall\, p \in [1,\infty),$$

and in particular

$$(f_n) \text{ is bounded in } L^3(0,T;L^{3/2}(\mathbb{R}^2)).$$

Introducing the splitting

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_\infty, \quad \mathcal{K}_0 := \mathcal{K}\,\mathbf{1}_{B_1} \in L^{3/2}, \quad \mathcal{K}_\infty := \mathcal{K}\,\mathbf{1}_{B_1^c} \in L^{5/2},$$

and using the Young inequality

$$\|u * v\|_{L^r} \le \|u\|_{L^p}\,\|v\|_{L^q}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$$

we obtain

$$\bar{\mathcal{K}} \in L^3_t(L^3_x + L^{15}_x) \subset L^3_t(L^3_{loc,x}).$$

Step 2. We observe that for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^2} f_n(t,x)\,\varphi(x)\,dx = \int_{\mathbb{R}^2} f_n\,(\Delta\varphi - \bar{\mathcal{K}}_n\cdot\nabla\varphi)\,dx$$

where the RHS term is bounded in $L^{6/5}(0,T)$ (by Holder inequality and the two bounds $f_n \in L^2$, $\bar{\mathcal{K}}_n \in L^3_{loc}$). We deduce that $\langle f_n\varphi\rangle$ is bounded in $W^{1,6/5}(0,T) \subset C^{0,1/6}([0,T])$ and

$$\langle f_n\varphi\rangle \to \langle f\varphi\rangle \quad \text{strongly in} \quad L^\infty(0,T).$$

We immediately deduce that for any $\varphi \in \mathcal{D}(\mathbb{R}^2)\otimes\mathcal{D}(\mathbb{R}^2)$, the space of linear combination of functions of separable variables $\phi(x,y) = \phi^1(x)\,\phi^2(y)$, we also have

$$\int_{\mathbb{R}^2} f_n(t,y)\,\varphi(x,y)\,dx \to \int_{\mathbb{R}^2} f(t,y)\,\varphi(x,y)\,dx \quad \text{in} \quad L^2((0,T)\times B_R),\ \forall\,R>0. \tag{1.3}$$

Step 3. We fix now $\varphi \in L^2(B_R\times\mathbb{R}^2)$ and we recall the density result : there exists a sequence (φ_k) of functions of $\mathcal{D}(\mathbb{R}^2)\otimes\mathcal{D}(\mathbb{R}^2)$ such that

$$\varphi_k \to \varphi \quad \text{in} \quad L^2(B_R\times\mathbb{R}^2).$$

We write

$$\int f_n\varphi - \int f\varphi \;\;=\;\; \int f_n(\varphi-\varphi_k) + \int f_n\varphi_k - \int f\varphi_k + \int f(\varphi_k-\varphi).$$

We observe that by the Cauchy-Schwarz inequality

$$\begin{aligned}
\Big\|\int f(\varphi_k-\varphi)\Big\|^2_{L^2((0,T)\times B_R} &\le \int_0^T\int_{B_R}\Big(\int_{\mathbb{R}^2} f(t,y)(\varphi_k(x,y)-\varphi(x,y))\,dy\Big)^2\,dxdt\\
&\le \int_0^T\int_{B_R}\int_{\mathbb{R}^2} f^2(t,y)\,dydxdt \times \int_0^T\int_{B_R}\int_{\mathbb{R}^2}(\varphi_k(x,y)-\varphi(x,y))^2\,dydxdt\\
&\le T\,|B_R|\,\|f\|^2_{L^2}\,\|\varphi_k-\varphi\|^2_{L^2} \to 0,
\end{aligned}$$

and that we have a similar result (uniformly in n) for the first term. We then classically deduce that (1.3) also holds for such a function φ.

Step 4. We define

$$\varphi_\varepsilon(x,y) := K(x-y)\,\mathbf{1}_{\varepsilon<|x-y|<1/\varepsilon}$$

so that $\varphi_\varepsilon \in L^2(B_R\times\mathbb{R}^2)$ for any $\varepsilon\in(0,1)$.

We write

$$\bar{\mathcal{K}}_n - \bar{\mathcal{K}} = \int f_n(k(x-y) - \varphi_\varepsilon(x,y)) + \int f_n \varphi_\varepsilon - \int f \varphi_\varepsilon + \int f(\varphi_\varepsilon(x,y) - k(x-y)).$$

We note $\Omega := (0,T) \times B_R$ and we define $K_{0,\varepsilon} := K\,\mathbf{1}_{B_\varepsilon}$, $K_{\infty,\varepsilon} := K\,\mathbf{1}_{B^c_{1/\varepsilon}}$. For the last term we have

$$\begin{aligned}\Big\| \int f(\varphi_\varepsilon(x,y) - k(x-y)) \Big\|_{L^2(\Omega)} &\leq \|k_{0,\varepsilon} * f\|_{L^2(\Omega)} + \|k_{\infty,\varepsilon} * f\|_{L^2(\Omega)} \\ &\leq C\,(\|k_{0,\varepsilon}\|_{3/2} + \|k_{\infty,\varepsilon}\|_{5/2})\,\|f\|_{L^3_t(L^{3/2}_x)} \to 0,\end{aligned}$$

and we conclude to (1.2) in the same way as in the preceding step. $\square$