Université Paris Dauphine-PSL November 17, 2025

Academic Master MATH

An introduction to evolution PDEs

CHAPTER 3 - MORE ABOUT THE HEAT EQUATION

Contents

We present some qualitative properties of the heat equation and more particularly we present several results on the self-similar behavior of the solutions in large time. These results are deduced from several functional inequalities, among them the Poincaré inequality and the Log-Sobolev inequality. Let us emphasize that the approach lies on an interplay between evolution PDEs and functional inequalities and, although we only deal with (simple) linear situations, these methods are robust enough to be generalized to (some) nonlinear situations.

1. Self-similar solutions of the heat equation and the Fokker-Planck equation

We consider the heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\,\Delta f \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d. \tag{1.1}$$

We recall that $f(t,.) \to 0$ as $t \to \infty$, and more precisely, that for any $1 \le q \le p \le \infty$ and a constant $C_{p,d}$ the following rate of decay (ultracontractivity property) holds:

$$\|f(t,.)\|_{L^p} \le \frac{C_{p,d}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\,\|f_0\|_{L^q} \qquad \forall\, t > 0, \tag{1.2}$$

what can be obtained thanks to the representation formula or by using Nash argument presented in a previous chapter. It is in fact possible to describe in a more accurate way that the mere estimate (1.2) how the heat equation solution $f(t,.)$ converges to 0 as time goes on. In order to do so, the first step consists in looking for particular solutions to the heat equation that we will discover by identifying some good change of scaling. We thus look for a self-similar solution to (1.2), namely we look for a solution F with particular form

$$F(t,x) = t^\alpha\, G(t^\beta x),$$

for some $\alpha,\beta \in \mathbb{R}$ and a *"self-similar profile"* G. As F must be mass conserving, we have

$$\int_{\mathbb{R}^d} F(t,x)\,dx = \int_{\mathbb{R}^d} F(0,x)\,dx = t^\alpha \int_{\mathbb{R}^d} G(t^\beta\, x)\,dx,$$

and we get from that the first equation $\alpha = \beta\, d$. On the other hand, we easily compute

$$\partial_t F = \alpha\, t^{\alpha-1} G(t^\beta\, x) + \beta\, t^{\alpha-1}\,(t^\beta\, x)\cdot(\nabla G)(t^\beta\, x), \quad \Delta F = t^\alpha\, t^{2\beta}\,(\Delta G)(t^\beta\, x).$$

In order that (1.1) is satisfied, we have to take $2\,\beta + 1 = 0$. We conclude with

$$F(t,x) = t^{-d/2}\, G(t^{-1/2}\, x), \qquad \frac{1}{2}\Delta G + \frac{1}{2}\mathrm{div}(x\, G) = 0. \tag{1.3}$$

We observe (and that is not a surprise!) that a solution $G \in L^1(\mathbb{R}^d) \cap \mathbf{P}(\mathbb{R}^d)$ to (1.3) will satisfy $\nabla G + x\, G = 0$, it is thus unique and given by

$$G(x) := c_0\, e^{-|x|^2/2}, \quad c_0^{-1} = (2\,\pi)^{d/2} \quad \text{(normalized Gaussian function)}.$$

To sum up, we have proved that F is our favorite solution to the heat equation: that is the fundamental solution to the heat equation.

Changing of point view, we may now consider G as a stationary solution to the harmonic Fokker-Planck equation (sometimes also called the Ornstein-Uhlenbeck equation)

$$\frac{\partial}{\partial t} g = \frac{1}{2}\, L\, g = \frac{1}{2}\, \nabla \cdot (\nabla g + g\, x) \qquad \text{in } (0,\infty) \times \mathbb{R}^d. \tag{1.4}$$

The link between the heat equation (1.1) and the Fokker-Planck equation (**??**) is as follows. If f is a solution to the Fokker-Planck equation (**??**), some elementary computations permit to show that

$$f(t,x) = (1+t)^{-d/2}\, g(\log(1+t), (1+t)^{-1/2}\, x)$$

is a solution to the heat equation (1.1), with $f(0,x) = g(0,x)$. Reciprocally, if f is a solution to the heat equation (1.1) then

$$g(t,x) := e^{d\,t/2}\, f(e^t - 1, e^{t/2}\, x)$$

solves the Fokker-Planck equation (1.4). The last expression also gives the existence of a solution *in the sense of distributions* to the Fokker-Planck equation (1.4) for any initial datum $f_0 = \varphi \in L^1(\mathbb{R}^d)$ as soon as we know the existence of a solution to the heat equation for the same initial datum (what we get thanks to the usual representation formula for instance).

2. Fokker-Planck equation and Poincaré inequality

2.1. **Long time asymptotic behaviour of the solutions to the Fokker-Planck equation.** From now on in this chapter, we consider the Fokker-Planck equation

$$\frac{\partial}{\partial t} f = \mathcal{L}\, f = \Delta f + \nabla \cdot (f\, \nabla V) \quad \text{in } (0,\infty) \times \mathbb{R}^d \tag{2.1}$$

$$f(0,x) = f_0(x) \quad \text{on } \mathbb{R}^d, \tag{2.2}$$

and we assume that the *"confinement potential"* V is the harmonic potential

$$V(x) := \frac{|x|^2}{2} + V_0, \quad V_0 := \frac{d}{2}\, \log 2\pi.$$

We start observing that

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t,x)\, dx = \int_{\mathbb{R}^d} \nabla_x \cdot (\nabla_x f + f\, \nabla_x V)\, dx = 0,$$

so that the mass (of the solution) is conserved. We also have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} (f_+)^2\, dx &= \int_{\mathbb{R}^d} f_+ (\Delta f + \mathrm{div}(x f))\, dx \\ &= -\int_{\mathbb{R}^d} |\nabla f_+|^2 - \int_{\mathbb{R}^d} f_+\, x \cdot \nabla f_+\, dx \le \frac{d}{2} \int_{\mathbb{R}^d} (f_+)^2\, dx, \end{aligned}$$

and thanks to the Gronwall lemma, we conclude that the maximum principle holds. Moreover, the function $G = e^{-V} \in L^1(\mathbb{R}^d) \cap \mathbf{P}(\mathbb{R}^d)$ is nothing but the normalized Gaussian function, and since $\nabla G = -G\, \nabla V$, it is a stationary solution to the Fokker-Planck equation (2.1).

Theorem 2.1. *Let us fix $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$.*
(1) There exists a unique global solution $f \in C([0,\infty); L^p(\mathbb{R}^d))$ to the Fokker-Planck equation (2.1). This solution is mass conservative

$$\langle f(t,.) \rangle := \int_{\mathbb{R}^d} f(t,x)\, dx = \int_{\mathbb{R}^d} f_0(x)\, dx =: \langle f_0 \rangle, \quad \text{if } f_0 \in L^1(\mathbb{R}^d), \tag{2.3}$$

and the following maximum principle holds

$$f_0 \ge 0 \quad \Rightarrow \quad f(t,.) \ge 0 \quad \forall\, t \ge 0.$$

(2) Asymptotically in large time the solution converges to the unique stationary solution with same mass, namely

$$\|f(t,.) - \langle f_0\rangle\, G\|_E \le e^{-\lambda_P t}\, \|f_0 - \langle f_0\rangle\, G\|_E \quad \textit{as} \quad t\to\infty, \tag{2.4}$$

where $\|\cdot\|_E$ stands for the norm of the Hilbert space $E := L^2(G^{-1})$ defined by

$$\|f\|_E^2 := \int_{\mathbb{R}^d} f^2\, G^{-1}\, dx$$

and λ_P is the best (larger) constant in the Poincaré inequality.

More generally, for any weight function $m : \mathbb{R}^d \to \mathbb{R}_+$, we denote by $L^p(m)$ the Lebesgue space associated to the mesure $m(x)dx$ and by L^p_m the Lebesgue space associated to the norm $\|f\|_{L^p_m} := \|f\, m\|_{L^p}$. We will also write $L^p_k := L^p_m$, when $m := \langle x\rangle^k$.

We are going to give the main lines of the proof of point 2. Because the equation is linear, we may assume in the sequel that $\langle f_0\rangle = 0$.

Using that $G\,G^{-1} = 1$, we deduce that $\nabla V = -G^{-1}\,\nabla G = G\cdot\nabla(G^{-1})$. We can then write the Fokker-Planck equation in the equivalent form

$$\begin{aligned} \frac{\partial}{\partial t} f &= \mathrm{div}_x\big(\nabla_x f + G\, f\, \nabla_x G^{-1}\big) \\ &= \mathrm{div}_x\big(G\,\nabla_x (f\, G^{-1})\big). \end{aligned} \tag{2.5}$$

We then compute

$$\begin{aligned} \frac12 \frac{d}{dt}\int f^2\, G^{-1} &= \int_{\mathbb{R}^d} (\partial_t f)\, f\, G^{-1}\, dx = \int_{\mathbb{R}^d} \mathrm{div}_x \left(G\,\nabla_x\left(\frac{f}{G}\right)\right) \frac{f}{G}\, dx \\ &= -\int_{\mathbb{R}^d} G\, \left|\nabla_x \frac{f}{G}\right|^2 dx. \end{aligned} \tag{2.6}$$

Using the Poincaré inequality established in the next Theorem 2.2 with the choice of function $h := f(t,.)/G$ and observing that $\langle f/G\rangle_G = 0$, we obtain

$$\frac12 \frac{d}{dt}\int f^2\, G^{-1} \le -\lambda_P \int_{\mathbb{R}^d} G\left(\frac{f}{G}\right)^2 dx = -\lambda_P \int_{\mathbb{R}^d} f^2\, G^{-1}\, dx,$$

and we conclude using the Gronwall lemma.

Theorem 2.2 (Poincaré inequality)**.** *There exists a constant $\lambda_P > 0$ (which only depends on the dimension) such that for any $h\in\mathcal{D}(\mathbb{R}^d)$, there holds*

$$\int_{\mathbb{R}^d} |\nabla h|^2\, G\, dx \ge \lambda_P \int_{\mathbb{R}^d} |h - \langle h\rangle_G|^2\, G\, dx, \tag{2.7}$$

where we have defined

$$\langle h\rangle_\mu := \int_{\mathbb{R}^d} h(x)\,\mu(dx)$$

for any given (probability) measure $\mu\in\mathbf{P}(\mathbb{R}^d)$ and any function $h\in L^1(\mu)$.

We present below two different proofs of this important result.

2.2. **A first proof of the Poincaré inequality.** We split the proof into three steps.

2.2.1. *Poincaré-Wirtinger inequality (in an open and bounded set Ω).*

Lemma 2.3. *Let us denote $\Omega = B_R$ the ball of $\mathbb{R}^d$ with center 0 and radius $R>0$, and let us consider $\nu\in\mathbf{P}(\Omega)$ a probability measure such that (abusing notations) $\nu, 1/\nu \in L^\infty(\Omega)$. There exists a constant $\kappa\in(0,\infty)$, such that for any (smooth) function f, there holds*

$$\kappa \int_\Omega |f - \langle f\rangle_\nu|^2\, \nu \le \int_\Omega |\nabla f|^2\, \nu, \qquad \langle f\rangle_\nu := \int_\Omega f\,\nu, \tag{2.8}$$

and therefore

$$\int_\Omega f^2\,\nu \le \langle f\rangle_\nu^2 + \frac{1}{\kappa}\int_\Omega |\nabla f|^2\,\nu. \tag{2.9}$$

Proof of Lemma 2.3. We start with

$$f(x) - f(y) = \int_0^1 \nabla f(z_t) \cdot (x-y)\, dt, \quad z_t = t\, x + (1-t)\, y.$$

Multiplying that identity by $\nu(y)$ and integrating in the variable $y \in \Omega$ the resulting equation, we get

$$f(x) - \langle f \rangle_\nu = \int_\Omega \int_0^1 \nabla f(z_t) \cdot (x-y)\, dt\, \nu(y)\, dy.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_\Omega (f(x) - \langle f \rangle_\nu)^2\, \nu(x)\, dx &\le \int_\Omega \int_\Omega \int_0^1 |\nabla f(z_t)|^2\, |x-y|^2\, dt\, \nu(y)\, \nu(x) dy dx \\ &\le C_1 \int_\Omega \int_\Omega \int_0^{1/2} |\nabla f(z_t)|^2\, dt dy\, \nu(x) dx + C_1 \int_\Omega \int_\Omega \int_{1/2}^1 |\nabla f(z_t)|^2\, dt dx\, \nu(y) dy,\end{aligned}$$

with $C_1 := \|\nu\|_{L^\infty}\, \mathrm{diam}(\Omega)^2$. Performing the the changes of variables $(x,y) \mapsto (z,y)$ and $(x,y) \mapsto (x,z)$ and using the fact that $z_t \in [x,y] \subset \Omega$, we deduce

$$\begin{aligned}&\int_\Omega (f(x) - \langle f \rangle_\nu)^2\, \nu(x)\, dx \\ &\le C_1 \int_\Omega \int_0^{1/2} \int_\Omega |\nabla f(z)|^2\, \frac{dz}{(1-t)^d}\, dt\, \nu(x) dx + C_1 \int_\Omega \int_{1/2}^1 \int_\Omega |\nabla f(z)|^2\, \frac{dz}{t^d}\, dt\, \nu(y) dy \\ &\le 2 C_1 \int_\Omega |\nabla f(z)|^2\, dz.\end{aligned}$$

We have thus established that the Poincaré-Wirtinger inequality (2.8) holds with the constant $\kappa^{-1} := 2\, C_1\, \|1/\nu\|_{L^\infty}$. $\square$

2.2.2. *Weighted L^2 estimate through L^2 estimate on the derivative.*

Proposition 2.4. *There holds*

$$\frac{1}{4} \int_{\mathbb{R}^d} h^2\, |x|^2\, G dx \quad \le \quad \int_{\mathbb{R}^d} |\nabla h|^2\, G dx + \frac{d}{2} \int_{\mathbb{R}^d} h^2\, G dx,$$

for any $h \in C^1_b(\mathbb{R}^d)$.

Proof of Proposition 2.4. We define $\Phi := -\log G = |x|^2/2 + \log(2\pi)^{d/2}$. For a given function h, we denote $g = h G^{1/2}$, and we expand

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla h|^2\, G\, dx &= \int_{\mathbb{R}^d} \left| \nabla g\, G^{-1/2} + g\, \nabla G^{-1/2} \right|^2\, G\, dx \\ &= \int_{\mathbb{R}^d} \left\{ |\nabla g|^2 + g \nabla g \nabla \Phi + \frac{1}{4} g^2 |\nabla \Phi|^2 \right\} dx,\end{aligned}$$

because $\nabla G^{-1/2} = \frac{1}{2} \nabla \Phi\, G^{-1/2}$. Performing one integration by part, we get

$$\int_{\mathbb{R}^d} |\nabla h|^2\, G\, dx = \int_{\mathbb{R}^d} |\nabla g|^2\, dx + \int_{\mathbb{R}^d} h^2\, \left(\frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi \right) G dx.$$

We conclude by neglecting the first term and computing the second term at the RHS. $\square$

2.2.3. *End of the first proof of the Poincaré inequality.* We split the L^2 norm into two pieces

$$\int_{\mathbb{R}^d} h^2\, G\, dx = \int_{B_R} h^2\, G\, dx + \int_{B_R^c} h^2\, G\, dx,$$

for some constant $R > 0$ to be choosen later. One the one hand, we have

$$\begin{aligned}\int_{B_R} h^2\, G\, dx &\le C_R \int_{B_R} |\nabla h|^2\, G\, dx + \Big(\int_{B_R^c} h\, G\, dx \Big)^2 \\ &\le C_R \int |\nabla h|^2\, G\, dx + \Big(\int_{B_R^c} G\, dx \Big) \int h^2\, G\, dx,\end{aligned}$$

where in the first line, we have used the Poincaré-Wirtinger inequality (2.9) in B_R with

$$\nu := G(B_R)^{-1} G_{|B_R}, \quad G(B_R) := \int_{B_R} G\,dx,$$

and the fact that $\langle hG \rangle = 0$, and in the second line, we have used the Cauchy-Schwarz inequality. One the other hand, we have

$$\begin{aligned} \int_{B_R^c} h^2\, G\, dx &\leq \frac{1}{R^2} \int_{\mathbb{R}^d} h^2\, |x|^2\, G\, dx \\ &\leq \frac{4}{R^2} \int_{\mathbb{R}^d} |\nabla h|^2\, G dx + \frac{2d}{R^2} \int_{\mathbb{R}^d} h^2\, G dx, \end{aligned}$$

by using Proposition 2.4. All together, we get

$$\int_{\mathbb{R}^d} h^2\, G\, dx \leq \big(C_R + \frac{4}{R^2}\big) \int_{\mathbb{R}^d} |\nabla h|^2\, G dx + \Big(\frac{2d}{R^2} + \int_{B_R^c} G\, dx\Big) \int h^2\, G\, dx,$$

and we choose $R > 0$ large enough in such a way that the constant in front of the last term at the RHS is smaller than 1. $\square$

2.3. **A second proof of the Poincaré inequality.** From (2.5), introducing the unknown $h := f/G$, we have

$$\begin{aligned} \partial_t h &= G^{-1} \mathrm{div}(G \nabla h) \\ &= \Delta h - x \cdot \nabla h =: Lh. \end{aligned}$$

On the one hand, we have

$$h(Lh) = L(h^2/2) - |\nabla h|^2,$$

L is self-adjoint in $L^2(G)$ and $L^* 1 = 0$. We then recover the identity (2.6), namely

$$\frac{1}{2} \frac{d}{dt} \int h^2 G\, dx = - \int |\nabla h|^2 G\, dx. \tag{2.10}$$

We fix $h_0 \in L^2(G)$ with $\langle h_0 G \rangle = 0$. We accept that $h_T \to 0$ in $L^2(G)$ as $T \to \infty$, what it has been already established during the proofs 1 and 2 or can be established without rate using softer argument (as it will be explained in the chapter about Lyapunov techniques). By time integration of (2.10), we thus have

$$\|h_0\|^2 = - \lim_{T \to \infty} \Big[\|h_t\|^2 \Big]_0^T = \lim_{T \to \infty} \int_0^T 2 \|\nabla h_t\|^2\, dt,$$

where here and below $\| \cdot \|$ denotes the $L^2(G)$ norm, and therefore

$$\|h_0\|^2 = \int_0^\infty 2 \|\nabla h_t\|^2\, dt. \tag{2.11}$$

On the other hand, we compute

$$\begin{aligned} \nabla h \cdot \nabla L h &= \nabla h \cdot \Delta \nabla h - \nabla h \cdot \nabla (x \cdot \nabla h) \\ &= \Delta(|\nabla h|^2/2) - |D^2 h|^2 - |\nabla h|^2 - x D h : D^2 h \\ &= L(|\nabla h|^2/2) - |D^2 h|^2 - |\nabla h|^2. \end{aligned}$$

We deduce

$$\frac{1}{2} \frac{d}{dt} \int |\nabla h|^2 G\, dx = - \int |D^2 h|^2 G\, dx - \int |\nabla h|^2 G\, dx \leq - \int |\nabla h|^2 G\, dx.$$

Similarly, as above, we have

$$\|\nabla h_0\|^2 - \|\nabla h_T\|^2 = - \int_0^T \frac{d}{dt} \|\nabla h_t\|^2\, dt \geq \int_0^T \|\nabla h_t\|^2\, dt,$$

and therefore

$$\|\nabla h_0\|^2 \geq \int_0^\infty 2 \|\nabla h_t\|^2\, dt. \tag{2.12}$$

Gathering (2.11) and (2.12), we conclude with the following Poincaré inequality with optimal constant.

Proposition 2.5 (Poincaré inequality with optimal constant)**.** *For any $h \in \mathcal{D}(\mathbb{R}^d)$ with $\langle hG\rangle = 0$,*

$$\|\nabla h\|_{L^2(G)} \ge \|h\|_{L^2(G)}.$$

We deduce from the above Poincaré inequality with optimal constant, the identity (2.10) and the Gronwall lemma, the following optimal decay estimate

$$\|h_t\|_{L^2(G)} \le e^{-t}\|h_0\|_{L^2(G)}, \quad \forall\, t \ge 0,$$

for any $h_0 \in L^2(G)$ such that $\langle h_0 G\rangle = 0$.

3. Log Sobolev inequality.

The estimate (2.4) gives a satisfactory (optimal) answer to the convergence to the equilibrium issue for the Fokker-Planck equation (2.1). However, we may formulate two criticisms. The proof is "completely linear" (in the sense that it can not be generalized to a nonlinear equation) and the considered initial data are very confined/localized (in the sense that they belong to the strong weighted space E, and again that it is not always compatible with the well posedness theory for nonlinear equations).

We present now a series of results which apply to more general initial data but, above all, which can be adapted to nonlinear equations. On the way, we will establish several functional inequalities of their own interest, among them the famous Log-Sobolev (or logarithmic Sobolev) inequality.

3.1. **Fisher information.** We are still interested in the harmonic Fokker-Planck equation (2.1)-(2.2). We define

$$D := \left\{ f \in L^1(\mathbb{R}^d); \quad f \ge 0, \quad \int f = 1, \quad \int f\, x = 0, \quad \int f\, |x|^2 = d \right\}$$

and

$$D_\le := \left\{ f \in L^1(\mathbb{R}^d); \quad f \ge 0, \quad \int f = 1, \quad \int f\, x = 0, \quad \int f\, |x|^2 \le d \right\}.$$

We observe that D (and $D_\le$) are invariant set for the flow of Fokker-Planck equation (2.1). We also observe that G is the unique stationary solution which belongs to D. Indeed, the equations for the first moments are

$$\partial_t \langle f\rangle = 0, \quad \partial_t\langle f x\rangle = -\langle f x\rangle, \quad \partial_t \langle f|x|^2\rangle = 2d\langle f\rangle - 2\langle f|x|^2\rangle. \tag{3.1}$$

It is therefore quite natural to think that any solution to the Fokker-Planck equation (2.1)-(2.2) with initial datum $f_0 \in D$ converges to G. It is what we will establish in the next paragraphs.

We define the Fisher information (or Linnik functional) $I(f)$ by

$$I(f) = \int \frac{|\nabla f|^2}{f} = 4\int |\nabla \sqrt{f}|^2 = \int f |\nabla \log f|^2$$

and the relative Fisher information $I(f|G)$ by

$$I(f|G) = I(f) - I(G) = I(f) - d.$$

Lemma 3.1. *For any $f \in D_\le$, there holds*

$$I(f|G) \ge 0, \tag{3.2}$$

with equality if, and only if, $f = G$.

Proof of Lemma 3.1. We define $V := \{f \in D_\le$ and $\nabla\sqrt{f} \in L^2\}$. We start with the proof of (3.2). For any $f \in V$, we have

$$\begin{aligned} 0 \le J(f) &:= \int \left| 2\nabla\sqrt{f} + x\,\sqrt{f}\right|^2 dx \\ &= \int \left(4\,|\nabla\sqrt{f}|^2 + 2\,x\cdot\nabla f + |x|^2\, f\right) dx = I(f) + \langle f|x|^2\rangle - 2\,d \\ &\le I(f) - d = I(f) - I(G) = I(f|G). \end{aligned}$$

We consider now the case of equality. If $I(f|G) = 0$ then $J(f) = 0$ and $2\nabla\sqrt{f} + x\,\sqrt{f} = 0$ a.e.. By a bootstrap argument, using Sobolev inequality, we deduce that $\sqrt{f} \in C^0$. Consider $x_0 \in \mathbb{R}^d$ such that $f(x_0) > 0$ (which exists because $f \in V$) and then $\mathcal{O}$ the open and connected to x_0 component of the set $\{f > 0\}$. We deduce from the preceding identity that $\nabla(\log \sqrt{f} + |x|^2/4) = 0$ in $\mathcal{O}$ and then $f(x) = e^{C-|x|^2/2}$ on $\mathcal{O}$ for some constant $C \in \mathbb{R}$. By continuity of f, we deduce that $\mathcal{O} = \mathbb{R}^d$, and then $C = -\log(2\,\pi)^{d/2}$ (because of the normalized condition imposed by the fact that $f \in V$). □

In some sense (see below) the relative Fisher information measure the distance to the steady state G. We also observe that

$$\frac{d}{dt} I(f) = I'(f) \cdot \mathcal{L} f, \tag{3.3}$$

with

$$I'(f) \cdot h = 2 \int \frac{\nabla f}{f} \nabla h - \int \frac{|\nabla f|^2}{f^2}\, h, \tag{3.4}$$

and we wish to establish that $I(f)$ decreases and converges to 0 with exponential decay.

Lemma 3.2. *For any smooth probability measure f, we have*

$$\frac{1}{2} I'(f) \cdot \Delta f = -\sum_{ij} \int \Big(\frac{1}{f^2}\, \partial_i f\, \partial_j f - \frac{1}{f} \partial_{ij} f \Big)^2 f, \tag{3.5}$$

$$\frac{1}{2} I'(f) \cdot (\nabla \cdot (f\, x)) = I(f), \tag{3.6}$$

$$\frac{1}{2} I'(f) \cdot \mathcal{L}\, f = -\sum_{ij} \int \Big(\frac{1}{f^2}\, \partial_i f\, \partial_j f - \frac{1}{f} \partial_{ij} f - \delta_{ij} \Big)^2 f - I(f|G). \tag{3.7}$$

As a consequence, there holds

$$\frac{1}{2} I'(f) \cdot \mathcal{L}(f) \le -I(f|G) \le 0.$$

Proof of Lemma 3.2. *Proof of* (3.5). Starting from (3.4) and integrating by part with respect to the x_i variable, we have

$$\begin{aligned} \frac{1}{2} I'(f) \cdot \Delta f &= \int \frac{1}{f} \partial_j f\, \partial_{iij} f - \int \frac{1}{2\, f^2} (\partial_j f)^2 \partial_{ii} f \\ &= \int \Big(\frac{\partial_i f}{f^2} \partial_j f\, \partial_{ij} f - \frac{1}{f} \partial_{ij} f\, \partial_{ij} f \Big) + \int \Big(\frac{1}{f^2} \partial_i f\, \partial_j f\, \partial_{ij} f - \frac{\partial_i f}{f^3} \partial_i f\, (\partial_j f)^2 \Big) \\ &= -\sum_{ij} \int \Big(\frac{1}{f^2}\, \partial_i f\, \partial_j f - \frac{1}{f} \partial_{ij} f \Big)^2 f. \end{aligned}$$

Proof of (3.6). We write

$$\frac{1}{2} I'(f) \cdot (\nabla \cdot (f\, x)) = \int \frac{\partial_j f}{f} \partial_{ij} (f\, x_i) - \frac{(\partial_j f)^2}{2\, f^2}\, \partial_i (f\, x_i).$$

We observe that

$$\begin{aligned} \partial_{ij}(f\, x_i) - \frac{(\partial_j f)}{2\, f}\, \partial_i (f\, x_i) &= (\partial_{ij} f)\, x_i + d\, \partial_j f + \delta_{ij}\, \partial_j f - \partial_i f\, \partial_j f\, \frac{x_i}{2\, f} - \frac{d}{2} \partial_j f \\ &= (\partial_{ij} f)\, x_i + (\frac{d}{2} + 1)\, \partial_j f - \partial_i f\, \partial_j f\, \frac{x_i}{2\, f}. \end{aligned}$$

Gathering the two preceding equalities, we obtain

$$\frac{1}{2} I'(f) \cdot (\nabla \cdot (f\, x)) = (\frac{d}{2} + 1)\, I(f) + \int \frac{\partial_j f}{f}\, \partial_{ij} f\, x_i - \int \frac{\partial_j f}{f}\, \partial_i f\, \partial_j f\, \frac{x_i}{2\, f}.$$

Last, we remark that thank to an integration by parts

$$-\frac{d}{2}\, I(f) = \frac{1}{2} \int \partial_i \left(\frac{(\partial_j f)^2}{f} \right) x_i = \int \frac{\partial_j f\, \partial_{ij} f}{f}\, x_i - \frac{1}{2} \frac{(\partial_j f)^2}{f^2}\, \partial_i f\, x_i,$$

and we then conclude

$$\frac{1}{2} I'(f) \cdot (\nabla \cdot (f\, x)) = I(f).$$

Proof of (3.7). Developing the expression below and using (3.5), we have

$$\begin{aligned} 0 &\leq \sum_{ij} \int \Big(\frac{1}{f^2}\, \partial_i f \,\partial_j f - \frac{1}{f} \partial_{ij} f - \delta_{ij} \Big)^2 f \\ &= -\frac{1}{2} I'(f) \cdot \Delta f + 2 \sum_i \int \Big(\partial_{ii} f - \frac{1}{f}\, (\partial_i f)^2 \Big) + d \int f. \end{aligned}$$

From $\int f = 1$, $\int \partial_{ii} f = 0$ and (3.6), we then deduce

$$0 \leq -\frac{1}{2} I'(f) \cdot \Delta f - 2 I(f) + d = -\frac{1}{2} I'(f) \cdot \mathcal{L} f + d - I(f),$$

which ends the proof of (3.7). □

Theorem 3.3. *The Fisher information I is decreasing along the flow of the Fokker-Planck equation, i.e. I is a Lyapunov functional, and more precisely*

$$I(f(t,.)|G) \leq e^{-2\,t}\, I(f_0|G). \tag{3.8}$$

That implies the convergence in large time to G of any solution to the Fokker-Planck equation associated to any initial condition $f_0 \in D \cap V$. More precisely,

$$\forall\, f_0 \in D \cap V \qquad f(t,.) \to G \quad in \quad L^p \cap L^1_2 \quad as \quad t \to \infty, \tag{3.9}$$

for any $p \in [1, 2^\sharp)$ where $2^\sharp = \infty$ when $d = 1, 2$ and $2^\sharp = d/(d-2)$ when $d \geq 3$.

During the proof of Theorem 3.3, we will need the following result (see **Excercise 4.3**).

Lemma 3.4. *A sequence (f_n) which is bounded in $L^1_2 \cap L^q$, $q > 1$, and is such that $f_n \to g$ a.e. in $\mathbb{R}^d$, also satisfies*

$$f_n \to g \quad in \quad L^p \cap L^1_k, \quad \forall\, k \in [0, 2),\ \forall\, p \in [1, q).$$

If furthemore, $\|f_n\|_{L^1_2} = \|g\|_{L^1_2}$ for any $n \geq 1$, then $f_n \to g$ in L^1_2.

Proof of Theorem 3.3. We only consider the case $d \geq 3$. On the one hand, thanks to (3.7), we have

$$\frac{d}{dt} I(f|G) \leq -2\, I(f|G), \tag{3.10}$$

and we conclude to (3.8) thanks to the Gronwall lemma. On the other hand, thanks to the Sobolev inequality, we have

$$\|f\|_{L^{2^*/2}} = \|\sqrt{f}\|^2_{L^{2^*}} \leq C\, \|\nabla \sqrt{f}\|^2_{L^2} = C\, I(f) \leq C\, I(f_0).$$

Consider now an increasing sequence (t_n) which converges to $+\infty$. Thanks to estimate (3.8) and the Rellich Theorem, we may extract a subsequence still denoted as (t_n) such that $\sqrt{f(t_n)}$ converges a.e. and weakly in $\dot H^1$ to a limit denoted by $\sqrt{g}$. As a consequence, $f(t_n) \to g$ a.e. and $(f(t_n))$ is bounded in $L^{2^*/2} \cap L^1_2$, so that $f_n \to g$ in $L^p \cap L^1_k$, $\forall\, k \in [0, 2)$, $\forall\, p \in [1, q)$, thanks to Lemma 3.6. From the lower semicontinuity of the norms, we have g is bounded in $L^{2^*/2} \cap L^1_2$, $\langle |v|^2 g \rangle \leq \liminf \langle |v|^2 f(t_n) \rangle = d$ and $I(g) \leq \liminf I(f(t_n)) < \infty$, so that $g \in D_{\leq} \cap V$. Finally, since $2\nabla \sqrt{f(t_n)} - x \sqrt{f(t_n)} \to 2\nabla \sqrt{g} - x\sqrt{g}$ weakly in L^2_{loc} (for instance) and (3.8), we have

$$0 \leq J(g) \leq \liminf_{k \to \infty} J(f(t_n, .) = \liminf_{k \to \infty} I(f(t_n, .)|G) = 0.$$

From $J(g) = 0$ and $g \in V \cap D_{\leq}$, we get $g = G$ as a consequence of Lemma 3.1, and it is then the all family $(f(t))_{t \geq 0}$ which converges to G as $t \to \infty$. The L^1_2 convergence is a consequence of the fact that $\langle f(t)\, |v|^2 \rangle = \langle G\, |v|^2 \rangle$ for any time $t \geq 0$ together with Lemma 3.6. □

3.2. **Entropy and Log-Sobolev inequality.** For a function $f \in D$, we define the entropy $H(f) \in \mathbb{R} \cup \{+\infty\}$ and the relative entropy $H(f|G) \in \mathbb{R} \cup \{+\infty\}$ by

$$H(f) = \int_{\mathbb{R}^d} f \, \log f \, dx, \quad H(f|G) = H(f) - H(G) = \int_{\mathbb{R}^d} j(f/G) \, G \, dx,$$

where $j : \mathbb{R}+ \to \mathbb{R}_+$, $j(s) := s \, \log s - s + 1$. It is worth emphasizing that the last integral is always defined in $\mathbb{R} \cup \{+\infty\}$ because $j(f) \geq 0$ and that for establishing the last equality we use that

$$\int_{\mathbb{R}^d} f \, \log G \, dx = \int_{\mathbb{R}^d} G \, \log G \, dx$$

because $f \in D$.

We start observing that for $f \in \mathbf{P}(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d)$, there holds

$$\begin{aligned} H'(f) \cdot \mathcal{L} f &:= \int_{\mathbb{R}^d} (1 + \log f) \, [\Delta f + \nabla \cdot (x \, f)] \\ &= -\int_{\mathbb{R}^d} \nabla f \cdot \nabla \log f - \int_{\mathbb{R}^d} x \, f \cdot \nabla \log f \\ &= -I(f) + d \, \langle f \rangle = -I(f|G). \end{aligned}$$

As a consequence, the entropy is a Lyapunov functional for the Fokker-Planck equation and more precisely

$$\frac{d}{dt} H(f) = -I(f|G) \leq 0. \tag{3.11}$$

Theorem 3.5. (Logarithmic Sobolev inequality). *For any $f \in D \cap V$, the following Log-Sobolev inequality holds*

$$H(f|G) \leq \frac{1}{2} I(f|G). \tag{3.12}$$

That one also writes equivalently as

$$\int_{\mathbb{R}^d} f \ln f - \int_{\mathbb{R}^d} G \ln G \leq \frac{1}{2} \Big(\int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f} - d \Big)$$

or also as

$$\int_{\mathbb{R}^d} u^2 \, \log(u^2) \, G(dx) \leq 2 \int_{\mathbb{R}^d} |\nabla u|^2 \, G(dx).$$

For some applications, it is worth emphasizing that the Log-Sobolev inequality depends on a nicer way of the dimension than the Poincaré inequality.

During the proof of Theorem 3.5, we will need the following result (see **Excercise 4.5**).

Lemma 3.6. *Consider a sequence (f_n) such that $0 \leq f_n \to f$ in $L^q \cap L^1_k$, $q > 1$, $k > 0$, then $H(f_n) \to H(f)$.*

Proof of Theorem 3.5. We denote by f_t the solution to the Fokker-Planck equation (2.1) associated to the initial datum $f_0 := f$. On the one hand, from (3.9), Lemma 3.6 and (3.11), we get

$$\begin{aligned} H(f) - H(G) &= \lim_{T \to \infty} [H(f) - H(f_T)] = \lim_{T \to \infty} \int_0^T \left[-\frac{d}{dt} H(f) \right] \, dt \\ &= \lim_{T \to \infty} \int_0^T [I(f|G)] \, dt. \end{aligned}$$

From that identity and (3.10), we deduce

$$\begin{aligned} H(f) - H(G) &\leq \lim_{T \to \infty} \int_0^T \left[-\frac{1}{2} \frac{d}{dt} I(f|G) \right] \, dt \\ &= \lim_{T \to \infty} \frac{1}{2} \, [I(f|G) - I(f_T|G)] = \frac{1}{2} I(f|G), \end{aligned}$$

thanks to (3.8). ◻

Lemma 3.7. (Csiszár-Kullback inequality). *Consider probability measure μ and a nonnegative measurable function g such that $g\,\mu$ is also a probability measure. Then*

$$\|g-1\|^2_{L^1(d\mu)} \le 2\int g\log g\,d\mu. \tag{3.13}$$

Proof of Lemma 3.7. Thanks to the Taylor-Laplace expansion formula, there holds

$$\begin{aligned} j(g) &:= g\log g - g + 1 = j(1) + (g-1)\,j'(1) + (g-1)^2\int_0^1 j''(1+s\,(g-1))\,(1-s)\,ds \\ &= (g-1)^2\int_0^1 \frac{1-s}{1+s\,(g-1)}\,ds. \end{aligned}$$

Using Fubini theorem, we get

$$H(g) := \int (g\log g - g + 1)\,d\mu = \int_0^1 (1-s)\int \frac{(g-1)^2}{1+s\,(g-1)}d\mu\,ds.$$

For any $s\in[0,1]$, we use the Cauchy-Schwarz inequality and the fact that both μ and $g\,\mu$ are probability measures in order to deduce

$$\left(\int |g-1|\,d\mu\right)^2 \le \left(\int \frac{(g-1)^2}{1+s\,(g-1)}d\mu\right)\left(\int [1+s\,(g-1)]d\mu\right) = \int \frac{(g-1)^2}{1+s\,(g-1)}d\mu.$$

As a conclusion, we obtain

$$H(g) \ge \int_0^1 \left(\int |g-1|\,d\mu\right)^2 (1-s)\,ds = \frac{1}{2}\left(\int |g-1|\,d\mu\right)^2,$$

which ends the proof of the Csiszár-Kullback inequality. □

Putting together (3.11), (3.12) and (3.13) with $G := \mu$ and $g := f/G$, we immediately obtain the following convergence result.

Theorem 3.8. *For any $f_0 \in D$ such that $H(f_0) < \infty$, the associated solution f to the Fokker-Planck equation* (2.1)-(2.2) *satisfies*

$$H(f|G) \le e^{-2t}\,H(f_0|G),$$

and then

$$\|f-G\|_{L^1} \le \sqrt{2}\,e^{-t}\,H(f_0|G)^{1/2}.$$

4. Exercises and Complements

Exercise 4.1. *Observe that the function $H := x_k$ satisfies*

$$H \in L^2(G), \quad \langle HG \rangle = 0, \quad LH = -H, \quad \|\nabla H\|^2_{L^2(G)} = \|H\|^2_{L^2(G)}.$$

Conclude that the constant $\lambda_P = 1$ in the Poincaré inequality established in Proposition 2.5 is optimal.

Exercise 4.2. *Generalize the Poincaré inequality to a general superlinear potential $V(x) = \langle x \rangle^\alpha/\alpha + V_0$, $\alpha \geq 1$, in the following strong (weighted) formulation*

$$\int |\nabla g|^2 \, \mathcal{G} \geq \kappa \int |g - \langle g \rangle_{\mathcal{G}}|^2 \, (1 + |\nabla V|^2) \, \mathcal{G} \qquad \forall \, g \in \mathcal{D}(\mathbb{R}^d),$$

where we have defined $\mathcal{G} := e^{-V} \in \mathbf{P}(\mathbb{R}^d)$ (for an appropriate choice of $V_0 \in \mathbb{R}$).

Exercise 4.3. *Establish Lemma 3.6.*

A possible solution of Exercise 4.3. We first establish that $f_n \to g$ strongly in L^1. We write

$$\|f_n - g\|_{L^1} \quad \leq \quad \int_{B_R} |f_n - g| \wedge M + 2R^{-2} \sup_n \int_{B_R^c} |f_n| |x|^2 + 2M^{1-q} \sup_n \int_{B_R} |f_n|^q,$$

for any $R, M > 0$ and any $k \geq 1$, and by using the dominated convergence theorem of Lebesgue for the first term. Thanks to the interpolation inequalities

$$\|h\|_{L^p} \leq \|h\|^{1-\alpha}_{L^1} \|h\|^\alpha_{L^q} \quad \text{and} \quad \|h\|_{L^1_k} \leq \|h\|^{k/2}_{L^1_2} \|h\|^{1-k/2}_{L^1},$$

with $1/p = 1 - \alpha + \alpha/q$, we next get that $f_n \to g$ strongly in $L^p \cap L^1_k$, for any $p \in [1, q)$, $k \in [0, 2)$. When we furthermore assume $f_n, g \geq 0$ and $\langle g|x|^2 \rangle = \langle f_n |x|^2 \rangle = d$ for any $n \geq 1$, from Fatou lemma, we may first deduce

$$\begin{aligned} \limsup_n \int_{B_R^c} f_n |x|^2 \quad &= \quad d - \liminf_n \int_{B_R} f_n |x|^2 \\ &\leq \quad d - \int_{B_R} f|x|^2 = \int_{B_R^c} f|x|^2, \end{aligned}$$

for any $R > 0$. On the other hand, we have

$$\int |f_n - f| \, |x|^2 \, dx \leq \int_{B_R} |f_n - f| \, |x|^2 \, dx + \int_{B_R^c} f_n \, |x|^2 \, dx + \int_{B_R^c} f \, |x|^2 \, dx.$$

From the two above informations together with the convergence $f_n \to f$ in L^1, we deduce

$$\limsup_n \|f - f_n\|_{L^1_2} \leq 2 \int_{B_R^c} f \, |x|^2 \, dx,$$

for any $R > 0$, and thus the conclusion by letting $R \to \infty$. □

Exercise 4.4. *Prove the convergence* (3.8) *for any $f_0 \in \mathbf{P}(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ such that $I(f_0) < \infty$. (Hint. Compute the equations for the moments of order* 1 *and* 2 *and introduce the relative Fisher information $I(f|M_{1,u,\theta})$ associated to a normalized Gaussian with mean velocity $u \in \mathbb{R}^d$ and temperature $\theta > 0$).*

Exercise 4.5. *Prove that $0 \leq f_n \to f$ in $L^q \cap L^1_k$, $q > 1$, $k > 0$, implies that $H(f_n) \to H(f)$. (Hint. Use the splitting*

$$s \, |\log s| \leq \sqrt{s} \, \mathbf{1}_{0 \leq s \leq e^{-|x|^k}} + s \, |x|^k \, \mathbf{1}_{e^{-|x|^k} \leq s \leq 1} + s (\log s)_+ \, \mathbf{1}_{s \geq 1} \quad \forall \, s \geq 0$$

and the dominated convergence theorem).

Exercise 4.6. *Generalize Theorem 3.8 to the case when $f_0 \in \mathbf{P}(\mathbb{R}^d) \cap L^1_q(\mathbb{R}^d)$, $q = 2$ or $q > 1$, such that $H(f_0) < \infty$. (Hint. Proceed along the same line as in Exercise 4.4).*

Exercise 4.7. *Generalize Theorem 3.5 and Theorem 3.8 to the case of a super-harmonic potential $V(x) = \langle x \rangle^\alpha / \alpha$, $\alpha \geq 2$, and to an initial datum $\varphi \in \mathbf{P}(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ such that $H(\varphi) < \infty$.*