Université Paris Dauphine-PSL
Academic Master MATH — November 17, 2025
An introduction to evolution PDEs

CHAPTER 2 - THE (SPACE HOMOGENEOUS) LANDAU EQUATION

Contents

Are written in blue color some changes (corrections) with respect to what has been taught during the classes.

1. Introduction

1.1. **The Landau equation.** In this chapter, we consider the (space homogeneous) Landau equation

$$\partial_t f(t,v) = Q(f,f)(t,v), \quad f(0,v) = f_0(v), \tag{1.1}$$

on the density function $f = f(t,v) \geq 0$, $t \geq 0$, $v \in \mathbb{R}^d$, $d \geq 2$, where the Landau kernel is defined by the formula

$$Q(g,f)(v) := \frac{\partial}{\partial v_i}\Big\{ \int_{\mathbb{R}^d} a_{ij}(v-v_*)\Big(g(v_*)\frac{\partial f}{\partial v_j}(v) - f(v)\frac{\partial g}{\partial v_j}(v_*)\Big)\, dv_*\Big\}. \tag{1.2}$$

Here and in the sequel we use Einstein's convention of sommation of repeated indices. The matrix $a = (a_{ij})$ is defined by

$$a(z) = |z|^{2+\gamma}\Pi(z), \quad \Pi_{ij}(z) := \delta_{ij} - \hat z_i \hat z_j, \quad \hat z_k := \frac{z_k}{|z|}, \tag{1.3}$$

so that Π is the orthogonal projection on the hyperplan $z^\perp := \{y \in \mathbb{R}^d;\, y\cdot z = 0\}$, in particular $\Pi^2 = \Pi$. Most of the time, we will restrict ourself to the more interesting and only physically meaningful case

$$d := 3, \quad \gamma := -3.$$

In that case, the Landau equation (1.1) models in a space homogeneous regime the evolution of a plasma where gas particles interact through the Coulomb force in a binary collisions regime.

1.2. **Physical properties.**
Let us start explaining some simple but fundamental features about the equation.
On the one hand, we observe that $\Pi(z)\xi = \xi - \hat z(\hat z \cdot \xi)$ for any $z, \xi \in \mathbb{R}^d$, so that

$$\Pi(z)z = 0 \quad \text{and} \quad \Pi(z)\xi\xi = |\xi|^2(1-(\hat z \cdot \hat \xi)^2) \geq 0,$$

and thus

$$a(z)\xi\xi \geq 0 \quad \text{and} \quad (a(z)\xi\xi = 0 \text{ iff } \xi \propto z). \tag{1.4}$$

Here and below, we use the bilinear form notation $auv = {}^t vau = v\cdot au$. In particular, the symmetric matrix a is positive but not strictly positive.
On the other hand, for any nice functions $f, \varphi : \mathbb{R}^d \to \mathbb{R}$, $f \geq 0$, we have

$$\begin{aligned}\int Q(f,f)\varphi\, dv &= \iint a(v-v_*)\big(f\nabla f_* - f_*\nabla f\big)\nabla\varphi\, dvdv_* \\ &= -\iint a(v-v_*)\big(f\nabla f_* - f_*\nabla f\big)\nabla_*\varphi_*\, dvdv_*,\end{aligned}$$

where we use the shorthands $f = f(v)$, $\nabla\psi = (\nabla\psi)(v)$, $f_* = f(v_*)$, $\nabla\psi_* = (\nabla\psi)(v_*)$, where the first formula comes from (1.2) and an integrating by part and where the second formula comes by just interchanging the variables v and v_* and observing that $a(-z) = a(z)$. Summing up the two formulas gives the more symmetric formula

$$\int Q(f,f)\varphi\, dv = \frac{1}{2}\iint a(v-v_*)\big(f\nabla f_* - f_*\nabla f\big)\big(\nabla\varphi - \nabla_*\varphi_*\big)\, dvdv_*. \tag{1.5}$$

Observing that $a(v-v_*)(\nabla\varphi - \nabla\varphi_*) = 0$ for $\varphi = 1, v_i, |v|^2$ because of (1.4), we deduce that

$$\int Q(f,f)\varphi\, dv = 0, \quad \text{for } \varphi = 1, v_i, |v|^2,$$

that we may rephrase by saying that 1, v and $|v|^2$ are collisional invariants of the Landau kernel and that reflects at this statistical level the fact that each collision leaves invariant the number of particles involved in a binary collisional system, leaves invariant its momentum and leaves invariant its energy. As a consequence, a solution $f = f(t,v)$ to the the Landau equation (1.1) satisfies

$$\frac{d}{dt}\int f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0,$$

so that

$$\int f(t,v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = \int f_0 \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv, \quad \forall\, t \geq 0,$$

meaning that the total mass, momentum and energy are constant of time through the Landau evolution. In order to simplify the presentation, we define the set of functions

$$\mathcal{E}^= := \Big\{ f \in L^1(\mathbb{R}^d);\, f \geq 0,\ \int f \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \int f\,|v|^2\, dv = d \Big\}$$

and we will always restrict ourself to case when $f_0 \in \mathcal{E}^=$ and thus, at least formally, $f(t,\cdot) \in \mathcal{E}^=$ for any $t \geq 0$. It turns out that we may come down to that situation from a mere change of variables and as a consequence of the Galilean invariance of the model, but we do not discuss further that issue.
Choosing now $\varphi := \log f$ in (2.3), we see that

$$D_H(f) := -\int Q(f,f)\log f\, dv = \frac{1}{2}\iint a f f_* \big(\frac{\nabla f}{f} - \frac{\nabla f_*}{f_*}\big)\big(\frac{\nabla f}{f} - \frac{\nabla f_*}{f_*}\big)\, dvdv_* \geq 0,$$

the nonnegativity coming from the positivity property in (1.4). Defining the entropy functional

$$H(f) := \int f \log f dv,$$

we deduce that any solution $f = f(t,v)$ to the the Landau equation (1.1) satisfies

$$\frac{d}{dt} H(f) = \int Q(f,f)(1+\log f)\, dv = -D_H(f) \le 0,$$

what is nothing but the famous H-Theorem of Boltzmann which reveals the non-reversibility property of the Landau dynamic. Integrating in time, we equivalently have

$$H(f(t,\cdot)) + \int_0^t D_H(f(s,\cdot))\, ds = H(f_0), \quad \forall\, t \ge 0.$$

We finally consider the stationary problem and for that purpose we define the normalized gaussian

$$\mathscr{M} := \frac{1}{(2\pi)^{d/2}} \exp\big(-\frac{|v|^2}{2}\big),$$

which in particular satisfies $\mathscr{M} \in \mathcal{E}^=$. It is in fact the only stationary state of the Landau equation which belongs to $\mathcal{E}^=$ as a consequence of the following result.

Lemma 1.1. *For $f \in \mathcal{E}^=$ smooth and positive, the three following assertions are equivalent*

$$f = \mathscr{M}, \quad Q(f,f) = 0, \quad D_H(f) = 0.$$

Proof of Lemma 1.1. We assume here $d = 3$. The two first direct implications being straightforward, we only have to show that

$$\big(f \in \mathcal{E} \text{ and } D_H(f) = 0\big) \quad \text{imply} \quad f = \mathscr{M}.$$

On the one hand, because $D_H(f) = 0$ and $f > 0$ imply

$$a(\nabla\sqrt{f f_*} - \nabla_*\sqrt{f f_*})(\nabla\sqrt{f f_*} - \nabla_*\sqrt{f f_*}) = 0,$$

we deduce from (1.4) that

$$\sqrt{f_*}\nabla\sqrt{f} - \sqrt{f}\nabla\sqrt{f_*} = \lambda(v,v_*)(v - v_*), \quad \forall\, v, v_* \in \mathbb{R}^d, \tag{1.6}$$

for a scalar function $(v,v_*) \mapsto \lambda(v,v_*)$. By permuting v and v_* in the equation, we see that $\lambda(v,v_*) = \lambda(v_*,v)$. For any independent vectors (v_1,v_2,v_3), $v_i \in \mathbb{R}^3$, the equation for different choices of values of v and v_* gives

$$\begin{aligned}
\sqrt{f_2}\nabla\sqrt{f_1} - \sqrt{f_1}\nabla\sqrt{f_2} &= \lambda_{12}(v_1 - v_2)\\
\sqrt{f_3}\nabla\sqrt{f_2} - \sqrt{f_2}\nabla\sqrt{f_3} &= \lambda_{23}(v_2 - v_3)\\
\sqrt{f_1}\nabla\sqrt{f_3} - \sqrt{f_3}\nabla\sqrt{f_1} &= \lambda_{31}(v_3 - v_1),
\end{aligned}$$

with $\lambda_{ij} := \lambda(v_i - v_j)$. Multiplying the first equation by $\sqrt{f_3}$, the second equation by $\sqrt{f_1}$, the third equation by $\sqrt{f_2}$ and summing up, we get

$$0 = \sqrt{f_3}\lambda_{12}(v_1 - v_2) + \sqrt{f_1}\lambda_{23}(v_2 - v_3) + \sqrt{f_2}\lambda_{31}(v_3 - v_1).$$

Taking the scalar product of that last equation with the vector $v_1 \wedge v_2$, we deduce

$$\sqrt{f_1}\lambda_{23} - \sqrt{f_2}\lambda_{31} = 0,$$

in particular

$$\lambda_{31} = \frac{\lambda_{23}}{\sqrt{f_2}}\sqrt{f_1} = \mu(v_3)\sqrt{f_1}, \quad \mu(v) := \frac{1}{|B|}\int_B \frac{\lambda(v_2 - v)}{\sqrt{f(v_2)}} dv_2.$$

By symmetry, we get that $\lambda_{ij} = \mu(v_i)\sqrt{f_j} = \mu(v_j)\sqrt{f_i}$, and thus

$$\lambda_{ij} = \lambda\sqrt{f_i}\sqrt{f_j}, \quad \text{for some} \quad \lambda \in \mathbb{R}.$$

Coming back to (1.6), we deduce

$$\nabla(\log f - \lambda|v|^2/2) = \nabla_*(\log f_* - \lambda|v_*|^2/2), \quad \forall\, v, v_* \in \mathbb{R}^3,$$

thus

$$\nabla(\log f - \lambda|v|^2/2) = u \in \mathbb{R}^3, \quad \forall\, v \in \mathbb{R}^3,$$

and finally

$$\log f = \lambda|v|^2/2 + u \cdot v + \log\varrho, \quad \forall\, v \in \mathbb{R}^3$$

for some $\varrho > 0$. In other words, f is the gaussian function

$$m(v) = \varrho \exp(-\lambda |v|^2/2 - u \cdot v), \quad \forall\, v \in \mathbb{R}^3,$$

from what we immediately conclude because of the moment conditions $m \in \mathcal{E}^=$. $\square$

1.3. **Other representation and notations.**
We define the following quantities

$$b_i(z) := \partial_j a_{ij}(z), \quad c(z) := \partial_{ij} a_{ij}(z), \tag{1.7}$$

so that in the Coulomb case

$$b_i(z) = -2\,|z|^{-3}\, z_i, \quad c(z) := -8\pi \delta_0, \tag{1.8}$$

from what we are able to rewrite the Landau operator (1.2) into the shorter forms

$$\begin{aligned} Q(g,f) &= \partial_i \{a^g_{ij} \partial_j f - b^g_i f\} \\ &= \partial_{ij} \{a^g_{ij} f\} - 2\partial_i \{b^g_i f\} \\ &= a^g_{ij} \partial_{ij} f + 8\pi g f, \end{aligned} \tag{1.9}$$

where we have defined

$$a^g_{ij} := a_{ij} * g, \quad b^g_i := b_i * g, \quad c^g := c * g = -8\pi g.$$

The first formulation is nothing but (1.2) after one integration by parts, the second and third formulations are then obtained by playing with derivative expansions.

2. H-solution and their stability

This section is dedicated to the formulation of weak solutions for which we establish the stability.

2.1. **A priori estimates, Villani's inequality and weak formulation.** For $H_0 \in \mathbb{R}$, we define $\mathcal{E}_{H_0}$ the set of functions

$$\mathcal{E}_{H_0} := \Big\{ f \in L^1(\mathbb{R}^d);\ f \ge 0,\ \int f \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \int f\,|v|^2\,dv \le d,\ H(f) \le H_0 \Big\}.$$

From the conservations law and the H-Theorem presented in Section 1.2, any solution to the Landau equation satisfies

$$f(t,\cdot) \in \mathcal{E}_{H_0}, \quad \forall\, t \ge 0, \tag{2.1}$$

with $H_0 := H(f_0)$. Let us be a bit more precise. Introducing the notations

$$H_\pm(g) := \int g (\log g)_\pm dv, \quad M_k(g) := \int g |v|^k\, dv,$$

the integral form of the H-Theorem writes

$$H_+(f(T,\cdot)) + \int_0^T D_H(f(t,\cdot)) dt = H(f_0) + H_-(f(T,\cdot)), \quad \forall\, T > 0.$$

As an immediate consequence of the inequality

$$s\,(\log s)_- \le \sqrt{s}\, \mathbf{1}_{0 \le s \le e^{-|v|^2}} + s\,|v|^2\, \mathbf{1}_{e^{-|v|^2} < s \le 1}, \quad \forall\, s \ge 0,$$

we obtain

$$H_-(g) \le (2\pi)^{d/2} + M_2(g), \quad \forall\, g \in L^1_+,$$

and thus

$$H_-(g) \le C_0 := (2\pi)^{d/2} + d, \quad \forall\, g \in \mathcal{E}_{H_0}.$$

We deduce in particular that any solution f to the Landau equation satisfies

$$\int_0^\infty D_H(f(t,\cdot)) dt \le D_0, \tag{2.2}$$

with $D_0 := C_0 + H_0$.

We are now going to see why the collision kernel $Q(f,f)$ is meaningful for a function f satisfying the above *natural a priori bounds*. Starting from (2.3) and recalling (1.3), for a function $\varphi \in C^2_c(\mathbb{R}^d)$, we write

$$\int Q(f,f)\varphi\, dv = \iint X \cdot Y\, dv_* dv, \tag{2.3}$$

with

$$\begin{aligned} X &:= 2^{-1/2}|z|^{1+\gamma/2}\sqrt{ff_*}\Pi(\nabla \log f - \nabla_* \log f_*) \\ Y &:= 2^{-1/2}|z|^{1+\gamma/2}\sqrt{ff_*}\Pi(\nabla\varphi - \nabla_*\varphi_*), \end{aligned}$$

where $z := v_* - v$. We observe that

$$\|X\|^2_{L^2} = D_H(f),$$

and

$$\|Y\|^2_{L^2} = \mathcal{B}(f;\varphi) := \frac{1}{2}\iint a f f_* (\nabla\varphi - \nabla_*\varphi_*)(\nabla\varphi - \nabla_*\varphi_*).$$

Because $|\nabla\varphi - \nabla_*\varphi_*| \le \|D^2\varphi\|_{L^\infty}|v - v_*|$, we have

$$\begin{aligned} & a(\nabla\varphi - \nabla_*\varphi_*)(\nabla\varphi - \nabla_*\varphi_*) \\ & = a(\nabla\varphi - \nabla_*\varphi_*)(\nabla\varphi - \nabla_*\varphi_*)\mathbf{1}_{|v-v_*|\le 1} + a(\nabla\varphi - \nabla_*\varphi_*)(\nabla\varphi - \nabla_*\varphi_*)\mathbf{1}_{|v-v_*|\ge 1} \\ & \le \|D^2\varphi\|_{L^\infty}|v-v_*|^{\gamma+4}\mathbf{1}_{|v-v_*|\le 1} + |v-v_*|^{\gamma+2}4\|\nabla\varphi\|_{L^\infty}\mathbf{1}_{|v-v_*|\ge 1}, \\ & \le 4\|\varphi\|_{W^{2,\infty}}\langle v\rangle\langle v_*\rangle, \end{aligned}$$

where in the last line we have particularize the discussion to the Coulomb exponent $\gamma := 3$, so that

$$\mathcal{B}(f;\varphi) \le 2\|\varphi\|_{W^{2,\infty}}\|f\|^2_{L^1_1}.$$

It thus turns out that when $f \in \mathcal{E}_{H_0}$ and $D_H(f) < \infty$, we have both $X, Y \in L^2(\mathbb{R}^{2d})$, so that the RHS term in (2.3) is well defined. In other words, using the Cauchy-Schwarz inequality, the following Villani's estimate holds true

$$|\langle Q(f,f),\varphi\rangle| \le D_H(f)^{1/2}\mathcal{B}(f;\varphi)^{1/2} \le \sqrt{2}\|\varphi\|^{1/2}_{W^{2,\infty}} D_H(f)^{1/2}\|f\|_{L^1_1}, \tag{2.4}$$

what gives a meaning to $Q(f,f)$ in the distributional sense.

Thanks to the above discussion, we may introduce the definition of weak solutions we will deal with. For $T > 0$, we define the functional set

$$\mathcal{F}_T := \Big\{ g \in C([0,T]; L^1_w);\ g(t) \in \mathcal{E}_{H_0}, \forall\, t \in (0,T),\ \int_0^T D_H(g(t))\, dt \le D_0 \Big\}.$$

Definition 2.1. *For $H_0 \in \mathbb{R}$ and $f_0 \in \mathcal{E}_{H_0} \cap \mathcal{E}^=$, we say that a function f a weak solution (or H-solution) to the Landau equation with initial datum f_0 if $f \in \mathcal{F}_T$ for any $T > 0$ and* (1.1) *holds in the distributional sense, or more precisely*

$$-\int_0^T\!\!\int f\partial_t\varphi - \int f_0\varphi(0,\cdot) = \int_0^\infty \langle Q(f,f),\varphi\rangle dt, \tag{2.5}$$

for any $\varphi \in C^2_c([0,\infty) \times \mathbb{R}^d)$.

It is worth emphasizing that because of (2.4) and $f \in \mathcal{F}_\infty$, we have $\langle Q(f,f),\varphi\rangle \in L^2(\mathbb{R}_+)$ or more precisely recalling the definition (2.3), we have $X \in L^2(\mathbb{R}_+ \times \mathbb{R}^6)$ and $Y \in L^2(\mathbb{R}_+ \times \mathbb{R}^6)$, what makes the RHS term well defined in (2.5).

2.2. **Weak stability.**

Theorem 2.2. *Assume that (f_n) is a sequence in $\mathcal{F}_T$ of solutions to the Landau equation. Then, up to the extraction of a subsequence, $f_n \rightharpoonup f$ $\sigma(L^1, L^\infty)$, where f belongs to $\mathcal{F}_T$ and is a weak solution to the Landau equation.*

Proof. Step 1. From the a priori bound and the Dunford-Pettis Lemma, there exist a subsequence still denoted by (f_n) and a function $f \in L^1((0,T)\times\mathbb{R}^d)$ such that $f_n \rightharpoonup f$ in the weak sense $\sigma(L^1, L^\infty)$. Because of the equation (2.5), for any $\varphi \in C^2_c(\mathbb{R}_+ \times \mathbb{R}^3)$, we have

$$\frac{d}{dt}\int \varphi f_n = \langle Q(f_n,f_n),\varphi\rangle$$

which is bounded in $L^2(0,T)$. From the above piece of information, we may deduce that

$$\int f_n\psi \to \int f\psi \quad \text{a.e. on} \quad (0,T), \quad \text{bounded in } L^\infty(0,T),$$

for any $\psi \in L^\infty((0,T)\times\mathbb{R}^3)$, and next that for any $K \in L^\infty(\mathbb{R}^3)$, we have

$$(f_n * K)f_n \rightharpoonup (f*K)f \quad \text{weakly in} \quad L^1((0,T)\times\mathbb{R}^3). \tag{2.6}$$

The proof of this claim is left as an exercise.

Step 2. We split $a := a_\varepsilon + a^c_\varepsilon$, $a_\varepsilon := a\psi_\varepsilon$, $\psi_\varepsilon \in \mathcal{D}(\mathbb{R}^3)$, $\mathbf{1}_{|z|\le\varepsilon} \le \psi_\varepsilon \le \mathbf{1}_{|z|\le 2\varepsilon}$, $a^c_\varepsilon := a\psi^c_\varepsilon$, $\psi^c := 1-\psi_\varepsilon$, $\varepsilon \in (0,1]$, and we next write, with obvious notations,

$$Q(f,f) = Q_\varepsilon(f,f) + Q^c_\varepsilon(f,f). \tag{2.7}$$

In order to deal with the first term, we define

$$\mathcal{B}_\varepsilon(f;\varphi) := \frac{1}{2}\iint a_\varepsilon f f_* (\nabla\varphi - \nabla_*\varphi_*)(\nabla\varphi - \nabla_*\varphi_*),$$

and coming back to the proof of (2.4), we first observe that

$$a_\varepsilon(\nabla\varphi - \nabla_*\varphi_*)(\nabla\varphi - \nabla_*\varphi_*) \le \|D^2\varphi\|_{L^\infty}|v-v_*|^{\gamma+4}\mathbf{1}_{|v-v_*|\le 2\varepsilon} \le 2\|D^2\varphi\|_{L^\infty}\varepsilon,$$

so that

$$|\langle Q_\varepsilon(f,f),\varphi\rangle| \le \mathcal{D}_H(f)^{1/2}\mathcal{B}_\varepsilon(f;\varphi)^{1/2} \le \|D^2\varphi\|^{1/2}_{L^\infty} D_H(f)^{1/2}\|f\|_{L^1_1}\varepsilon \to 0,$$

as $\varepsilon \to 0$ for any fixed $\varphi \in C^2_c(\mathbb{R}_+\times\mathbb{R}^3)$. On the other hand, observing that

$$Q^c_\varepsilon(f,f)(v) = D^2 : \{(a^c_\varepsilon * f)f\} - 2\mathrm{div}\{(b^c_\varepsilon * f)f\},$$

with $b^c_\varepsilon(z) = \mathrm{div} a^c_\varepsilon(z)$, and taking advantage of the fact that $a^c_\varepsilon, b^c_\varepsilon \in L^\infty$, we deduce from (2.6) that

$$\begin{aligned}&\langle Q^c_\varepsilon(f_n,f_n),\varphi\rangle - \langle Q^c_\varepsilon(f,f),\varphi\rangle\\ &\quad = \int [(a^c_\varepsilon * f_n)f_n - (a^c_\varepsilon * f)f] : D^2\varphi - 2[(b^c_\varepsilon * f_n)f_n - (b^c_\varepsilon * f)f]\cdot\nabla\varphi \to 0,\end{aligned}$$

as $n\to\infty$, for any fixed $\varepsilon \in (0,1]$ and $\varphi \in C^2_c(\mathbb{R}_+\times\mathbb{R}^3)$. Using the splitting (2.7) and the two above convergences, we classically deduce

$$Q(f_n,f_n) \rightharpoonup Q(f,f) \text{ in } \mathcal{D}'(\mathbb{R}_+\times\mathbb{R}^3) \text{ as } n\to\infty. \tag{2.8}$$

We immediately conclude by passing to the limit in the weak formulation (2.5). ☐

It is worth emphasizing that for proving (2.8), we need to use that $D_H(f) \in L^1(0,T)$, what is a consequence of the lsc property

$$\int_0^T D_H(f) \le \liminf \int_0^T D_H(f_n). \tag{2.9}$$

We may prove that result using a convexity argument and we refer to the exercises sheet for such an argument. An alternative way is presented in Lemma 3.6 below.

3. Ellipticity and additional estimates

We show that the Landau equation has a parabolic nature, from what we deduce some regularity properties for the solutions.

3.1. **On the ellipticity of a^f.**
We start with an elementary estimate about the localization of the positivity of elements of $\mathcal{E}_{H_0}$.

Lemma 3.1. *There exist some constants $R, \lambda, \eta > 0$ depending only of H_0 such that*

$$|\{f \ge \lambda\} \cap B_R| \ge \eta, \quad \forall\, f \in \mathcal{E}_{H_0}.$$

Proof of Lemma 3.1. For any $R, \lambda > 0$ and $\Lambda > 1$, we write

$$\begin{aligned} \Lambda |\{f \ge \lambda\} \cap B_R| &\ge \int_{B_R} f \mathbf{1}_{\lambda < f \le \Lambda} \\ &\ge \int f - \int_{B_R} f \mathbf{1}_{f < \lambda} - \int_{B_R^c} f - \int f \mathbf{1}_{f > \Lambda}. \end{aligned}$$

Observing that we have

$$\int_{B_R} f \mathbf{1}_{f<\lambda} \le \lambda |B_R|, \quad \int_{B_R^c} f \le \frac{1}{R^2}, \quad \int f \mathbf{1}_{f > \Lambda} \le \int f \frac{\log_+ f}{\log_+ \Lambda} \le \frac{H_0^+}{\log \Lambda},$$

we deduce

$$\Lambda |\{f \ge \lambda\} \cap B_R| \ge 1 - \frac{1}{R^2} - \frac{H_0^+}{\log \Lambda} - \lambda |B_R|,$$

and we conclude by choosing first $R > 0$ large enough, next $\Lambda > 1$ large enough and finally $\lambda > 0$ small enough. $\square$

We define the truncated diffusion coefficient

$$\widetilde a_{ij} = \widetilde \psi(|z|)(\delta_{ij} - \hat z_i \hat z_j), \tag{3.1}$$

with $\widetilde\psi(s) = \widetilde\varphi(s)/s$, $\widetilde\varphi \in C^2(\mathbb{R}_+)$, $\widetilde\varphi(s) = s^3$ on $[0,1/2]$, $\widetilde\varphi(s) = 1$ on $[1,\infty)$ and $0 \le \widetilde\varphi' \le 2$ on $\mathbb{R}_+$, so that $a \ge \widetilde a$ as symmetric matrices, and we also define the truncated diffusion matrix $\widetilde a^f := \widetilde a * f$. We establish a coercivity estimate on this last one.

Proposition 3.2. *There exists a constant $a_0 > 0$ depending only on H_0 such that*

$$\widetilde a^f \ge a_0 \langle v \rangle^{-3} \mathbb{I}, \quad \forall\, f \in \mathcal{E}_{H_0}. \tag{3.2}$$

Proof of Proposition 3.2. For $e \in \mathbb{S}^2$, $\varepsilon > 0$ and with the notations of Lemma 3.1, we introduce the two sets

$$\begin{aligned} \mathscr{B} &:= \{z \in B_R(v) \cap B_{1/2}; f(v-z) \ge \lambda,\ 1 - (\hat z \cdot e)^2 > \varepsilon\} \\ \mathscr{C} &:= \{z \in B_R(v) \cap B_{1/2}^c; f(v-z) \ge \lambda,\ 1 - (\hat z \cdot e)^2 > \varepsilon\}. \end{aligned}$$

We have

$$\begin{aligned} \widetilde a^f(v) ee &= \int f(v-z) \widetilde\psi(z)(1 - (\hat z \cdot e)^2) dz \\ &\ge \lambda \varepsilon \int_{\mathscr{B} \cup \mathscr{C}} \widetilde\psi(z) dz \ge \lambda\varepsilon \Big(\int_{\mathscr{B}} |z|^2 dz + \int_{\mathscr{C}} \frac{dz}{8|z|} \Big). \end{aligned}$$

Observing that

$$\int_{\mathscr{B}} |z|^2 dz \ge \int_{B_\varrho(0)} |z|^2 dz = \varrho^5 \int_{B_1(0)} |u|^2 du = \frac{\pi}{3} \varrho^5,$$

with $\varrho \ge 0$ defined by $|\mathscr{B}| = B_\varrho(0) = \frac{4}{3}\pi \varrho^3$, we deduce

$$\widetilde a^f(v) ee \ge \lambda \varepsilon \Big(\frac{2}{3} |\mathscr{B}|^{5/3} + \frac{1}{8(|v| + R)} |\mathscr{C}| \Big). \tag{3.3}$$

In order to lower bound the RHS term, we introduce the set

$$\mathscr{A} := \{z \in B_R(v); 1 - (\hat z \cdot e)^2 \le \varepsilon\}.$$

Writing in euclidian coordinates

$$z = z_1 e + z',\ z_1 \in \mathbb{R},\ z' \perp e, \quad v = v_1 e + v',\ v_1 \in \mathbb{R},\ v' \perp e,$$

we see that $z \in \mathscr{A}$ implies

$$|v_1 - z_1| \le R \quad \text{and} \quad z_1^2 > (1 - \varepsilon)|z|^2 = (1-\varepsilon)(z_1^2 + |z'|^2).$$

Both equations together imply

$$|z'|^2 \le \frac{\varepsilon}{1-\varepsilon} z_1^2 \le 2\varepsilon (R+|v|)^2.$$

We deduce the following upper bound on its volume

$$|\mathscr{A}| \le (2R)\pi 2(R+|v|)^2 \varepsilon.$$

Defining

$$\varepsilon := [8\pi(R+|v|)^2 R]^{-1}\eta, \tag{3.4}$$

and using Lemma 3.1, we deduce

$$2\max(|\mathscr{B}|,|\mathscr{C}|) \ge |\mathscr{B}\cup\mathscr{C}| \ge |\{z\in B_R(v); f(v-z)\ge\lambda\}| - |\mathscr{A}| \ge \eta/2. \tag{3.5}$$

Gathering the three estimates (3.3), (3.4) and (3.5), we have established

$$\widetilde{a}^f(v)ee \ge \frac{\lambda\eta}{8\pi R}\min\Big(\frac{2}{3}\Big(\frac{\eta}{4}\Big)^{5/3}, \frac{\eta}{32}\Big)\frac{1}{(1+R+|v|)^3}, \quad \forall e \in \mathbb{S}^2,$$

what is nothing but the announced estimate. □

3.2. **Weighted Fisher, Sobolev and Lebesgue estimates.** We first establish an estimate on a weighted Firsher information.

Proposition 3.3. *For any $f\in\mathcal{E}_{H_0}$, there holds*

$$\|\nabla\sqrt{f}\|^2_{L^2_{-3/2}} \lesssim 1 + D_H(f).$$

Proof of Proposition 3.3. From the very definitions of a and $\widetilde{a}$, we have $D_H(f)\ge \widetilde{D}_H(f)$, with

$$\begin{aligned}
\widetilde{D}_H(f) &:= \frac12 \iint \widetilde{a}_{ij}(v-v_*)\big(f\partial_i f_* - f_*\partial_i f\big)\big(\partial_j \log f_* - \partial_j \log f\big)\, dvdv_* \\
&= \iint \widetilde{a}_{ij}(v-v_*)\big(f_* 4\partial_i\sqrt{f}\partial_j\sqrt{f} - \partial_i f\partial_j f_*\big)\, dvdv_* \\
&= 4\int \widetilde{a}^f_{ij}\partial_i\sqrt{f}\partial_j\sqrt{f} + \iint f f_* \partial^2_{ij}\widetilde{a}_{ij}(v-v_*)\, dvdv_*.
\end{aligned}$$

From the very definition (3.1), we compute

$$\partial_j\widetilde{a}_{ij} = \widetilde{\psi}(|z|)(-\hat{z}_i\partial_j\hat{z}_j) = -2\hat{z}_i\widetilde{\psi}(|z|)/|z|$$

and next

$$\partial_{ij}\widetilde{a}_{ij} = -2\widetilde{\psi}'(|z|)/|z| - 2\widetilde{\psi}(|z|)/|z|^2 = -2\widetilde{\varphi}'(|z|)/|z|^2 \ge -8.$$

Using Proposition 3.2, we deduce

$$\widetilde{D}_H(f) \ge 4a_0\int |\nabla\sqrt{f}|^2\langle v\rangle^{-3} - 8M_0^2,$$

from what we immediately conclude. □

Proposition 3.4. *For any $f\in\mathcal{E}_{H_0}$, there holds*

$$\|f\|^{r(p)}_{L^p_{k(p)}} \lesssim 1 + D_H(f), \tag{3.6}$$

for any $p\in(1,3]$ and with $k(p) := \frac{15}{2}\frac{1}{p} - \frac{11}{2}$, $\frac{1}{r(p)} := \frac32(1-\frac1p)$, in particular $(k,r)(3) = (-3,1)$, and

$$\|\nabla f\|^{r'(q)}_{L^q_{k'(q)}} \lesssim 1 + D_H(f), \tag{3.7}$$

for any $q\in[1,3/2]$ and with $k'(q) := \frac{15}{2}\frac{1}{q} - 8$, $\frac{1}{r'(q)} := \frac12(4-\frac3q)$, in particular $(k',r')(3/2) = (-3,1)$.

Proof of Proposition 3.4. On the one hand, thanks to the Sobolev embedding $\dot H^1 \subset L^6$, we have

$$\begin{aligned}\Big(\int f^3 \langle v\rangle^{-9}\Big)^{1/3} &\lesssim \int |\nabla(\sqrt{f}\langle v\rangle^{-3/2})|^2 \\ &\lesssim \int |\nabla\sqrt{f}|^2\langle v\rangle^{-3} + \int f|\nabla\langle v\rangle^{-3/2}|^2,\end{aligned}$$

and we deduce estimate (3.6) in the case $p = 3$ thanks to Proposition 3.3. We deduce the full range of exponents $p \in (1, 3]$ in (3.6) by interpolating (just using the Holder inequality) that first estimate together with the L^1_2 a priori bound.

On the other hand, thanks to the Holder inequality, we have

$$\begin{aligned}\|\nabla f\|^q_{L^q_{k'}} &= \int \left|\frac{\nabla f}{\sqrt{f\langle v\rangle^3}}\right|^q f^{q/2}\langle v\rangle^{q(k'+3/2)} \\ &\le \Big(\int \frac{|\nabla f|^2}{f\langle v\rangle^3}\Big)^{q/2}\Big(\int [f\langle v\rangle^{2(k'+3/2)}]^{q/(2-q)}\Big)^{(2-q)/2} = 2^q\|\nabla\sqrt{f}\|^q_{L^2_{-3/2}}\|f\|^{q/2}_{L^p_k},\end{aligned}$$

with $p := q/(2-q)$ and $k = 2k' + 3$. Because of (3.6) and Proposition 3.3, we deduce

$$\|\nabla f\|_{L^q_{k'}} \lesssim (1 + D_H(f))^{\frac12+\frac{1}{2r(p)}}$$

when $p \in [1, 3]$ and $k = k(p)$. The first condition implies $q \in [1, 3/2]$ and the second condition implies $k' = k(p)/2 - 3/2 = k'(q)$. We conclude to (3.7) by observing that $\frac{1}{r'(q)} = \frac12 + \frac{1}{2r(p)}$. □

As a consequence of Definition 2.1, we deduce that any H-solution f to the Landau equation also satisfies

$$f \in L^1(0, T; L^3_{-3} \cap W^{1,3/2}_{-3}), \quad \forall T > 0.$$

That implies that each term involved in the first and second formulations of $Q_L(f, f)$ in (1.9) can be defined separately in a distributional sense. In order to see this, we start establishing some estimates on the functions a^f and b^f.

Lemma 3.5. *For any $f \in L^1_2 \cap L^3_{-3}$, there hold*

$$\|a^f\|_{L^5} \lesssim \|f\|_{L^1_2\cap L^3_{-3}}, \tag{3.8}$$

$$\|b^f\|_{L^{15/8}} \lesssim \|f\|_{L^1_2\cap L^3_{-3}}. \tag{3.9}$$

Proof of Lemma 3.5. On the one hand, the Holder inequality tells us that

$$\|f\|_{L^{15/13}} \le \|f\|^{2/5}_{L^3_{-3}}\|f\|^{3/5}_{L^1_2}.$$

On the other hand, the Hardy-Littlewood-Sobolev inequality tells us

$$\|f * |z|^{-\lambda}\|_{L^r} \lesssim \|f\|_{L^p}, \quad \frac1r = \frac1p + \frac{\lambda}{3} - 1, \tag{3.10}$$

for $0 < \lambda < 3$, $1 < p, r < \infty$. In particular, we have

$$\|a^f\|_{L^5} \le \|f * |z|^{-1}\|_{L^5} \lesssim \|f\|_{L^{15/13}},$$

by making the choice $r = 5$, $\lambda = 1$, $p = 15/13$ in (3.10), from what we deduce (3.8). Similarly, we have

$$\|b^f\|_{L^{15/8}} \le \|f * |z|^{-2}\|_{L^{15/8}} \lesssim \|f\|_{L^{15/13}},$$

by making the choice $r = 15/8$, $\lambda = 2$, $p = 15/13$ in (3.10), from what we deduce (3.9). □

As a consequence, we may write

$$\begin{aligned}\langle Q(f, f), \varphi\rangle &= -\iint \big\{f b^f_i \partial_i\varphi + a^f_{ij}\partial_j f\partial_i\varphi\big\}\, dvdv_* \\ &= \iint f\big\{a^f_{ij}\partial_{ij}\varphi + 2b^f_i\partial_i\varphi\big\}\, dvdv_*,\end{aligned}$$

where both integrals are well-defined. In the first integral indeed, the first term is well-defined since $f \in L^3_{-3}$, $b^f_i\partial_i\varphi \in L^{3/2}_{\rm comp} \subset (L^3_{-3})'$ and the second term is well-defined since $\partial_j f \in L^{3/2}_{-3}$,

$a^f_{ij}\partial_i\varphi \in L^5_{\rm comp} \subset (L^{3/2}_{-3})'$. The new term in the second integral is also well-defined because $a^f_{ij}\partial_{ij}\varphi \in L^5_{\rm comp} \subset (L^3_{-3})'$.

We end this section by establishing the lsc of D_H and thus completing the proof of Theorem 2.2.

Lemma 3.6. *For any sequence (f_n) of H-solution to the Landau equation such that $f_n \rightharpoonup f$ weakly in L^1, there holds*

$$f_n \to f \text{ strongly in } L^1 \tag{3.11}$$

and

$$\int_0^T D_H(f) \le \liminf \int_0^T D_H(f_n). \tag{3.12}$$

Proof of Lemma 3.6. Step 1. We know that

$$\partial_t \int f_n \varphi dv \text{ bounded in } L^2(0,T)$$

and (f_n) is bounded in $(W^{1,1}_{-1} \cap L^1_2)(\mathcal{U})$. By the same arguments as in Aubin-Lions Lemma, we deduce (3.11).

Step 2. We observe that

$$D_H(f) = \iint a\xi\xi dv dv_*, \quad \xi := \nabla\sqrt{ff_*} - \nabla_*\sqrt{ff_*}.$$

Because of step 1, we have $\sqrt{f_n(t,v)f_n(t,v_*)} \to \sqrt{f_n(t,v)f_n(t,v_*)}$ in $L^2((0,T)\times\mathbb{R}^{2d})$. With obvious notations, we deduce that $\xi_n \rightharpoonup \xi$ weakly in L^2, because the convergence holds in $\mathcal{D}'$ and the sequence is bounded in L^2. We now write $a^{\varepsilon,c}(\xi_n - \xi)(\xi_n - \xi) \ge 0$, so that

$$\int_0^T D(f_n) \ge \int_0^T D^{\varepsilon,c}(f_n) \ge \int_0^T\int_{\mathbb{R}^{2d}} [2a^{\varepsilon,c}\xi_n\xi - a^{\varepsilon,c}\xi\xi].$$

Passing to the limit, we have

$$\liminf_{n\to\infty}\int_0^T D(f_n) \ge \lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^{2d}} [2a^{\varepsilon,c}\xi_n\xi - a^{\varepsilon,c}\xi\xi] = \int_0^T D^{\varepsilon,c}(f),$$

for any $\varepsilon > 0$. We conclude to (3.12) by observing that $a^{\varepsilon,c}\xi\xi \nearrow a\xi\xi$ as $\varepsilon \to 0$. □