UNIQUENESS AND LONG TIME ASYMPTOTIC FOR THE PARABOLIC-ELLIPTIC KELLER-SEGEL EQUATION

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Abstract. The present paper deals with the parabolic-elliptic Keller-Segel equation in the plane in the general framework of weak (or "free energy") solutions associated to initial datum with finite mass M, finite second moment and finite entropy. The aim of the paper is threefold:

(1) We prove the uniqueness of the "free energy" solution on the maximal interval of existence $[0, T^*)$ with $T^* = \infty$ in the case when $M \le 8\pi$ and $T^* < \infty$ in the case when $M > 8\pi$. The proof uses a DiPerna-Lions renormalizing argument which makes possible to get the "optimal regularity" as well as an estimate of the difference of two possible solutions in the critical $L^{4/3}$ Lebesgue norm similarly as for the $2d$ vorticity Navier-Stokes equation.

(2) We prove immediate smoothing effect and, in the case $M < 8\pi$, we prove Sobolev norm bound uniformly in time for the rescaled solution (corresponding to the self-similar variables).

(3) In the case $M < 8\pi$, we also prove weighted $L^{4/3}$ linearized stability of the self-similar profile and then universal optimal rate of convergence of the solution to the self-similar profile. The proof is mainly based on an argument of enlargement of the functional space for semigroup spectral gap.

Keywords: Keller-Segel model; chemotaxis; weak solutions; free energy; entropy method; logarithmic Hardy-Littlewood-Sobolev inequality; Hardy-Littlewood-Sobolev inequality; subcritical mass; uniqueness; large time behavior; self-similar variables.

AMS Subject Classification: 35B45, 35B30, 35D05, 35K15, 35B40, 35D10, 35K60

Contents

1. Introduction

The aim of the paper is to prove uniqueness of "free energy" weak solutions to the the so-called parabolic-elliptic Keller-Segel equation in the plane associated to initial datum with finite mass $M \ge 0$, finite polynomial moment and finite entropy, and in the subcritical case $M < 8\pi$ to prove optimal rate of convergence to self-similarity of these solutions. In [19] our analysis will be extended to the parabolic-parabolic Keller-Segel equation in a similar context.

The Keller-Segel (KS) system for chemotaxis describes the collective motion of cells that are attracted by a chemical substance that they are able to emit ([34, 27]). We refer to [8] and the references quoted therein for biological motivation and mathematical introduction. In this paper we are concerned with the parabolic-elliptic KS model in the plane which takes the form

$$
\begin{aligned}
\partial_t f &= \Delta f - \nabla(f\,\nabla c) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2, \\
c &:= -\bar\kappa = -\kappa * f \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2,
\end{aligned}
\tag{1.1}
$$

with $\kappa := \frac{1}{2\pi} \log |z|$, so that in particular

$$-\nabla c = \bar{\mathcal{K}} := \mathcal{K} * f, \quad \mathcal{K} := \nabla \kappa = \frac{1}{2\pi} \frac{z}{|z|^2}.$$

Here $t \geq 0$ is the time variable, $x \in \mathbb{R}^2$ is the space variable, $f = f(t,x) \geq 0$ stands for the *mass density of cells* while $c = c(t,x) \in \mathbb{R}$ is the *chemo-attractant concentration* which solves the (elliptic) Poisson equation $-\Delta c = f$ in $(0,\infty) \times \mathbb{R}^2$.

The evolution equation (1.1) is complemented with an initial condition

$$f(0,.) = f_0 \quad \text{in} \quad \mathbb{R}^2, \tag{1.2}$$

where throughout this paper, we shall assume that

$$0 \leq f_0 \in L^1_2(\mathbb{R}^2), \quad f_0 \log f_0 \in L^1(\mathbb{R}^2). \tag{1.3}$$

Here and below we define the weight function $\langle x \rangle := (1+|x|^2)^{1/2}$ and the weighted Lebesgue space $L^p_k(\mathbb{R}^2)$ for $1 \leq p \leq \infty$, $k \geq 0$, by

$$L^p_k(\mathbb{R}^2) := \{f \in L^1_{loc}(\mathbb{R}^2);\ \|f\|_{L^p_k} := \|f \, \langle x \rangle^k\|_{L^p} < \infty\},$$

as well as $L^1_+(\mathbb{R}^2)$ the cone of nonnegative functions of $L^1(\mathbb{R}^2)$.

The fundamental identities are that any solution to the Keller-Segel equation (1.1) satisfies at least formally the conservation of mass

$$M(t) := \int_{\mathbb{R}^2} f(t,x)\, dx = \int_{\mathbb{R}^2} f_0(x)\, dx =: M, \tag{1.4}$$

the second moment equation

$$M_2(t) := \int_{\mathbb{R}^2} f(t,x)\, |x|^2\, dx = C_1(M)\, t + M_{2,0}, \quad M_{2,0} := \int_{\mathbb{R}^2} f_0(x)\, |x|^2\, dx, \tag{1.5}$$

$C_1(M) := 4M\left(1 - \frac{M}{8\pi}\right)$, and the free energy-dissipation of the free energy identity

$$\mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s)\, ds = \mathcal{F}_0, \tag{1.6}$$

where the free energy $\mathcal{F}(t) = \mathcal{F}(f(t))$, $\mathcal{F}_0 = \mathcal{F}(f_0)$ is defined by

$$\mathcal{F} = \mathcal{F}(f) := \int_{\mathbb{R}^2} f \log f\, dx + \frac{1}{2} \int_{\mathbb{R}^2} f\, \bar{\kappa}\, dx,$$

and the dissipation of free energy is defined by

$$\mathcal{D}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F}}(f) := \int_{\mathbb{R}^2} f\, |\nabla(\log f) + \nabla \bar{\kappa}|^2\, dx.$$

It is worth emphasizing that the critical mass $M_* := 8\pi$ is a threshold because one sees from (1.5) that there does not exist nonnegative and mass preserving solution when $M > 8\pi$ (the identity (1.5) would imply that the second moment becomes negative in a finite time shorter than $T^{**} := 2\pi M_{2,0}/[M(8\pi - M)]$).

On the one hand, in the subcritical case $M < 8\pi$, thanks to the logarithmic Hardy-Littlewood Sobolev inequality (see e.g. [3, 18])

$$\forall\, f \geq 0, \quad \int_{\mathbb{R}^2} f(x) \log f(x)\, dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)\, f(y)\, \log|x-y|\, dxdy \geq C_2(M), \tag{1.7}$$

with $C_2(M) := M\,(1 + \log \pi - \log M)$, one can easily check (see [8, Lemma 7]) that for subcritical mass $M < 8\pi$, there holds

$$\mathcal{H} := \mathcal{H}(f) = \int f \log f \leq C_3(M)\, \mathcal{F} + C_4(M), \tag{1.8}$$

with $C_3(M) := 1/\big(1-\frac{M}{8\pi}\big)$, $C_4(M) := C_3(M)\, C_2(M)\, M/(8\pi)$. Then from (1.8) and the very classical functional inequality (see for instance [8, Lemma 8])

$$\mathcal{H}^+ := \mathcal{H}^+(f) = \int f(\log f)_+ \le \mathcal{H} + \frac{1}{4} M_2 + C_5(M), \tag{1.9}$$

with $C_5(M) := 2M\log(2\pi) + 2/e$, one concludes that (1.4), (1.5) and (1.6) provide a convenient family of a priori estimates in order to define weak solutions, namely

$$\begin{aligned} \mathcal{H}^+(f(t)) + M_2(f(t)) + C_3(M)\int_0^t \mathcal{D}_{\mathcal{F}}(f(s))\, ds \le \\ \le C_3(M)\,\mathcal{F}_0 + \frac{5}{4} M_{2,0} + 2C_1(M)t + C_4(M) + C_5(M), \end{aligned} \tag{1.10}$$

where the RHS term is finite under assumption (1.3) on f_0, since

$$\begin{aligned} \mathcal{F}_0 \ &\le\ \mathcal{H}_0 + \frac{1}{4\pi}\iint f_0(x)\, f_0(y)\, (\log|x-y|)_+ \, dxdy \\ &\le\ \mathcal{H}_0 + \frac{1}{4\pi}\iint f_0(x)\, f_0(y)\, |x-y|^2 \, dxdy \le \mathcal{H}_0 + \frac{1}{\pi}\, M\, M_{2,0}, \end{aligned} \tag{1.11}$$

with $\mathcal{H}_0 := \mathcal{H}(f_0)$. In other words, we have

$$\mathscr{A}_T(f) := \sup_{t\in[0,T]} \big\{\mathcal{H}^+(f(t)) + M_2(f(t))\big\} + \int_0^T \mathcal{D}_{\mathcal{F}}(f(s))\, ds \le C(T) \quad \forall\, T \in (0,T^*) \tag{1.12}$$

with $T^* = +\infty$ and a constant $C(T)$ which depends on M, $M_{2,0}$, $\mathcal{H}_0$ and the final time T.

On the other hand, in the critical case $M = 8\pi$ and the supercritical case $M > 8\pi$, the above argument using the logarithmic Hardy-Littlewood Sobolev inequality (1.7) fails, but one can however prove that (1.12) holds with $T^* = +\infty$ when $M = 8\pi$ and that (1.12) holds with some $T^* \in (0, T^{**}]$ when $M > 8\pi$ (see [6] for details as well as Remark 2.3 below).

Definition 1.1. *For any initial datum f_0 satisfying (1.3) and any final time $T^* > 0$, we say that*

$$0 \le f \in L^\infty(0,T; L^1(\mathbb{R}^2)) \cap C([0,T); \mathcal{D}'(\mathbb{R}^2)), \quad \forall\, T \in (0,T^*), \tag{1.13}$$

is a weak solution to the Keller-Segel equation in the time interval $(0,T^*)$ associated to the initial condition f_0 whenever f satisfies (1.4), (1.5) and

$$\mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s)\, ds \le \mathcal{F}_0 \quad \forall\, t \in (0,T^*), \tag{1.14}$$

as well as the Keller-Segel equation (1.1)-(1.2) in the distributional sense, namely

$$\int_{\mathbb{R}^2} f_0(x)\, \varphi(0,x)\, dx = \int_0^{T^*}\int_{\mathbb{R}^2} f(t,x) \Big\{ (\nabla_x(\log f) + \bar{\mathcal{K}}) \cdot \nabla_x\varphi - \partial_t\varphi \Big\}\, dxdt \tag{1.15}$$

for any $\varphi \in C^2_c([0,T) \times \mathbb{R}^2)$.

It is worth emphasizing that thanks to the Cauchy-Schwarz inequality, we have

$$\int_{\mathbb{R}^2} f\, |\nabla_x(\log f) + \bar{\mathcal{K}}|\, dx \le M^{1/2}\, \mathcal{D}_{\mathcal{F}}^{1/2},$$

and the RHS of (1.15) is then well defined thanks to (1.10).

This framework is well adapted for the existence theory.

Theorem 1.2. *For any initial datum f_0 satisfying (1.3) there exists at least one weak solution on the time interval $(0,T^*)$ in the sense of Definition 1.1 to the Keller-Segel equation (1.1)-(1.2) with $T^* = +\infty$ when $M \le 8\pi$ and $T^* < +\infty$ when $M > 8\pi$.*

We refer to [8, Theorem 1] for the subcritical case $M \in (0, 8\pi)$ and to [6] for the critical and supercritical cases $M \ge 8\pi$.

Our first main result establishes that this framework is also well adapted for the well-posedness issue.

Theorem 1.3. *For any initial datum f_0 satisfying* (1.3) *there exists at most one weak solution in the sense of Definition 1.1 to the Keller-Segel equation* (1.1)-(1.2).

Theorem 1.3 improves the uniqueness result proved in [20] in the class of solutions $f \in C([0,T]; L^1_2(\mathbb{R}^2)) \cap L^\infty((0,T)\times\mathbb{R}^2)$ which can be built under the additional assumption $f_0 \in L^\infty(\mathbb{R}^2)$ (see also [24] where a uniqueness result is established for a related model). Our proof follows a strategy introduced in [23] for the 2D viscous vortex model. It is based on a DiPerna-Lions renormalization trick (see [21]) which makes possible to get the optimal regularity of solutions for small time and then to follow the uniqueness argument introduced by Ben-Artzi for the 2D viscous vortex model (see [4, 10]).

Next we consider the smoothness issue and the long time behaviour of solution for subcritical mass issue. For that last purpose it is convenient to work with self-similar variables. We introduce the rescaled functions g and u defined by

$$g(t,x) := R(t)^{-2} f(\log R(t), R(t)^{-1}x), \quad u(t,x) := c(\log R(t), R(t)^{-1}x), \tag{1.16}$$

with $R(t) := (1+2t)^{1/2}$. The rescaled parabolic-elliptic KS system reads

$$\begin{aligned} \partial_t g &= \Delta g + \nabla(gx - g\,\nabla u) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2, \\ u &= -\kappa * g \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2. \end{aligned} \tag{1.17}$$

Our second main result concerns the regularity of the solutions.

Theorem 1.4. *For any initial datum f_0 satisfying* (1.3) *the associated solution f is smooth for positive time, namely $f \in C^\infty((0,T^*)\times\mathbb{R}^2)$, and satisfies the identity* (1.6) *on $(0,T^*)$. Moreover, when $M < 8\pi$, the rescaled solution g defined by* (1.16) *satisfies*

$$\sup_{t\ge 0} M_k(g(t)) \le \max((k-1)^{k/2}M, M_k(f_0)) \quad \forall\, k \ge 2, \tag{1.18}$$

as well as

$$\sup_{t\ge\varepsilon} \|g(t,.)\|_{W^{2,\infty}} \le \mathcal{C} \quad \forall\,\varepsilon > 0, \tag{1.19}$$

for some explicit constant $\mathcal{C}$ which depends on ε, M, $\mathcal{F}_0$ and $M_{2,0}$.

It is worth mentioning that L^p bound on g for positive time and for $p \in [1,\infty)$ was known but non uniformly in time and as an a priori bound, while (1.19) is proved as an a posteriori estimate. Our proof is merely based on the same estimates as those established in [8], on a bootstrap argument and on the remark that the rescaled free energy provides uniform in time estimates.

From now on in this introduction, we definitively restrict ourself to the subcritical case $M < 8\pi$ and we focus on the long time asymptotic of the solutions. It has been proved in [8, Theorem 1.2] that the solution given by Theorem 1.2 satisfies

$$g(t,.) \to G \quad \text{in} \quad L^1(\mathbb{R}^2) \quad \text{as} \quad t\to\infty, \tag{1.20}$$

where G is a solution to the rescaled stationary problem

$$\begin{aligned} &\Delta G + \nabla(Gx - G\,\nabla U) = 0 \quad \text{in} \quad \mathbb{R}^2, \\ &0 \le G, \quad \int_{\mathbb{R}^2} G\,dx = M, \quad U = -\mathcal{K} * G. \end{aligned} \tag{1.21}$$

Moreover, the uniqueness of the solution G to (1.21) has been proved in [8, 5], see also [15, 16, 17], so that $G = G_M$ stands for the unique self-similar profile with same mass M as f_0 and it is given in implicit form by

$$G = M \, \frac{e^{-G*\kappa - |x|^2/2}}{\int_{\mathbb{R}^2} e^{-G*\kappa-|x|^2/2}\, dx} \tag{1.22}$$

and that $U = -G*\kappa$ satisfies

$$\Delta U + \frac{M}{\int_{\mathbb{R}^2} e^{U-|x|^2/2}\, dx} \, e^{U-|x|^2/2} = 0. \tag{1.23}$$

Our third main result is about the convergence to self-similarity.

Theorem 1.5. *For any $M \in (0, 8\pi)$, and any finite real numbers $\mathcal{F}_0$, $M_{4,0}$, there exists a (non explicit) constant C such that for any initial datum f_0 satisfying* (1.3) *with*

$$M_0(f_0) = M, \quad M_4(f_0) \le M_{4,0}, \quad \mathcal{F}(f_0) \le \mathcal{F}_0,$$

the associated solution in self-similar variables g defined by (1.16)-(1.1) *satisfies the optimal rate convergence*

$$\|g(t,.) - G\|_{L^{4/3}} \le C\, e^{-t} \quad \forall\, t \ge 1,$$

where G stands for the self-similar profile with same mass M as f_0.

Let us emphasize that the strong assumption $M_4(f_0) < \infty$ can be weaken. For instance, assuming only $M_{k'}(f_0) < \infty$ for some $k' > 3$, the same proof leads to the same optimal rate, and with the sole assumption $M_2(f_0) < \infty$, one can get a not optimal rate of convergence to the self-similar profile in the sense that $\|g(t,.) - G\|_{L^{4/3}} \le C_\eta\, e^{-\eta t}$ for all $t \ge 1$ and for some $\eta \in (0,1)$, $C_\eta \in (0,\infty)$.

For some particular class of initial data (essentially for an initial datum f_0 which is close enough to the self-similar profile G in the sense of the strongly confining norm $L^2(e^{\nu\,|x|^2}dx)$, $\nu > 0$) it has been proved that the associated solution converges with exponential rate when M is small enough in [7] and for any $M \in (0, 8\pi)$ in [16, 17]. In these last works, the linear stability of the linearized rescaled equation around the self-similar profile is established and that is the cornerstone of these nonlinear stability results. Our proof follows a strategy of "enlarging the functional space of semigroup spectral gap" initiated in [32] for studying long time convergence to the equilibrium for the homogeneous Boltzmann equation, and then developed in [30, 25, 12, 11, 29] (see also [31]) in the framework of kinetic equations and growth-fragmentation equations. More precisely, taking advantage of the linear stability of the linearized rescaled equation established in [17] in the small space $L^2(e^{\nu\,|x|^2}dx)$ we prove that the same result holds in the more larger space $L^{4/3}_k$, $k > 3/2$. Then gathering the long time convergence (without rate) to self-similarity (1.20) with the estimates of Theorem 1.4, we get that any solution reaches a small $L^{4/3}_k$-neighborhood of G in finite time and we conclude to Theorem 1.5 by nonlinear stability in $L^{4/3}_k \cap L^1_4$.

Let us end the introduction by describing the plan of the paper. In Section 2 we present some functional inequalities which will be useful in the sequel of the paper, we establish several a posteriori bounds satisfied by any weak solution, and we prove Theorem 1.4. Section 3 is dedicated to the proof of the uniqueness result stated in Theorem 1.3. In Section 4 we prove the long time behaviour result as stated in Theorem 1.5.

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2. A posteriori estimates - Proof of Theorem 1.4

We start by presenting some elementary functional inequalities which will be of main importance in the sequel. The two first estimates are picked up from [23, Lemma 3.2] but are probably classical and the third one is a variant of the Gagliardo-Niremberg-Sobolev inequality.

Lemma 2.1. *For any $0 \le f \in L^1(\mathbb{R}^2)$ with finite mass M and finite Fisher information*

$$I = I(f) := \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f},$$

there holds

$$\forall\, p \in [1,\infty), \quad \|f\|_{L^p(\mathbb{R}^2)} \le C_p\, M^{1/p}\, I(f)^{1-1/p}, \tag{2.1}$$

$$\forall\, q \in [1,2), \quad \|\nabla f\|_{L^q(\mathbb{R}^2)} \le C_q\, M^{1/q-1/2}\, I(f)^{3/2-1/q}. \tag{2.2}$$

For any $0 \le f \in L^1(\mathbb{R}^2)$ with finite mass M, there holds

$$\forall\, p \in [2,\infty) \quad \|f\|_{L^{p+1}(\mathbb{R}^2)} \le C_p\, M^{1/(p+1)}\, \|\nabla (f^{p/2})\|_{L^2}^{2/(p+1)}. \tag{2.3}$$

For the sake of completeness we give the proof below.

Proof of Lemma 2.1. We start with (2.2). Let $q \in [1,2)$ and use the Hölder inequality:

$$\|\nabla f\|_{L^q}^q = \int \left| \frac{\nabla f}{\sqrt{f}} \right|^q f^{q/2} \le \left(\int \frac{|\nabla f|^2}{f} \right)^{q/2} \left(\int f^{q/(2-q)} \right)^{(2-q)/2} = I(f)^{q/2}\, \|f\|_{L^{q/(2-q)}}^{q/2}.$$

Denoting by $q^* = 2q/(2-q) \in [2,\infty)$ the Sobolev exponent associated to q in dimension 2, we have, thanks to a standard interpolation inequality and to the Sobolev inequality,

$$\begin{aligned} \|f\|_{L^{q/(2-q)}} &= \|f\|_{L^{q^*/2}} \le \|f\|_{L^1}^{1/(q^*-1)}\, \|f\|_{L^{q^*}}^{(q^*-2)/(q^*-1)} \\ &\le C_q\, \|f\|_{L^1}^{1/(q^*-1)}\, \|\nabla f\|_{L^q}^{(q^*-2)/(q^*-1)}. \end{aligned} \tag{2.4}$$

Gathering these two inequalities, it comes

$$\|\nabla f\|_{L^q} \le C_q\, I(f)^{1/2}\, \|f\|_{L^1}^{1/(2(q^*-1))}\, \|\nabla f\|_{L^q}^{(q^*-2)/(2(q^*-1))},$$

from which we deduce (2.2).

We now establish (2.1). For $p \in (1,\infty)$, write $p = q^*/2 = q/(2-q)$ with $q := 2p/(1+p) \in [1,2)$ and use (2.4) and (2.2):

$$\|f\|_{L^p} \le C_p\, \|f\|_{L^1}^{\frac{1}{q^*-1} + \frac{q^*-2}{q^*-1}\left(\frac{1}{q} - \frac{1}{2}\right)} I(f)^{\frac{q^*-2}{q^*-1}\left(\frac{3}{2} - \frac{1}{q}\right)},$$

from which one easily concludes.

We verify (2.3). From the Sobolev inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|w^{2(1+1/p)}\|_{L^1(\mathbb{R}^2)} &= \|w^{1+1/p}\|_{L^2(\mathbb{R}^2)}^2 \le \|\nabla (w^{1+1/p})\|_{L^1(\mathbb{R}^2)}^2 \\ &\le (1+1/p)^2\, \|w^{1/p}\|_{L^2}^2\, \|\nabla w\|_{L^2(\mathbb{R}^2)}^2 \end{aligned} \tag{2.5}$$

and we conclude to (2.3) by taking $w := f^{p/2}$. □

The proof of (1.19) in Theorem 1.4 is split in several steps that we present as some intermediate autonomous a posteriori bounds.

Lemma 2.2. *For any weak solution f and any finale time $T \in (0, T^*)$ there exists a constant $C := C(M, \mathscr{A}_T(f))$ such that*

$$\frac{1}{2} \int_0^T I(f(t))\, dt \le C. \tag{2.6}$$

In particular, in the subcritical case $M < 8\pi$ the constant C only depends on M, $\mathcal{H}_0$, $M_{2,0}$ and $T \in (0,\infty)$.

Proof of Lemma 2.2. We write

$$\begin{aligned}\mathcal{D}_{\mathcal{F}}(f) &= \int f\,|\nabla(\log f + \bar\kappa)|^2 \\ &\geq \int f\,|\nabla \log f|^2 + 2\int \nabla f\cdot\nabla\bar\kappa = I(f) - 2\int f^2.\end{aligned}$$

On the other hand, for any $A > 1$, using the Cauchy-Schwarz inequality and the inequality (2.1) for $p = 3$, we have

$$\begin{aligned}\int f^2\,\mathbf{1}_{f\geq A} &\leq \Big(\int f\,\mathbf{1}_{f\geq A}\Big)^{1/2}\Big(\int f^3\Big)^{1/2} \\ &\leq \Big(\int f\,\frac{(\log f)_+}{\log A}\Big)^{1/2}\Big(C_3^3\,M\,I(f)^2\Big)^{1/2},\end{aligned}$$

from what we deduce for $A = A(M, \mathcal{H}^+(f))$ large enough, and more precisely taking A such that $\log A = 16\,\mathcal{H}^+(f)\,C_3^3\,M$,

$$\int f^2\,\mathbf{1}_{f\geq A} \leq C_3^{3/2}\,M^{1/2}\,\frac{\mathcal{H}^+(f)^{1/2}}{(\log A)^{1/2}}\,I(f) \leq \frac{1}{4}\,I(f). \tag{2.7}$$

Together with the first estimate, we find

$$\begin{aligned}\frac{1}{2}I(f) &\leq \mathcal{D}_{\mathcal{F}}(f) + 2\int f^2\,\mathbf{1}_{f\leq A} \\ &\leq \mathcal{D}_{\mathcal{F}}(f) + 2\,M\,\exp(16\,\mathcal{H}^+(f)\,C_3^3\,M),\end{aligned}$$

and we conclude thanks to (1.12) in the general case and thanks to (1.4)–(1.11) in the subcritical case $M < 8\pi$. □

Remark 2.3. *As we have already mentioned we are not able to use the logarithmic Hardy-Littlewood Sobolev inequality* (1.7) *in the critical and supercritical cases. However, introducing the Maxwell function* $\mathscr{M} := M\,(2\pi)^{-1}\exp(-|x|^2/2)$ *of mass* M *and the relative entropy*

$$H(h|\mathscr{M}) := \int_{\mathbb{R}^2} (h\,\log(h/\mathscr{M}) - h + \mathscr{M})\,dx$$

one classically shows that any solution f *to the Keller-Segel equation* (1.1) *formally satisfies*

$$\begin{aligned}\frac{d}{dt}H(f(t)|\mathscr{M}) &= -I(f(t)) + \int f(t)^2 + 2M \\ &\leq -I(f(t)) + MA + C_3^{3/2}\,M^{1/2}\,\frac{\mathcal{H}^+(f(t))^{1/2}}{(\log A)^{1/2}}\,I(f(t)) + 2M \qquad (\forall\,A > 0) \\ &= -I(f(t)) + M\,\exp\big(4C_3^3\,M\,\mathcal{H}^+(f(t))\big) + 2M \\ &= -I(f(t)) + M\,\exp\big\{C_6\,H(f(t)|\mathscr{M})\big\} + 2M,\end{aligned}$$

for a constant $C_6 = C_6(M)$*, where we have used* (2.7)*, we have made the choice* $\log A := 4C_3^3\,M\,\mathcal{H}^+(f(t))$ *and we have used a variant of inequality* (1.9)*. This differential inequality provides a local a priori estimate on the relative entropy which is the key estimate in order to prove local existence result for supercritical mass as well as global existence result for critical mass in* [6].

As an immediate consequence of Lemmas 2.1 and 2.2, we have

Lemma 2.4. *For any* $T \in (0, T^*)$*, any weak solution* f *satisfies*

$$f \in L^{p/(p-1)}(0,T; L^p(\mathbb{R}^2)), \quad \forall\,p \in (1,\infty), \tag{2.8}$$

$$\bar{\mathcal{K}} \in L^{p/(p-1)}(0,T; L^{2p/(2-p)}(\mathbb{R}^2)), \quad \forall\,p \in (1,2), \tag{2.9}$$

$$\nabla_x\bar{\mathcal{K}} \in L^{p/(p-1)}(0,T; L^p(\mathbb{R}^2)), \quad \forall\,p \in (2,\infty). \tag{2.10}$$

Proof of Lemma 2.4. The bound (2.8) is a direct consequence of (2.6) and (2.1). The bound (2.9) then follows from the definition of K, the Hardy-Littlewood-Sobolev inequality (see e.g. [28, Theorem 4.3])

$$\Big\| \frac{1}{|z|} * f \Big\|_{L^{2r/(2-r)}(\mathbb{R}^2)} \le C_r \, \|f\|_{L^r(\mathbb{R}^2)}, \quad \forall\, r \in (1,2), \tag{2.11}$$

with $r = p$ and (2.8). Finally, from (2.6) and (2.2) we have

$$\nabla f \in L^{\frac{2q}{3q-2}}(0,T; L^q(\mathbb{R}^2)), \quad \forall\, q \in (1,2).$$

Applying the above Hardy-Littlewood-Sobolev inequality to $\nabla_x \bar{\mathcal{K}} = \mathcal{K} * (\nabla_x f)$ with $r = q$, we get

$$\nabla_x \bar{\mathcal{K}} \in L^{\frac{2q}{3q-2}}(0,T; L^{\frac{2q}{2-q}}(\mathbb{R}^2)), \quad \forall\, q \in (1,2),$$

which is nothing but (2.10). □

Lemma 2.5. *Any weak solution f satisfies*

$$\begin{aligned} &\int_{\mathbb{R}^2} \beta(f_{t_1})\, dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta''(f_s)\, |\nabla f_s|^2 \, dxds \\ &\qquad \le \int_{\mathbb{R}^2} \beta(f_{t_0})\, dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} (\beta'(f_s)\, f_s^2 - \beta(f_s)\, f_s)_+ \, dxds, \end{aligned} \tag{2.12}$$

for any times $0 \le t_0 \le t_1 < T^*$ and any renormalizing function $\beta : \mathbb{R} \to \mathbb{R}$ which is convex, piecewise of class C^1 and such that

$$|\beta(u)| \le C\,(1 + u\,(\log u)_+), \quad (\beta'(u)\, u^2 - \beta(u)\, u)_+ \le C\,(1 + u^2) \quad \forall\, u \in \mathbb{R}.$$

Proof of Lemma 2.5. We write

$$\partial_t f - \Delta_x f = \bar{\mathcal{K}} \cdot \nabla_x f + f^2,$$

and we split the proof into three steps.

Step 1. Continuity. Consider a mollifier sequence (ρ_n) on $\mathbb{R}^2$, that is $\rho_n(x) := n^2 \rho(nx)$, $0 \le \rho \in \mathcal{D}(\mathbb{R}^2)$, $\int \rho = 1$, and introduce the mollified function $f_t^n := f_t *_x \rho_n$. Clearly, $f^n \in C([0,T), L^1(\mathbb{R}^2))$. Using (2.8) and (2.10), a variant of the commutation Lemma [21, Lemma II.1 and Remark 4] tells us that

$$\partial_t f^n - \bar{\mathcal{K}} \cdot \nabla_x f^n - \Delta_x f^n = r^n, \tag{2.13}$$

with

$$r^n := (f^2) * \rho_n + (\bar{\mathcal{K}} \cdot \nabla_x f) * \rho_n - \bar{\mathcal{K}} \cdot \nabla_x f^n \to f^2 \quad \text{in} \quad L^1(0,T; L^1_{loc}(\mathbb{R}^2)).$$

The important point here is that $f^2, |\nabla_x \bar{\mathcal{K}}|\, f \in L^1((0,T) \times \mathbb{R}^2)$, thanks to (2.10) and (2.8).

As a consequence, the chain rule applied to the smooth function f^n reads

$$\partial_t \beta(f^n) = \bar{\mathcal{K}} \cdot \nabla_x \beta(f^n) + \Delta_x \beta(f^n) - \beta''(f^n)\, |\nabla_x f^n|^2 + \beta'(f^n)\, r^n, \tag{2.14}$$

for any $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside of a compact set. Because the equation (2.13) with $\bar{\mathcal{K}}$ fixed is linear, the difference $f^{n,k} := f^n - f^k$ satisfies (2.13) with r^n replaced by $r^{n,k} := r^n - r^k \to 0$ in $L^1(0,T; L^1_{loc}(\mathbb{R}^2)$ and then also (2.14) (with again f^n and r^n changed in $f^{n,k}$ and $r^{n,k}$). In that last equation, we choose $\beta(s) = \beta_1(s)$ where $\beta_A(s) = s^2/2$ for $|s| \le A$, $\beta_A(s) = A\,|s| - A^2/2$ for $|s| \ge A$ and we obtain for any non-negative function $\chi \in C^2_c(\mathbb{R}^d)$,

$$\begin{aligned} &\int_{\mathbb{R}^2} \beta_1(f^{n,k}(t,x))\, \chi(x)\, dx \le \\ &\le \int_{\mathbb{R}^2} \beta_1(f^{n,k}(0,x))\, \chi(x)\, dx + \int_0^t \int_{\mathbb{R}^2} |r^{n,k}(s,x)|\, \chi(x)\, dxds \\ &+ \int_0^t \int_{\mathbb{R}^2} \beta_1(f^{n,k}(s,x)) \Big| -f\, \chi + \Delta \chi(x) - \bar{\mathcal{K}}(s,x) \cdot \nabla \chi(x) \Big| \, dxds, \end{aligned}$$

where we have used that $\mathrm{div}_x \bar{\mathcal{K}} = f$, that $|\beta_1'| \le 1$ and that $\beta_1'' \ge 0$. Because $f_0 \in L^1$, we have $f^{n,k}(0) \to 0$ in $L^1(\mathbb{R}^2)$, and we deduce from the previous inequality, the convergence $r^{n,k} \to 0$ in $L^1(0,T;L^1_{loc}(\mathbb{R}^2))$, the convergence $\beta_1(f^{n,k})\bar{\mathcal{K}} \to 0$ in $L^1(0,T;L^1_{loc}(\mathbb{R}^2))$ (because $\beta_1(s) \le |s|$, because $f^{n,k} \to 0$ in $L^3(0,T,L^{3/2}(\mathbb{R}^2))$ by (2.8) with $p = 3/2$ and because $\bar{\mathcal{K}} \in L^6(0,T;L^3(\mathbb{R}^2)) \subset L^{3/2}(0,T;L^3(\mathbb{R}^2))$ by (2.9) with $p = 6/5$) and the convergence $\beta_1(f^{n,k})f \to 0$ in $L^1(0,T;L^1(\mathbb{R}^2))$ that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^2} \beta_1(f^{n,k}(t,x))\,\chi(x)\,dx \underset{n,k\to\infty}{\longrightarrow} 0.$$

Since χ is arbitrary, we deduce that there exists $\bar{f} \in C([0,\infty);L^1_{loc}(\mathbb{R}^2))$ so that $f^n \to \bar{f}$ in $C([0,T];L^1_{loc}(\mathbb{R}^2))$, $\forall T > 0$. Together with the convergence $f^n \to f$ in $C([0,\infty);\mathcal{D}'(\mathbb{R}^2))$ and the a priori bound (1.10), we deduce that $f = \bar{f}$ and

$$f^n \to f \quad \text{in} \quad C([0,T];L^1(\mathbb{R}^2)), \quad \forall T > 0. \tag{2.15}$$

Step 2. Linear estimates. We come back to (2.14), which implies, for all $0 \le t_0 < t_1$, all $\chi \in C^2_c(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \beta(f^n_{t_1})\,\chi\,dx + \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \beta''(f^n_s)\,|\nabla_x f^n_s|^2\,\chi\,dxds = \int_{\mathbb{R}^2} \beta(f^n_{t_0})\,\chi\,dx \tag{2.16}$$
$$+ \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \Big\{\beta'(f^n_s)\,r^n\,\chi + \beta(f^n_s)\,\Delta\chi - \beta(f^n_s)\,\mathrm{div}_x(\bar{\mathcal{K}}\chi)\Big\}\,dxds.$$

Choosing $0 \le \chi \in C^2_c(\mathbb{R}^2)$ and $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is non-negative and vanishes outside of a compact set, and passing to the limit as $n \to \infty$, we get

$$\int_{\mathbb{R}^2} \beta(f_{t_1})\,\chi\,dx + \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \beta''(f_s)\,|\nabla_x f_s|^2\,\chi\,dxds \le \int_{\mathbb{R}^2} \beta(f_{t_0})\,\chi\,dx \tag{2.17}$$
$$+ \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \Big\{\Big[\beta'(f)\,f^2 - \beta(f)\,f\Big]\,\chi + \beta(f)\,\Big[\Delta\chi - \bar{\mathcal{K}}\cdot\nabla\chi\Big]\Big\}\,dxds.$$

By approximating $\chi \equiv 1$ by the sequence (χ_R) with $\chi_R(x) = \chi(x/R)$, $0 \le \chi \in \mathcal{D}(\mathbb{R}^2)$, we see that the last term in (2.17) vanishes and we get (2.12) in the limit $R \to \infty$ for any renormalizing function β with linear growth at infinity.

Step 3. superlinear estimates. Finally, for any β satisfying the growth condition as in the statement of the Lemma, we just approximate β by an increasing sequence of smooth renormalizing functions β_R with linear growth at infinity, and pass to the limit in (2.12) in order to conclude. □

Lemma 2.6. *For any weak solution f, any time $T \in (0,T^*)$ and any $p \ge 2$, there exists a constant $C := C(M,\mathscr{A}_T,T,p)$ such that for any $0 \le t_0 < t_2 \le T$*

$$\|f(t_1)\|^p_{L^p} + \frac{1}{2}\int_{t_0}^{t_1} \|\nabla_x(f^{p/2})\|^2_{L^2}\,dt \le \|f(t_0)\|^p_{L^p} + C. \tag{2.18}$$

Proof of Lemma 2.6. We define the renormalizing function $\beta_K : \mathbb{R}_+ \to \mathbb{R}_+$, $K \ge 2$, by

$$\beta_K(u) := u^p \ \text{ if } \ u \le K, \quad \beta_K(u) := \frac{K^{p-1}}{\log K}\,u\log u \ \text{ if } \ u \ge K,$$

so that β_K is convex and piecewise of class C^1, and moreover there holds

$$\beta_K'(u)u^2 - \beta_K(u)u = (p-1)\,u^{p+1}\,\mathbf{1}_{u<K} + \frac{K^{p-1}}{\log K}\,u^2\,\mathbf{1}_{u>K},$$

and

$$\beta_K''(u) = p(p-1)\,u^{p-2}\,\mathbf{1}_{u<K} + \frac{K^{p-1}}{\log K}\,\frac{1}{u}\,\mathbf{1}_{u>K}.$$

Thanks to Lemma 2.5, we may write

$$\int_{\mathbb{R}^2} \beta_K(f_{t_1})\,dx + \frac{4}{p'}\int_{t_0}^{t_1}\int_{\mathbb{R}^2} |\nabla_x (f^{p/2})|^2\,\mathbf{1}_{f\le K}\,dxds + \frac{K^{p-1}}{\log K}\int_{t_0}^{t_1}\int_{\mathbb{R}^2} \frac{|\nabla_x f|^2}{f}\,\mathbf{1}_{f\ge K}\,dxds$$
$$\le \int_{\mathbb{R}^2} \beta_K(f_{t_0})\,dx + (p-1)\int_{t_0}^{t_1}\int_{\mathbb{R}^2} f^{p+1}\,\mathbf{1}_{f\le K}\,dxds + \frac{K^{p-1}}{\log K}\int_{t_0}^{t_1}\int_{\mathbb{R}^2} f^2\,\mathbf{1}_{f\ge K}\,dxds,$$

where $p' \in [1,\infty]$ stands for the conjugate exponent associated to $p \in [1,\infty]$ and defined by $1/p + 1/p' = 1$.

On the one hand, using the splitting $f = (f\wedge A) + (f-A)_+$, we have

$$\begin{aligned}\mathcal{T}_1 &:= (p-1)\int_{t_0}^{t_1}\int_{\mathbb{R}^2} f^{p+1}\,\mathbf{1}_{f\le K}\,dxds\\ &\le (p-1)\,2^p\,A^p\,M\,T + (p-1)\,2^p\int_{t_0}^{t_1}\int_{\mathbb{R}^2} f_{A,K}^{p+1}\,dxds,\end{aligned}$$

where we have defined $f_{A,K} := \min((f-A)_+, K-A)$, $K > A > 0$. Moreover, thanks to inequality (2.3) and the same trick as in the proof of Lemma 2.2, we have

$$\begin{aligned}\int_{\mathbb{R}^2} f_{A,K}^{p+1}\,dx &\le C_p\int_{\mathbb{R}^2} f_{A,K}\,dx\int_{\mathbb{R}^2} |\nabla (f_{A,K}^{p/2})|^2\,dx\\ &\le C_p\,\frac{\mathcal{H}^+(f)}{\log A}\int_{\mathbb{R}^2} |\nabla_x (f^{p/2})|^2\,\mathbf{1}_{f\le K}\,dx.\end{aligned}$$

As a consequence, we obtain

$$\mathcal{T}_1 \le (p-1)\,2^p\,A^p\,M\,T + \frac{1}{p'}\int_{t_0}^{t_1}\int_{\mathbb{R}^2} |\nabla_x (f^{p/2})|^2\,\mathbf{1}_{f\le K}\,dxds,$$

for $A = A(p,\mathscr{A}_T) > 1$ large enough.

On the other hand, thanks to the Sobolev inequality (line 2) and the Cauchy-Schwarz inequality (line 3), we have

$$\begin{aligned}\mathcal{T}_2 &:= \frac{K^{p-1}}{\log K}\int_{\mathbb{R}^2} f^2\,\mathbf{1}_{f\ge K}\,dx \le 4\,\frac{K^{p-1}}{\log K}\int_{\mathbb{R}^2}(f-K/2)_+^2\,dx\\ &\le 4\,\frac{K^{p-1}}{\log K}\Big(\int_{\mathbb{R}^2}|\nabla (f-K/2)_+|\,dx\Big)^2 = 4\,\frac{K^{p-1}}{\log K}\Big(\int_{\mathbb{R}^2}|\nabla f|\,\mathbf{1}_{f\ge K/2}\,dx\Big)^2\\ &\le 4\,\frac{K^{p-1}}{\log K}\int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f}\,\mathbf{1}_{f\ge K/2}\,dx\int f\,\mathbf{1}_{f\ge K/2}\,dx\\ &\le 4\,\frac{K^{p-1}}{\log K}\Big\{\frac{4}{p^2}\int_{\mathbb{R}^2}|\nabla (f^{p/2})|^2\,\big(\frac{2}{K}\big)^{p-1}\,\mathbf{1}_{f\le K} + \int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f}\,\mathbf{1}_{f\ge K}\Big\}\frac{\mathcal{H}^+(f)}{\log(K/2)}\\ &\le \frac{1}{p'}\int_{\mathbb{R}^2}|\nabla (f^{p/2})|^2\,\mathbf{1}_{f\le K}\,dx + \frac{1}{2}\frac{K^{p-1}}{\log K}\int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f}\,\mathbf{1}_{f\ge K}\,dx,\end{aligned}$$

for any $K \ge K^* = K^*(p,\mathscr{A}_T) > \max(A,2)$ large enough.

All together, we have proved that for some constant A and K^* only depending on p, T and the initial datum f_0, and for any $K \ge K^*$ there holds

$$\int_{\mathbb{R}^2} \beta_K(f_{t_1})\,dx + \frac{2}{p'}\int_{t_0}^{t_1}\int_{\mathbb{R}^2} |\nabla_x (f^{p/2})|^2\,\mathbf{1}_{f\le K}\,dxds \le \int_{\mathbb{R}^2}\beta_K(f_{t_0})\,dx + (p-1)\,2^p\,A^p\,M\,T.$$

We conclude to (2.18) by passing to the limit $K\to\infty$. □

Lemma 2.7. *Any weak solution f is smooth, that is*

$$f \in C_b^\infty((\varepsilon,T)\times\mathbb{R}^2),\ \forall\,\varepsilon,T,\ 0<\varepsilon<T<T^*,$$

so that in particular it is a "classical solution" for positive time.

Proof of Lemma 2.7. For any time $t_0 \in (0,T)$ and any exponent $p \in (1,\infty)$, there exists $t_0' \in (0,t_0)$ such that $f(t_0') \in L^p(\mathbb{R}^2)$ thanks to (2.8), from what we deduce using (2.18) on the time interval (t_0',T) that

$$f \in L^\infty(t_0,T;L^p(\mathbb{R}^2)) \quad \text{and} \quad \nabla_x f \in L^2((t_0,T)\times\mathbb{R}^2). \tag{2.19}$$

Next, by writing $\mathcal{K} = \mathcal{K}\,\mathbf{1}_{|z|\le 1} + \mathcal{K}\,\mathbf{1}_{|z|\ge 1} \in L^{3/2} + L^\infty$, it is easily checked $\|\mathcal{K} * f\|_{L^\infty} \le C\,(\|f\|_{L^3} + \|f\|_{L^1})$, and then $\bar{\mathcal{K}} \in L^\infty(t_0,T;L^\infty(\mathbb{R}^2))$ because of (2.19) and (1.13). We thus have

$$\partial_t f + \Delta_x f = f^2 + \bar{\mathcal{K}}\cdot\nabla_x f \in L^2((t_0,T)\times\mathbb{R}^2), \quad \forall\, t_0 > 0, \tag{2.20}$$

so that the maximal regularity of the heat equation in L^2-spaces (see Theorem X.11 stated in [9] and the quoted reference) provides the bound

$$f \in L^2(t_0,T;H^2(\mathbb{R}^2)) \cap L^\infty(t_0,T;H^1(\mathbb{R}^2)), \quad \forall\, t_0 > 0. \tag{2.21}$$

Thanks to (2.21), an interpolation inequality and the Sobolev inequality, we deduce that $\nabla_x f \in L^p((t_0,T)\times\mathbb{R}^2)$ for any $1 < p < \infty$, whence $\bar{\mathcal{K}}\cdot\nabla_x f \in L^p((t_0,T)\times\mathbb{R}^2)$, for all $t_0 > 0$. Then the maximal regularity of the heat equation in L^p-spaces (see Theorem X.12 stated in [9] and the quoted references) provides the bound

$$\partial_t f, \nabla_x f \in L^p((t_0,T)\times\mathbb{R}^2), \quad \forall\, t_0 > 0 \tag{2.22}$$

and then the Morrey inequality implies the Holderian regularity $f \in C^{0,\alpha}((t_0,T)\times\mathbb{R}^2)$ for any $0 < \alpha < 1$, and any $t_0 > 0$. Observing that the RHS term in (2.20) has then also an Holderian regularity, we deduce that

$$\partial_t f, \partial_x f, \partial^2_{x_i x_j} f \in C^{0,\alpha}_b((t_0,T)\times\mathbb{R}^2),\ \forall\, T, t_0; \quad 0 < t_0 < T < T^*,$$

thanks to the classical Holderian regularity result for the heat equation (see Theorem X.13 stated in [9] and the quoted references). We conclude by (weakly) differentiating in time and space the equation (2.20), observing that the resulting RHS term is still a function with Holderian regularity, applying again [9, Theorem X.13] and iterating the argument. □

Proof of Theorem 1.4. We split the proof into seven steps.

Step 1. The regularity of f has been yet established in Lemma 2.7.

Step 2. First, we claim that the free energy functional $\mathcal{F}$ is lsc in the sense that for any bounded sequence (f_n) of nonnegative functions of $L^1_2(\mathbb{R}^2)$ with same mass $M < 8\pi$ and such that $\mathcal{F}(f_n) \le A$ and $f_n \rightharpoonup f$ in $\mathcal{D}'(\mathbb{R}^2)$, there holds

$$0 \le f \in L^1_2(\mathbb{R}^2) \quad \text{and} \quad \mathcal{F}(f) \le \liminf \mathcal{F}(f_n). \tag{2.23}$$

The proof of (2.23) is classical (see [13, 14, 8]) and we just sketch it for the sake of completeness. Because of (1.8) and (1.9), we have $\mathcal{H}^+(f_n) + M_2(f_n) \le \mathscr{A}_T$ for any $n \ge 1$, and we may apply the Dunford-Pettis lemma which implies that $f_n \rightharpoonup f$ in $L^1(\mathbb{R}^2)$ weak. Now, introducing the splitting $\mathcal{F} = \mathcal{F}_\varepsilon + \mathcal{R}_\varepsilon$, $\mathcal{F}_\varepsilon = \mathcal{H} + \mathcal{V}_\varepsilon$, with

$$\begin{aligned} \mathcal{V}_\varepsilon(g) &:= \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x)\,g(y)\,\kappa(x-y)\,\mathbf{1}_{|x-y|\ge\varepsilon}, \\ \mathcal{R}_\varepsilon(g) &:= \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x)\,g(y)\,\kappa(x-y)\,\mathbf{1}_{|x-y|\le\varepsilon}, \end{aligned}$$

we clearly have that $\mathcal{F}_\varepsilon(f) \le \liminf \mathcal{F}_\varepsilon(f_n)$ because $\mathcal{H}$ is lsc and $\mathcal{V}_\varepsilon$ is continuous for the L^1 weak convergence. On the other hand, using the convexity inequality $uv \le u\log u + e^v$ $\forall\, u > 0$, $v \in \mathbb{R}$

and the elementary inequality $(\log u)_- \le u^{-1/2}\ \forall\, u \in (0,1)$, we have for $\varepsilon \in (0,1)$ and $\lambda > 1$

$$\begin{aligned} |\mathcal{R}_\varepsilon(g)| &= \frac{1}{4\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x)\, \mathbf{1}_{g(x)\le\lambda}\, g(y)\, (\log|x-y|)_-\, \mathbf{1}_{|x-y|\le\varepsilon} \\ &\quad + \frac{1}{4\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x)\, \mathbf{1}_{g(x)\ge\lambda}\, g(y)\, \log(|x-y|^{-1})\, \mathbf{1}_{|x-y|\le\varepsilon} \\ &\le \frac{\lambda}{4\pi} \int_{\mathbb{R}^2} g(y)\, dy \int_{|z|\le\varepsilon} (\log|z|)_-\, dz \\ &\quad + \frac{1}{4\pi} \int_{\mathbb{R}^2} g(x)\, \mathbf{1}_{g(x)\ge\lambda} \int \big\{ g(y) \log g(y) + |x-y|^{-1} \big\}\, dy \\ &\le \frac{\lambda}{3}\, M\, \varepsilon^{3/2} + \frac{1}{4\pi}\, \frac{\mathcal{H}^+(g)}{\log\lambda}\, \big\{ \mathcal{H}^+(g) + 2\pi\varepsilon \big\}, \end{aligned}$$

and we get that $\sup_n |\mathcal{R}_\varepsilon(f_n)| \to 0$ as $\varepsilon \to 0$ from which we conclude that $\mathcal{F}$ is lsc. Now, we easily deduce that the free energy identity (1.6) holds. Indeed, since f is smooth for positive time, for any fixed $t \in (0, T^*)$ and any given sequence (t_n) of positive real numbers which decreases to 0, we clearly have

$$\mathcal{F}(f(t_n)) = \mathcal{F}(t) + \int_{t_n}^t \mathcal{D}_\mathcal{F}(f(s))\, ds.$$

Then, thanks to the Lebesgue convergence theorem, the lsc property of $\mathcal{F}$ and the fact that $f(t_n) \rightharpoonup f_0$ weakly in $\mathcal{D}'(\mathbb{R}^2)$, we deduce from the above free energy identity for positive time that

$$\mathcal{F}(f_0) \le \liminf_{n\to\infty} \mathcal{F}(f(t_n)) \le \lim_{n\to\infty} \big\{ \mathcal{F}(t) + \int_{t_n}^t \mathcal{D}_\mathcal{F}(f(s))\, ds \big\} = \mathcal{F}(t) + \int_0^t \mathcal{D}_\mathcal{F}(f(s))\, ds.$$

Together with the reverse inequality (1.14) we conclude to (1.6).

Step 3. From now on, we assume that $M < 8\pi$ is subcritical and we prove the uniform in time estimates (1.18) and (1.19). We start with the a priori additional moment estimate (1.18). Because we will show the uniqueness of solution without using that additional moment estimates, these ones are rigorously justified thanks to a standard approximation argument, see [8] for details. Denoting g the rescaled solution (1.16) and

$$M_k := \int_{\mathbb{R}^2} g(x)\, |x|^k\, dx$$

we compute with $\Phi(x) = |x|^k$, $k \ge 2$, thanks to the antisymmetry of the kernel and the Holder inequality

$$\begin{aligned} \frac{d}{dt} M_k &= k^2\, M_{k-2} - k\, M_k - \frac{1}{2\pi} \int_{\mathbb{R}^2} \Phi'(x) g(t,x) \int_{\mathbb{R}^2} g(t,y)\, \frac{x-y}{|x-y|^2}\, dydx \\ &= k^2\, M_{k-2} - k\, M_k \\ &\quad - \frac{1}{4\pi} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(t,y) g(t,x)\, (\Phi'(x) - \Phi'(y)) \frac{x-y}{|x-y|^2}\, dydx \\ &\le k^2\, M^{2/k}\, M_k^{1-2/k} - k\, M_k, \end{aligned}$$

from which we easily conclude that (1.18) holds.

Step 4. Defining the rescaled free energy $\mathcal{E}(g)$ and the associated dissipativity of rescaled free energy $\mathcal{D}(g)$ by

$$\begin{aligned} \mathcal{E}(g) &:= \int g(1 + \log g) + \frac{1}{2} \int g|x|^2 + \frac{1}{4\pi} \iint g(x) g(y) \log|x-y|\, dxdy \\ \mathcal{D}_\mathcal{E}(g) &:= \int g |\nabla(\log g + \frac{|x|^2}{2} + \kappa * g)|^2, \end{aligned}$$

we have that any solution g to the rescaled equation (1.17) satisfies

$$\frac{d}{dt}\mathcal{E}(g) + \mathcal{D}_{\mathcal{E}}(g) = 0 \quad \text{on} \quad [0,\infty). \tag{2.24}$$

On the one hand, as for (1.8), the following functional inequality

$$\int g \log g + \frac{1}{2}\int g|x|^2 \le C_3(M)\,\mathcal{E}(g) + C_4(M) \quad \forall\, g \in L^1_+(\mathbb{R}^2) \tag{2.25}$$

holds, and together with (1.9), we find

$$\int g(\log g)_+ + \frac{1}{4}\int g|x|^2 \le C_3(M)\,\mathcal{E}(g) + C_7 \quad \forall\, g \in L^1_+(\mathbb{R}^2), \tag{2.26}$$

where $C_7 := C_4 + C_5$. As a consequence of (2.24) and (2.26), we get the uniform in time upper bound on the rescaled free energy for the solution g of (1.17)

$$\sup_{t\ge 0} \int g_t(\log g_t)_+ + \frac{1}{4}\int g_t|x|^2 \le C_3(M)\,\mathcal{E}(f_0) + C_7(M). \tag{2.27}$$

Step 5. As in the proof of Lemma 2.6, we easily get that the rescalled solution g of the rescaled equation (1.17) satisfies for any $p \in [2,\infty)$

$$\begin{aligned}\frac{d}{dt}\|g\|^p_{L^p} + \frac{4}{p'}\|\nabla(g^{p/2})\|^2_{L^2} &= 2\,(p-1)\,\|g\|^p_{L^p} + (p-1)\|g\|^{p+1}_{L^{p+1}} \\ &\le 2\,(p-1)\,M + 3(p-1)\|g\|^{p+1}_{L^{p+1}}.\end{aligned}$$

Writing $s = s\wedge A + (s-A)_+$, so that $s^{p+1} \le 2^{p+1}(s\wedge A)^{p+1} + 2^{p+1}(s-A)^{p+1}_+$, and using the Gagliardo-Nirenberg-Sobolev type inequality (2.5) in order to get

$$\begin{aligned}\int (g-A)^{p+1}_+ &\le C_p \int |\nabla(g-A)^{p/2}_+|^2 \int (g-A)_+ \\ &\le C_p \int |\nabla(g^{p/2})|^2\,\frac{\mathcal{H}^+(g)}{\log A}\end{aligned}$$

for any $A > 1$, we deduce

$$\begin{aligned}\frac{d}{dt}\|g\|^p_{L^p} + \|\nabla(g^{p/2})\|^2_{L^2} &\le 2pM + 3p2^{p+1}A^p\,M + 3p2^{p+1}\int (g-A)^{p+1}_+ \\ &\le C_8(M,p,A) + C_p\,\frac{\mathcal{H}_+(g)}{\log A}\|\nabla(g^{p/2})\|^2_{L^2}.\end{aligned}$$

Taking A large enough, we obtain

$$\frac{d}{dt}\|g\|^p_{L^p} + \frac{1}{2}\|\nabla(g^{p/2})\|^2_{L^2} \le C_9(M,p,\mathcal{E}_0). \tag{2.28}$$

Using the Nash inequality

$$\|w\|^2_{L^2(\mathbb{R}^2)} \le C_N\,\|w\|_{L^1(\mathbb{R}^2)}\,\|\nabla w\|_{L^2(\mathbb{R}^2)}$$

with $w := g^{p/2}$, we conclude with

$$\frac{d}{dt}\|g\|^p_{L^p} + \frac{1}{C_N^2}\,\|g\|^{-p}_{L^{p/2}}\,\|g\|^{2p}_{L^p} \le C_9(M,p,\mathcal{E}_0).$$

Defining $u(t) := \|g(t)\|^p_{L^p}$ first with $p = 2$, so that $\|g(t)\|^{p/2}_{L^{p/2}} = M$, we recognize the classical nonlinear ordinary differential inequality

$$u' + c\,u^2 \le C \quad \text{on} \quad (0,\infty),$$

for some constants c and C (which only depend on M and $\mathcal{E}_0$) from which we deduce the bound

$$\forall\,\varepsilon > 0\ \exists\,\mathcal{C} = \mathcal{C}(\varepsilon,c,C) \quad \sup_{t\ge\varepsilon}\|g(t)\|^p_{L^p} \le \mathcal{C}, \tag{2.29}$$

with $p = 2$. In order to get the same uniform estimate (2.29) in all the Lebesgue spaces L^p, $p \in (2, \infty)$, we may proceed by iterating the same argument as above with the choice $p = 2^k$, $k \in \mathbb{N}^*$. Coming back to (2.28) with $p = 2$, we also deduce that for any $\varepsilon, T > 0$ there exists $\mathcal{C} = \mathcal{C}(\varepsilon, T, \mathcal{E}_0)$ so that

$$\sup_{t_0 \ge \varepsilon} \int_{t_0}^{t_0+T} \|\nabla g(s)\|^2_{L^2(\mathbb{R}^2)}\, ds \le \mathcal{C}.$$

Step 6. The function $g_i := \partial_{x_i} g$ satisfies

$$\partial_t g_i - \Delta g_i - \nabla(x g_i) = g_i + 2\, g\, g_i - \partial_{x_i}(\nabla u \cdot \nabla g),$$

from which we deduce that

$$\begin{aligned} \frac{d}{dt}\int |g_i|^p + p(p-1)\int |\nabla g_i|^2 |g_i|^{p-2} \le \\ \le (3p-2)\int |g_i|^p + 2p\int g\,|g_i|^p + p\int \partial_{x_i}(\nabla u \cdot \nabla g)\, g_i\, |g_i|^{p-2}. \end{aligned} \tag{2.30}$$

For $p = 2$, we have for any $t \ge \varepsilon$

$$\begin{aligned} \mathcal{T}(t) \quad &:= \quad 4\int g\,|g_i|^2 + 2\int \partial_{x_i}(\nabla u \cdot \nabla g)\, g_i \\ &\le \quad 4\|g\|_{L^3}\,\|g_i\|^2_{L^3} + 2\,\|\nabla u \cdot \nabla g\|^2_{L^2} + \frac{1}{2}\,\|\partial_i g_i\|^2_{L^2} \end{aligned}$$

thanks to the Holder inequality, an integration by part and the Young inequality. Next, we have for any $t \ge \varepsilon$

$$\mathcal{T}(t) \quad \le \quad C_1\,\|g_i\|^{4/3}_{L^2}\|\nabla g_i\|^{2/3}_{L^2} + C_2\|\nabla g\|^2_{L^2} + \frac{1}{2}\,\|\nabla g_i\|^2_{L^2}$$

where we have used the classical Gagliardo-Niremberg inequality (see (85) in [9, Chapter IX] and the quoted references)

$$\|w\|_{L^r(\mathbb{R}^2)} \le C_{GN}\,\|w\|^{1-a}_{L^q(\mathbb{R}^2)}\,\|\nabla w\|^a_{L^2(\mathbb{R}^2)}, \quad a = 1 - \frac{q}{r}, \;\; 1 \le q \le r < \infty, \tag{2.31}$$

with $w := g_i$, $r = 3$, $q = 2$, the uniform bound established in step 5 and the fact that $\nabla u = -\mathcal{K} * g \in L^\infty((\varepsilon, \infty) \times \mathbb{R}^2)$ thanks to the same argument as in the proof of Lemma 2.7. Last, by the Young inequality we get for any $t \ge \varepsilon$

$$\mathcal{T}(t) \quad \le \quad \frac{2}{3}\,C_1^{3/2}\,\|g_i\|^2_{L^2} + \frac{1}{3}\,\|\nabla g_i\|^2_{L^2} + C_2\|\nabla g\|^2_{L^2} + \frac{1}{2}\,\|\nabla g_i\|^2_{L^2},$$

from which we deduce from (2.30)

$$\frac{d}{dt}\int |g_i|^2 + \int |\nabla g_i|^2 \le C_3\,\|\nabla g\|^2_{L^2} \quad \text{on} \quad (\varepsilon, \infty),$$

with $C_3 := 4 + \frac{2}{3}\,C_1^{3/2} + C_2$. Remarking that for any fixed $\varepsilon \in (0,1)$ and any $t_1 \ge 2\varepsilon$, we may define $t_0 \in (t_1 - \varepsilon, t_1)$ so that

$$\|\nabla g(t_0)\|^2_{L^2} \quad = \quad \inf_{(t_1-\varepsilon, t_1)} \|\nabla g\|^2_{L^2} \le \frac{2}{\varepsilon}\int_{t_1-\varepsilon}^{t_1} \|\nabla g(s)\|^2_{L^2}\, ds \le C_4$$

thanks to the bound established at the end of step 5, we deduce from the above differential inequality that

$$\|g_i(t_1)\|^2_{L^2} \le \|g_i(t_0)\|^2_{L^2} + C_3\int_{t_0}^{t_1} \|\nabla g(s)\|^2_{L^2}\, ds \le C_5,$$

where again $C_5 := C_4 + C_3\, C_4\, \varepsilon/2$ only depends on ε, M and $\mathcal{E}_0$. Coming back to the above differential inequality again, we easily conclude that for any $\varepsilon > 0$, there exists a constant $\mathcal{C}_\varepsilon = \mathcal{C}(\varepsilon, M, \mathcal{E}_0)$ so that

$$\sup_{t \ge \varepsilon}\left\{\|\nabla g(t)\|^2_{L^2} + \int_t^{t+1} \|D^2 g(s)\|^2_{L^2}\right\} \le \mathcal{C}_\varepsilon. \tag{2.32}$$

Step 7. Starting from the differential inequality (2.30) for $p \in (2,\infty)$ and using the Morrey-Sobolev inequalities

$$\|g\|_{L^\infty} \le C\,\|g\|_{H^2} \quad \text{and} \quad \|D^2u\|_{L^\infty} \le C\,\|D^2u\|_{H^2} \le C\,\|g\|_{H^2},$$

we easily get

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int |\nabla g|^p &\le C\,(1+\|g\|_{L^\infty}+\|D^2u\|_{L^\infty})\int |\nabla g|^p \\ &\le C\,(1+\|g\|_{H^2})\int |\nabla g|^p \quad \text{on} \quad (\varepsilon,\infty),\end{aligned}$$

from which we deduce for any $t_1 \ge t_0 \ge \varepsilon$

$$\|\nabla g(t_1)\|_{L^p} \le \|\nabla g(t_0)\|_{L^p}\,\exp\Big(\int_{t_0}^{t_1} C\,(1+\|g(s)\|_{H^2})\,ds\Big).$$

Now, arguing similarly as in step 6, we deduce from the above time integral inequality, the Sobolev inequality $\|\nabla g\|_{L^p} \le C_p\,\|g\|_{H^2}$ for $p \in [2,\infty)$ and the already established bound (2.32), that for any $\varepsilon > 0$, there exists a constant $\mathcal{C}_\varepsilon = \mathcal{C}(\varepsilon, M, \mathcal{E}_0, p)$ so that

$$\sup_{t\ge\varepsilon}\|\nabla g(t)\|_{L^p} \le \mathcal{C}_\varepsilon. \tag{2.33}$$

Step 8. Iterating twice the arguments we have presented in steps 6 and 7, it is no difficult to prove

$$\sup_{t\ge\varepsilon}\|g(t,.)\|_{W^{3,p}} \le \mathcal{C} \quad \forall\,\varepsilon>0,\ p\in[2,\infty),$$

for some constant $\mathcal{C} = \mathcal{C}(\varepsilon, p, M, \mathcal{F}_0, M_{2,0})$ from which (1.19) immediately follows. □

3. Uniqueness - Proof of Theorem 1.3

We split the proof into two steps. We recall that from Theorem 1.5 we already know that $\|f\|_{L^2} \in C^1(0,T)$ and $\|f\|_{L^p} \in L^\infty(t_0,T)$ for any $0 < t_0 < T < T^*$ and any $p \in [1,\infty]$.

Step 1. We establish our new main estimate, namely that any weak solution satisfies

$$t^{1/4}\|f(t,.)\|_{L^{4/3}} \to 0 \quad \text{as } t \to 0. \tag{3.1}$$

First, from (1.1) and the regularity of the solution, we have

$$\frac{d}{dt}\|f\|^2_{L^2} + 2\|\nabla_x f\|^2_{L^2} = \|f\|^3_{L^3} \quad \text{on} \quad (0,T).$$

As in the proof of Lemma 2.6, we deduce that

$$\frac{d}{dt}\|f\|^2_{L^2} + \frac{1}{2}\|\nabla_x f\|^2_{L^2} \le A^2 M \quad \text{on} \quad (0,T)$$

for A large enough. Thanks to Nash inequality

$$\|f\|^2_{L^2} \le C\,M\,\|\nabla f\|_{L^2},$$

we thus obtain

$$\frac{d}{dt}\|f\|^2_{L^2} + c_M\|f\|^4_{L^2} \le A^2 M \quad \text{on} \quad (0,T).$$

It is a classical trick of ordinary differential inequality to deduce that there exists a constant $\mathcal{C}$ (which only depends on c_M, A^2M and T) so that

$$t\,\|f(t,.)\|^2_{L^2} \le \mathcal{C} \quad \forall\,t\in(0,T). \tag{3.2}$$

We now prove (3.1) from (3.2) and an interpolation argument. On the one hand, we use the Holder inequality in order to get

$$\begin{aligned}\int f^{4/3} &= \int f^{2/3}\,\langle \log f\rangle^{2/3}\,f^{2/3}\,\langle \log f\rangle^{-2/3} \\ &\le \Big(\int f\,\langle \log f\rangle\Big)^{2/3}\Big(\int f^2\,\langle \log f\rangle^{-2}\Big)^{1/3},\end{aligned}$$

or in other words

$$\|f\|_{L^{4/3}} \le C \left(\int f^2 \, \langle \log f \rangle^{-2} \right)^{1/4}. \tag{3.3}$$

On the other hand, we observe that

$$\begin{aligned} t \int f^2 \, \langle \log f \rangle^{-2} &\le t \int_{f \le R} f^2 \, \langle \log f \rangle^{-2} + t \int_{f \ge R} f^2 \, \langle \log f \rangle^{-2} \\ &\le t \, \frac{R}{\langle \log R \rangle^2} \int_{f \le R} f + \frac{t}{\langle \log R \rangle^2} \int_{f \ge R} f^2 \\ &\le t \, \frac{MR}{\langle \log R \rangle^2} + \frac{K}{\langle \log R \rangle^2} \le \frac{M+K}{\langle \log 1/t \rangle^2} s \to 0, \end{aligned} \tag{3.4}$$

where we have used the mass conservation and the estimate (3.2) in the third line and we have chosen $R := t^{-1}$ in order to get the last inequality. We conclude to (3.1) by gathering (3.3) and (3.4).

Step 3. Conclusion. We consider two weak solutions f_1 and f_2 to the Keller-Segel equation (1.1) that we write in the mild form

$$f_i(t) = e^{t\Delta} f_i(0) + \int_0^t e^{(t-s)\Delta} \nabla (V_i(s) \, f_i(s)) \, ds, \quad V_i = \mathcal{K} * f_i,$$

where $e^{t\Delta}$ stands for the heat semigroup defined in $\mathbb{R}^2$ by $e^{t\Delta} f := \gamma_t * f$, $\gamma_t(x) := (2\pi t)^{-1} \, \exp(-|x|^2/(2t))$. When we assume $f_1(0) = f_2(0)$, the difference $F := f_2 - f_1$ satisfies

$$F(t) = \int_0^t \nabla \cdot e^{(t-s)\Delta} (V_2(s) \, F(s)) \, ds + \int_0^t \nabla \cdot e^{(t-s)\Delta} (W(s) \, f_1(s)) \, ds = I_1 + I_2,$$

with $W := V_2 - V_1$. For any $t > 0$, we define

$$Z_i(t) := \sup_{0<s\le t} s^{1/4} \, \|f_i(s)\|_{L^{4/3}}, \quad \Delta(t) := \sup_{0<s\le t} s^{1/4} \, \|F(s)\|_{L^{4/3}}.$$

We observe that thanks to the Hardy-Littlewood-Sobolev inequality (2.11), we have

$$\|\mathcal{K} * g\|_{L^4} \le C \, \|g\|_{L^{4/3}}, \tag{3.5}$$

and that the regularizing effect of the heat equation reads

$$\|\nabla (e^{t\Delta} g)\|_{L^{4/3}} \le \|\nabla \gamma_t\|_{L^{4/3}} \, \|g\|_{L^1} \le \frac{C}{t^{3/4}} \, \|g\|_{L^1}. \tag{3.6}$$

We then compute

$$\begin{aligned} J_1 &:= t^{1/4} \, \|I_1(t)\|_{L^{4/3}} \\ &\le t^{1/4} \int_0^t \|\nabla \cdot e^{(t-s)\Delta} (V_2(s) \, F(s))\|_{L^{4/3}} \, ds \\ &\le t^{1/4} \int_0^t \frac{C}{(t-s)^{3/4}} \, \|V_2(s) \, F(s)\|_{L^1} \, ds \\ &\le t^{1/4} \int_0^t \frac{C}{(t-s)^{3/4}} \, \|V_2(s)\|_{L^4} \, \|F(s)\|_{L^{4/3}} \, ds \\ &\le t^{1/4} \int_0^t \frac{C}{(t-s)^{3/4}} \, \|f_2(s)\|_{L^{4/3}} \, \|F(s)\|_{L^{4/3}} \, ds \\ &\le \int_0^t \frac{C}{(t-s)^{3/4}} \, \frac{t^{1/4}}{s^{1/2}} \, ds \, Z_2(t) \, \Delta(t) \\ &= \int_0^1 \frac{C}{(1-u)^{3/4}} \, \frac{du}{u^{1/2}} \, Z_2(t) \, \Delta(t), \end{aligned}$$

where we have used the regularizing effect of the heat equation (3.6) at the third line, the Holder inequality at the fourth line, the Hardy-Littlewood-Sobolev inequality (3.5) at the fifth line.

Similarly, we have

$$
\begin{aligned}
J_2 &:= t^{1/4}\,\|I_2(t)\|_{L^{4/3}} \\
&\le \int_0^1 \frac{C}{(1-u)^{3/4}}\,\frac{du}{u^{1/2}}\,\Delta(t)\,Z_1(t).
\end{aligned}
$$

All together, we conclude thanks to (3.1) with the inequality

$$
\Delta(t) \quad \le \quad (Z_1(t)+Z_2(t)) \int_0^1 \frac{du}{(1-u)^{3/4}}\,\Delta(t) \le \frac{1}{2}\Delta(t)
$$

for $t \in (0,T)$, $T>0$ small enough, which in turn implies $\Delta(t) \equiv 0$ on $[0,T)$. □

4. Self-similar behaviour - Proof of Theorem 1.5

In this section we restrict ourself to the subcritical case $M < 8\pi$ and we investigate the self-similar long time behaviour of generic solutions to the KS equation or more precisely, and equivalently, we investigate the long time convergence to the self-similar profile of the rescaled solution g defined through (1.16). We start by recalling some known results on the self-similar profile and its stability. First, we consider the stationary problem (1.21).

Theorem 4.1. *For any $M \in (0,8\pi)$, there exists a unique nonnegative self-similar profile $G = G_M$ of mass M with finite second moment and finite entropy of the KS equation* (1.1)*, it is the unique solution to the stationary problem* (1.21) *and it satisfies*

$$
G \in C^\infty(\mathbb{R}^2), \quad e^{-(1+\varepsilon)|x|^2/2 + C_{1,\varepsilon}} \le G \le e^{-(1-\varepsilon)|x|^2/2+C_{2,\varepsilon}},
$$

for any $\varepsilon \in (0,1)$ and some constants $C_{i,\varepsilon} \in (0,\infty)$. Moreover, the self-similar profile G is characterized as the unique solution to the optimization problem

$$
\tilde g \in \mathcal{Z}_M, \quad \mathcal{E}(\tilde g) = \min_{g \in \mathcal{Z}_M} \mathcal{E}(g), \tag{4.1}
$$

where $\mathcal{Z}_M := \{g \in L^1_+ \cap L^1_2;\ M_0(g) = M\}$, as well as the unique function $g \in \mathcal{Z}_M$ such that $\mathcal{D}_{\mathcal{E}}(g) = 0$.

That theorem follows by a combination of known results. First, as a consequence of the fact that $U := -\mathcal{K} * G$ satisfies (1.23) together with the elementary inequality

$$
\forall\, x \in \mathbb{R}^2 \quad \big|U(x) + \frac{M}{2\pi}\,(\log|x|)_+\big| \le C, \tag{4.2}
$$

where C only depends on M, $M_2(G)$ and $\mathcal{H}(G)$ (see [8, Lemma 23] and the argument presented in order to bound $\mathcal{R}_\varepsilon(g)$ in step 2 of the proof of Theorem 1.4), and the Naito's variant [33] of the famous Gidas, Ni, Niremberg radial symmetry result on solutions to Poisson type equations, it has been established in [8, Lemma 25] that U is radially symmetric. It follows that any self-similar profile G is radially symmetric. On the other hand, the uniqueness of radially symmetric self-similar profiles has been proved in [5, Theorem 3.1] (see also [15, Theorem 1.2]) and that concludes the proof of the uniqueness of the solution to the stationary problem (1.21). The smoothness property is established in [8, Lemma 25] and the behaviour for large values of $|x|$ is a immediate consequence of (4.2). It is clear from (2.24) that any solution to the minimization problem (4.1) also satisfies $\mathcal{D}_{\mathcal{E}}(\tilde g) = 0$ which in turns implies that $\log \tilde g + |x|^2/2 + \kappa * \tilde g = 0$ and then $\tilde g$ is a solution to the stationary problem (1.21).

Second, the profile G is a stationary solution to the evolution equation (1.17) and the associated linearized equation reads

$$
\partial_t h = \Lambda h := \mathrm{div}_x\big(\nabla h + x\,h + (\mathcal{K} * G)\,h + (\mathcal{K} * h)\,G\big).
$$

We briefly explain the spectral analysis of Λ in the Hilbert space E *of self-adjointeness* performed in [17]. To be consistent with our notations on weighted Lebesgue space, we define $E = L^2(G^{-1/2})$

as the Hilbert space associated to scalar product $(\cdot,\cdot)_E$ defined by

$$(f,g)_E := \int_{\mathbb{R}^2} f\,\bar g\, G^{-1}\,dx, \quad \|f\|_E^2 := (f,f)_E.$$

We also define $h_{0,0} := \partial G_M/\partial M$ and

$$E_0^\perp := \{f \in E;\ (f,h_{0,0})_E = 0\}.$$

It has been shown in [17, Lemma 8] that $h_{0,0}$ is the first eigenfunction of the operator Λ associated to the first eigenvalue $\lambda = 0$. Moreover, defining the quadratic form

$$Q_1[f] := \int_{\mathbb{R}^2} f^2\, G^{-1}\,dx + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)\,f(y)\,\kappa(x-y)\,dxdy,$$

it has been shown in [17, Lemma 10] that Q_1 is nonnegative, that $Q_1[h_{0,0}] = 0$ and that

$$Q_1[f] = 0 \text{ and } (f,h_{0,0})_E = 0 \quad \text{imply} \quad f = 0.$$

As a consequence $Q_1[\cdot]$ defines an Hilbert norm on $E_0^\perp$ which is equivalent to the initial norm $\|\cdot\|_E$, we denote by $\langle\cdot,\cdot\rangle$ the associated scalar product. That new norm is suitable for exhibiting a spectral gap for the operator Λ and to make the stability analysis of the associated semigroup $e^{t\Lambda}$.

Theorem 4.2 ([17])**.** *For any $g \in E_0^\perp$ which belongs to the domain of Λ, there holds*

$$\langle \Lambda g, g\rangle \le -Q_1[g]. \tag{4.3}$$

Moreover, there exists $a^* < -1$ and $C > 0$ so that

$$\|e^{t\Lambda}h - e^{-t}\,\Pi_1 h - \Pi_0 h\|_E \le C\,e^{a^* t}\,\|h - (\Pi_1+\Pi_0)h\|_E \qquad \forall\, t \ge 0,\ \forall\, h \in E, \tag{4.4}$$

where Π_0 is the (orthogonal) projection on Vect$(h_{0,0})$ and Π_1 is the (orthogonal) projection on Vect$(h_{1,1},h_{1,2})$ where $h_{1,i} := \partial_{x_i} G$.

Inequality (4.3) is nothing but [17, Theorem 15] and (4.4) is a consequence of the fact the spectrum of Λ is discrete and included in the real line and that the second (larger) eigenvalue of Λ is -1, see [17, Section 4].

Our first main result in this section is an linearized stability result in a large space $\mathcal{E}$, namely we consider

$$\mathcal{E} \;:=\; L^{4/3}_k(\mathbb{R}^2), \quad k > 3/2.$$

We consider that space because it is the larger space in terms of moment decay in which we are able to prove a (optimal) spectral gap on the linearized semigroup. For such a general Banach space framework and the associated spectral analysis issue, we adopt the classical notations of [35, 26] used in [25], for more details we refer to [25, Section 2.1] and the references therein (in particular [26, 35, 22]).

Theorem 4.3. *For any $k > 3/2$ and any $a > \bar a := \max(a^*, a(k))$, $a(k) := 1/2 - k$ (so that $a(k) < -1$) there exists a constant $C_{k,a}$ so that*

$$\|e^{t\Lambda}h - e^{-t}\,\Pi_1 h - \Pi_0 h\|_{\mathcal{E}} \le C\,e^{at}\,\|h - \Pi_1 h - \Pi_0 h\|_{\mathcal{E}} \quad \forall\, t \ge 0,\ \forall\, h \in \mathcal{E},$$

where again Π_0 stands for projection on the eigenspace Vect$(h_{0,0})$ associated to the eigenvalue 0 *and Π_1 stands for projection on the eigenspace Vect$(h_{1,1},h_{1,2})$ associated to the eigenvalue -1. Both operators are defined through the formula (see* [25, Section 2.1] *or better* [26, III-(6.19)]*)*

$$\Pi_\xi := -\frac{1}{2i\pi}\int_{|z-\xi|=r} (\Lambda - z)^{-1}\,dz, \quad \xi = 0,-1,\ \ r > 0\ (\textit{small enough}).$$

The proof is a straightforward adaptation of arguments of *"functional extension of semigroup spectral gap estimates"* developed in [25] for the Fokker-Planck equation.

Lemma 4.4. *For any $k \ge 0$ fixed, there exists a constant C_k such that for any $g \in D(\Lambda)$, there holds*

$$\langle \Lambda g, g^\dagger \rangle_{\mathcal E} \le C_k \int |g|^{4/3} \langle x \rangle^{\frac43 k - 1} + \big(\frac12 - k\big) \int |g|^{4/3} \langle x \rangle^{\frac43 k}, \tag{4.5}$$

where $g^\dagger := \bar g \, |g|^{-2/3}$ (here $\bar g$ stands for the complex conjugate of g).

Proof of Lemma 4.4. For the sake of simplicity we assume $g \ge 0$ so that $g^* = g^{1/3}$, we set $\ell := 4k/3$, we write

$$\langle \Lambda g, g^\dagger \rangle_{\mathcal E} = \int_{\mathbb R^2} (\Lambda g) \, g^{1/3} \, \langle x \rangle^\ell = T_1 + ... + T_4,$$

and we compute each term T_i separately. First, performing two integrations by part, we have

$$\begin{aligned} T_1 &:= \int_{\mathbb R^2} (\Delta g) \, g^{1/3} \, \langle x \rangle^\ell \, dx \\ &= -\frac13 \int_{\mathbb R^2} |\nabla g|^2 \, g^{-2/3} \, \langle x \rangle^\ell \, dx + \frac34 \int_{\mathbb R^2} g^{4/3} \, \Delta \langle x \rangle^\ell \, dx. \end{aligned}$$

Second, performing one integration by part, we have

$$\begin{aligned} T_2 &:= \int_{\mathbb R^2} (2g + x \cdot \nabla g) \, g^{1/3} \, \langle x \rangle^\ell \, dx \\ &= \int_{\mathbb R^2} \big\{ \frac12 \langle x \rangle^{\ell - 2} + \big(\frac12 - k\big) \langle x \rangle^\ell \big\} \, g^{4/3} \, dx. \end{aligned}$$

Third, performing one integration by part, we have

$$\begin{aligned} T_3 &:= \int_{\mathbb R^2} \big(2 \, G g + (\mathcal K * G) \cdot \nabla g\big) \, g^{1/3} \, \langle x \rangle^\ell \, dx \\ &= \frac54 \int_{\mathbb R^2} G \, g^{4/3} \, \langle x \rangle^\ell \, dx - \frac34 \int_{\mathbb R^2} g^{4/3} \, (\mathcal K * G) \cdot \nabla_x \langle x \rangle^\ell \, dx \\ &\le C \int_{\mathbb R^2} g^{4/3} \, \langle x \rangle^{\ell - 1} \, dx, \end{aligned}$$

for some constant $C \in (0, \infty)$.
Fourth and last, thanks to the Holder inequality and the Hardy-Littlewood-Sobolev inequality (3.5), we have

$$\begin{aligned} T_4 &:= \int_{\mathbb R^2} (\mathcal K * g) \cdot \nabla G \, g^{1/3} \, \langle x \rangle^\ell \, dx \\ &\le \|\nabla G \, \langle x \rangle^k\|_\infty \, \|g\|^{1/3}_{L^{4/3}} \, \|\mathcal K * g\|_{L^4} \le C \, \|g\|^{4/3}_{L^{4/3}}. \end{aligned}$$

Gathering all these estimates, we get (4.5). □

We define

$$\mathcal A g := N \chi_R \, g \quad \text{and} \quad \mathcal B g = \Lambda g - \mathcal A \, g.$$

From lemma 4.4 we easily have that for any $a > a(k)$ there exists M and R large enough so that $\mathcal B - a$ is dissipative in $\mathcal E$ (see [35, Chapter I, Definition 4.1]) in the sense that

$$\langle (\mathcal B - a) \, g, g^* \rangle_{\mathcal E} \le 0,$$

where $g^* := \bar g \, |g|^{-2/3} \, \|g\|^{2/3}_{\mathcal E}$.

Lemma 4.5. *There exists some constants $C > 0$ and $b \in \mathbb R$ so that*

$$\|e^{t \mathcal B} h\|_{L^2_1} \le \frac{C \, e^{bt}}{t^{1/2}} \, \|h\|_{L^{4/3}_1} \quad \forall \, h \in L^{4/3_1}, \ \forall \, t > 0. \tag{4.6}$$

Proof of Lemma 4.5. The proof of the hypercontractivity property as stated in Lemma 4.5 is a classical consequence of the Gagliardo-Niremberg inequality. For the sake of completeness we sketch it. Arguing similarly as in the proof of Lemma 2.6 and Lemma 4.4 and denoting $h_t := e^{t\mathcal{B}}h$, we compute

$$\frac{1}{2}\frac{d}{dt}\int |h_t|^2\langle x\rangle^2 = -\int |\nabla h_t\langle x\rangle|^2 + \int h_t\,(\mathcal{K}*h_t)\cdot\nabla G\langle x\rangle^2$$
$$+\int |h_t|^2\Big\{1-|\nabla\langle x\rangle|^2+\langle x\rangle^2\big(\frac{3}{2}G-N\,\chi_R\big)+\langle x\rangle\,\mathcal{K}*G\cdot\nabla\langle x\rangle\Big\}.$$

On the one hand, thanks to the Gagliardo-Niremberg inequality (2.31) with $q=4/3$, $r=2$ and $a=1/3$, we know that

$$\int |\nabla(h\langle x\rangle)|^2 \ge C_{GN}^{-6}\Big(\int |h\,\langle x\rangle|^2\Big)^3\Big(\int |h\,\langle x\rangle|^{4/3}\Big)^{-3}.$$

On the other hand, introducing the splitting $\mathcal{K}=\mathcal{K}_0+\mathcal{K}_\infty$ with $\mathcal{K}_0 := \mathcal{K}\,\mathbf{1}_{|z|\le 1}$ and $\mathcal{K}_\infty := \mathcal{K}\,\mathbf{1}_{|z|\ge 1}$ and using the Holder inequality and the Young inequality, we have

$$\begin{aligned}\int h_t\,(\mathcal{K}*h_t)\cdot\nabla G\langle x\rangle^2 &\le \|\nabla G\,\langle x\rangle^2\|_{L^\infty}\,\|h\|_2\,\|\mathcal{K}_0*h\|_{L^2}+\|\nabla G\,\langle x\rangle^2\|_{L^2}\,\|h\|_{L^2}\,\|\mathcal{K}_\infty*h\|_{L^\infty}\\ &\le C\,(\|\mathcal{K}_0\|_{L^1}\,\|h\|_{L^2}^2+\|h\|_{L^2}\,\|\mathcal{K}_0\|_{L^3}\|h\|_{L^{3/2}})\\ &\le C\,\|h\|_{L^2_1}^2.\end{aligned}$$

We also bound the last term by $C\,\|h_t\|_{L^2_1}^2$. All together and using the notations $X(t) := \|h_t\|_{L^2_1}^2$ and $Y(t) := \|h_t\|_{L^{4/3}_1}^{4/3}$ and the fact that $Y(t)\le Y(0)$ thanks to Lemma 4.4, we get

$$X' \le -\alpha\,(X/Y(0))^3+\beta\,X$$

for some constants $\alpha,\beta>0$. The estimate (4.6) is then a classical consequence to the above differential inequality. □

Proof of Theorem 4.3. The proof follows by a straightforward application of [25, Theorem 2.13] using Theorem 4.2, Lemma 4.4, Lemma 4.5 and [29, Lemma 2.4]. □

Before going to the proof of Theorem 1.5 we present two results that will be useful during the proof of that Theorem.

Lemma 4.6. *For any $M\in(0,8\pi)$, $k'>2\ge k>3/2$, $M_{k'}\ge(k'-1)^{k'/2}\,M$ and $\mathcal{C}>0$, there exists an increasing function $\eta:[0,\infty)\to[0,\infty)$, $\eta(0)=0$, $\eta(u)>0$ for any $u>0$, such that*

$$\forall\,g\in\mathcal{Z}\qquad \mathcal{D}(g)\ge\eta(\|g-G\|_{L^{4/3}_k}) \tag{4.7}$$

where

$$\mathcal{Z} := \{g\in L^1_+(\mathbb{R}^2),\ \int g=M, \int g|x|^{k'}\le M_{k'},\ \|g\|_{W^{2,\infty}}\le\mathcal{C}\}.$$

Proof of Lemma 4.6. We proceed by contradiction. If (4.7) does not hold, there exists a sequence (g_n) in $\mathcal{Z}$ and a real $\delta>0$ such that

$$\mathcal{D}(g_n)\to 0 \text{ as } n\to 0 \quad\text{and}\quad \|g-G\|_{L^{4/3}_k}\ge\delta.$$

Therefore, on the one hand, there exists $\bar g\in\mathcal{Z}$ such that, up to the extraction of the subsequence, there holds $g_n\to\bar g$ strongly in $L^{4/3}_k$, so that $\|\bar g-G\|_{L^{4/3}_k}\ge\delta$. One the other hand, using that $(\nabla\sqrt{g_n}+\sqrt{g_n}\,\mathcal{K}*g_n)$ is bounded in L^2, that implies that $2\nabla\sqrt{g_n}+\sqrt{g_n}\,\mathcal{K}*g_n\rightharpoonup 2\nabla\sqrt{\bar g}+\sqrt{\bar g}\,\mathcal{K}*\bar g$ weakly in $L^2(\mathbb{R}^2)$ and then

$$\mathcal{D}(\bar g) := \|2\nabla\sqrt{\bar g}+\sqrt{\bar g}\,\mathcal{K}*\bar g\|_{L^2}^2\le\liminf\mathcal{D}(g_n)=0.$$

We easily conclude thanks to the mass condition $M_0(\bar g)=M$ and the uniqueness Theorem 4.1 that $\bar g=G$. That is our contradiction. □

Lemma 4.7. *Define* $\mathcal{E}_2 := R(I-\Pi_0-\Pi_1)$ *the supplementary linear submanifold to the eigenspaces associated to the eigenvalues* 0 *and* -1. *There exists a norm* $|||\cdot|||$ *on* $\mathcal{E}_2$ *equivalent to the initial one* $\|\cdot\|_{\mathcal{E}}$ *so that*

$$\frac{d}{dt}|||e^{t\Lambda}f|||^2 \le -2\,|||e^{t\Lambda}f|||^2 \quad \forall\, t\ge 0,\ \forall\, f\in\mathcal{E}_2. \tag{4.8}$$

Proof of Lemma 4.7. This result is nothing but [25, Proposition 5.14]. For the sake of completeness and because we will need to use the same computation at the nonlinear level, we just check it below. First recall that from Theorem 4.3, we know that for any $a\in(\bar a,-1)$ there exists $C=C(a)$ such that

$$\|e^{\Lambda t}\,f\|_{\mathcal{E}} \le C\,e^{a\,t}\,\|f\|_{\mathcal{E}}, \quad \forall\, t\ge 0,\ \forall\, f\in\mathcal{E}_2,$$

and on the other hand, from Lemma 4.4 there exists some constant $b\in\mathbb{R}$ such that

$$\langle \Lambda f, f^*\rangle \le b\,\|f\|^2_{\mathcal{E}}.$$

We define

$$|||f|||^2 := \eta\,\|f\|^2_{\mathcal{E}} + \int_0^\infty \|e^{\tau\Lambda}\,e^{\tau}\,f\|^2_{\mathcal{E}}\,d\tau \tag{4.9}$$

with $\eta\in(0,(b+1)^{-1})$, the norm $|||\cdot|||$ is clearly well defined and it is equivalent to $\|\cdot\|_{\mathcal{E}}$ because

$$\forall\, f\in\mathcal{E}_2,\quad \eta\,\|f\|^2_{\mathcal{E}} \le \eta\,\|f\|^2_{\mathcal{E}} + \int_0^\infty \|e^{\Lambda\tau}\,e^{\tau}\,f\|^2_{\mathcal{E}}\,d\tau \le \Big(\eta + \int_0^\infty C^2\,e^{2\,(a+1)\tau}\,d\tau\Big)\,\|f\|^2_{\mathcal{E}}$$

Next, for $f\in\mathcal{E}_2$ and with the notation $f_t := e^{\Lambda t}f$, we compute

$$\begin{aligned}
\frac{d}{dt}|||e^{\Lambda t}f|||^2 &= \eta\,\frac{d}{dt}\|f_t\|^2 + \int_0^\infty \frac{d}{dt}\|e^{\Lambda\,(t+\tau)+\tau}\,f\|^2\,d\tau \\
&= 2\eta\,\langle f_t^*,\Lambda f_t\rangle + \int_0^\infty \Big\{\frac{d}{d\tau}\|e^{\Lambda\,(t+\tau)+\tau}\,f\|^2 - 2\,\|e^{\Lambda\,(t+\tau)+\tau}\,f\|^2\Big\}\,d\tau \\
&\le 2\eta\,b\,\|f_t\|^2 + \Big[\|e^{\Lambda\,(t+\tau)+\tau}\,f\|^2\Big]_0^\infty - 2\int_0^\infty \|e^{\Lambda\tau}\,e^{\tau}\,f_t\|^2\,d\tau \\
&= \Big\{2\eta\,(b-a)-1\Big\}\|f_t\|^2 - 2\,\Big\{\eta\|f_t\|^2 + \int_0^\infty \|e^{\Lambda\tau}\,e^{\tau}\,f_t\|^2\,d\tau\Big\} \\
&\le -2\,|||e^{\Lambda t}f|||^2,
\end{aligned}$$

so that (4.8) is proved. □

We conclude with the proof of the long time convergence Theorem.

Proof of Theorem 1.5. The proof follows the same strategy as in [32, 30, 25] (see also [2, 1, 36] where similar proof is carried on in the context of the Boltzmann equation). We split the proof into four steps.

Step 1. We consider a solution g to the rescaled equation (1.17) with initial datum $f_0\neq G$. Thanks to Theorem 1.4 there holds $g(t)\in\mathcal{Z}$ for any $t\ge 1$. For any $\delta>0$ and $T:=(\mathcal{E}(f_0)-\mathcal{E}(G))/\eta^{-1}(\delta)+1$ there exists $t_0\in[1,T]$ so that

$$\mathcal{D}_{\mathcal{E}}(g(t_0)) \le \eta^{-1}(\delta) \tag{4.10}$$

because on the contrary we would have

$$\frac{d}{dt}(\mathcal{E}(g(t))-\mathcal{E}(G)) \le -\eta^{-1}(\delta) \quad \text{on} \quad (1,T)$$

and then

$$\mathcal{E}(g(T))-\mathcal{E}(G) \le -(\mathcal{E}(f_0)-\mathcal{E}(G)) < 0$$

which is in contradiction with the fact that G satisfies $\mathcal{E}(G)<\mathcal{E}(f)\ \forall\, f\in\mathcal{Z}\backslash\{G\}$ from Theorem 4.1. We deduce from (4.10) and Lemma 4.6 that

$$\|g(t_0)-G\|_{L^{4/3}_{k'}} \le \delta.$$

Step 2. The function $h := g - G$ satisfies the equation

$$\partial_t h = \Lambda h + \mathrm{div}(h\,\mathcal{K} * h).$$

We introduce the splitting

$$h = h_0 + h_1 + h_2, \quad h_{12} = h_1 + h_2$$

with

$$h_0 := \Pi_0 h = \alpha_0\, h_{0,0}, \quad h_1 = \Pi_1 h,$$

so that the evolution of h_1 and h_2 are given by

$$\partial_t h_1 = -h_1 + \Pi_1[\mathrm{div}(h\,\mathcal{K} * h)] \tag{4.11}$$

and

$$\partial_t h_2 = \Lambda h_2 + \mathcal{Q}_2, \quad \mathcal{Q}_2 := \Pi_2[\mathrm{div}(h\,\mathcal{K} * h)]. \tag{4.12}$$

From the definition of $h_{0,0}$, which implies that the mass of $h_{0,0}$ is 1, and thanks to the mass conservation, we have

$$0 = \int h = \alpha_0 + \int h_{12}$$

so that

$$\|h_0\|_{\mathcal{E}} = |\alpha_0|\, \|h_{0,0}\|_{\mathcal{E}} \le C\, \|h_{12}\|_{\mathcal{E}}. \tag{4.13}$$

Moreover, from (4.11) and with the notation $h_1^* = h_1\, |h_1|^{-1/3}\, \|h_1\|^{2/3}_{L^{4/3}_k}$, we clearly have

$$\begin{aligned} \frac{d}{dt}\|h_1\|^2_{L^{4/3}_k} &= 2\,\langle -h_1 + \Pi_1[\mathrm{div}(h\,\mathcal{K} * h)], h_1^*\rangle \\ &\le -2\,\|h_1\|^2_{L^{4/3}_k} + 2\,\|h_1\|_{L^{4/3}_k}\, \|\Pi_1[\mathrm{div}(h\,\mathcal{K} * h)]\|_{L^{4/3}_k} \\ &= -2\,\|h_1\|^2_{L^{4/3}_k} + C\,\|h_1\|_{L^{4/3}_k}\, \|\mathrm{div}(h\,\mathcal{K} * h)\|_{L^{4/3}_k}. \end{aligned} \tag{4.14}$$

Step 3. Estimate on the nonlinear term. We make the splitting

$$\|\mathrm{div}(h\,\mathcal{K} * h)\|_{L^{4/3}_k} \le I_1 + I_2, \quad I_1 := \|h^2\|_{L^{4/3}_k}, \quad I_2 := \|\nabla h \cdot \mathcal{K} * h\|_{L^{4/3}_k},$$

and we compute each term separately. On the one hand, using the Holder inequality and the Galgliardo-Niremberg inequality (see [9, Chapter IX, inequality (86)]) in dimension 2

$$\|u\|_{L^p} \le C\,\|u\|^{1-a}_{L^q}\, \|u\|^a_{W^{1,r}}, \quad a = \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} + \frac{1}{2} - \frac{1}{r}},$$

with $r = p = \infty$, $q = 4/3$ and $a = 3/5$, we have

$$I_1 \quad \le \quad \|h\|_{L^\infty}\, \|h\|_{L^{4/3}_k} \le C\, \|h\|^{7/5}_{L^{4/3}_k}\, \|h\|^{3/5}_{W^{1,\infty}}.$$

On the other hand, thanks to the Hardy-Littlewood-Sobolev inequality

$$\|\mathcal{K} * u\|_{L^4} \le C\, \|u\|_{L^{4/3}},$$

the elementary inequality

$$\|\nabla u\|^2_{L^2_k} = -\int_{\mathbb{R}^2} u\, \mathrm{div}(\langle x\rangle^{2k}\, \nabla u) \le C\, \|u\|_{W^{2,\infty}}\, \|u\|_{L^1_{2k}},$$

and the Holder inequality

$$\|u\|_{L^1_{2k}} \le \|\langle x\rangle^{-1}\|^{1/\gamma}_{L^{4\gamma/\alpha}}\, \|u\|^\alpha_{L^{4/3}_k}\, \|u\|^{1-\alpha}_{L^1_{k'}}$$

with $0 < \alpha < 1$, $2\gamma > \alpha$ and $k' = k'(\alpha,\gamma) := ((2-\alpha)k + \gamma)/(1-\alpha)$, we have

$$I_2 \quad \le \quad \|\nabla h\|_{L^2_k}\, \|\mathcal{K} * h\|_{L^4} \le C_{\alpha,\gamma}\, \|h\|^{1+\alpha/2}_{L^{4/3}_k}\, \|h\|^{1/2}_{W^{2,\infty}}\, \|h\|^{(1-\alpha)/2}_{L^1_{k'}}.$$

When $k' = 4$, we can take $k = 8/5 > 3/2$, $\gamma/\alpha = 5/8 > 1/2$ and we get $\alpha = 32/121 \in (0,1)$ and $\alpha/2 < 2/5$. All together we find

$$\forall\, h \in \mathcal{Z}, \qquad \|\mathrm{div}(h\,\mathcal{K} * h)\|_{L^{4/3}_k} \le C\,\|h\|^{1+\alpha/2}_{L^{4/3}_k},$$

and thanks to Theorem 1.4 $h(t) \in \mathcal{Z}$ for all $t \ge 1$ (where in the definition $\mathcal{Z}$ the constant $\mathcal{C}$ is given by (1.19)), we conclude with

$$\forall\, t \ge 1, \qquad \|\mathrm{div}(h(t)\,\mathcal{K} * h(t))\|_{L^{4/3}_k} \le C\,\|h(t)\|^{1+\alpha/2}_{L^{4/3}_k}. \tag{4.15}$$

It is worth noticing that in the limit $k \to 3/2$, $\gamma/\alpha \to 1/2$ and $\alpha \to 0$, we find $k' = 3$ which is a strict lower bound in order that estimate (4.15) holds with $\alpha > 0$.

Step 4. Estimate on the remaining term and conclusion. From (4.12), using the norm $|\!|\!|\cdot|\!|\!|$ defined in (4.9) and the notation $S_\tau := e^{\tau\Lambda}\,e^{\tau}$, we compute

$$\begin{aligned}\frac{d}{dt}|\!|\!|h_2|\!|\!|^2 &= \eta\,\langle h_2^*, \Lambda h_2\rangle + \int_0^\infty \langle (S_\tau h_2)^*, S_\tau \Lambda h_2\rangle\,d\tau \\ &\quad +\eta\,\langle h_2^*, \mathcal{Q}_2\rangle + \int_0^\infty \langle (S_\tau h_2)^*, \mathcal{Q}_2\rangle\,d\tau \\ &\le -2\,|\!|\!|h_2|\!|\!|^2 + C\,\|h_2\|_{L^{4/2}_k}\,\|\mathrm{div}(h\,\mathcal{K} * h)\|_{L^{4/2}_k},\end{aligned} \tag{4.16}$$

where we have used Lemma 4.7 in order to bound the first (linear) term and the equivalence between the two norms $|\!|\!|\cdot|\!|\!|$ and $\|\cdot\|_{L^{4/2}_k}$ in order to estimate the second one (which involves the nonlinear quantity). Gathering (4.14), (4.16), (4.15), we clearly see that

$$u(t) := \|h_1\|^2_{\mathcal{E}} + |\!|\!|h_2|\!|\!|^2$$

satisfies the differential inequality

$$u' \le -2u + C\,\|h\|^{2+\alpha} \quad \text{on } (0,\infty)$$

and then thanks to and (4.13) and to the first step

$$u' \le -2u + C\,u^{1+\alpha/2} \quad \text{on } (t_0,\infty), \quad u(t_0) \le K_2\,\delta. \tag{4.17}$$

Taking $\delta > 0$ small enough in the first step, we classically deduce that

$$u(t) \le C_a\,e^{2a\,t} \quad \forall\, t \ge t_0 \tag{4.18}$$

for any $a > -1$, so that for a close enough to -1, we deduce from (4.17)-(4.18) that

$$u' \le -2u + K_2\,e^{-2t} \quad \text{on } (t_0,\infty),$$

from which we easily obtain $u \le C\,e^{-2t}$. We conclude the proof using once more (4.13). □

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