Université Paris Dauphine-PSL November 17, 2025
Academic Master MATH
An introduction to evolution PDEs

CHAPIER 6: MORE ABOUT REGULARITY & LONGTIME BEHAVIOUR

We present several complementary tools about the regularity of solutions and their longtime asymptotic. The last part, in blue color has not been yet presented during the classes (and maybe will not be presented this year).

1. De Giorgi, Nash, Moser

At least for the parabolic equation

$$\partial_t f = \mathrm{div}(A\,\nabla f) \quad \text{on} \quad \mathcal{U} := (0,T)\times\mathbb{R}^d, \tag{1.1}$$

where

$$0 < \nu I \le A \in L^\infty(\mathcal{U}), \tag{1.2}$$

we have already seen that we may establish the ultracontractivity estimate

$$\|f(t,\cdot)\|_{L^p} \le \frac{C}{t^{d/2(1/q-1/p)}}\|f_0\|_{L^p}, \tag{1.3}$$

for any $1 \le q \le p \le \infty$. Nash approach consists in observing that

$$\frac{d}{dt}\int \beta(f) = -\int \beta''(f) A\nabla f \nabla f \le 0, \tag{1.4}$$

for any convex function $\beta : \mathbb{R}\to\mathbb{R}$, in making the choice $\beta(s) = s^2$ and $\beta(s) = |s|$ and in using some interpolation and ODE estimates in order to establish the ultracontractivity estimate (1.3) for $p = 2$ and $q = 1$. Duality and interpolation arguments make possible to get the full range of exponents. We explain how some other choices of convex functions β lead to the same conclusion. These techniques are more suitable in some situations, in particular in order to establish the ultracontractivity estimate (1.3) for the most general parabolic equations.

1.1. **Moser approach.** In order to simplicity the notations, let us only consider the heat equation and thus a solution f to

$$\frac{\partial f}{\partial t} = \Delta f \quad \text{on} \quad \mathcal{U}, \qquad f(0,\cdot) = f_0 \quad \text{in} \quad \mathbb{R}^d. \tag{1.5}$$

• We recall that this one satisfies the energy identity

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2} = -\|\nabla f\|^2_{L^2}. \tag{1.6}$$

We integrate in time equation (1.6) in order to get

$$\frac{1}{2}\int f_t^2 + \int_s^t\int |\nabla f|^2 = \frac{1}{2}\int f_s^2,$$

for any $0 < s < t$. We fix $0 < t_0 < t_1 < t < T$ and we integrate in $s \in (t_0, t_1)$ the above equation. We obtain

$$(t_1 - t_0) \int f_t^2 + 2(t_1 - t_0) \int_{t_1}^t \int |\nabla f|^2 \le \int_{t_0}^T \int f^2.$$

Taking the supremum in $t \in (t_1, T)$ and throwing up the second term at the RHS, we deduce

$$\|f\|^2_{L^\infty(I_1;L^2)} \le \frac{1}{t_1 - t_0} \|f\|^2_{L^2(I_0;L^2)}, \tag{1.7}$$

where $I_i := [t_i, T]$. On the other hand, taking $t := T$, throwing up the first term at the RHS and using the Sobolev inequality

$$\|f\|_{L^{2^*}} \le C_S \|\nabla f\|_{L^2}, \quad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}, \tag{1.8}$$

for estimating by below the second term at the RHS, we deduce

$$\|f\|^2_{L^2(I_1;L^{2^*})} \le \frac{C_S^2}{2} \frac{1}{t_1 - t_0} \|f\|^2_{L^2(I_0;L^2)}. \tag{1.9}$$

We now recall the interpolation inequality

$$\|g\|_{L^{q_\theta} L^{r_\theta}} \le \|g\|^\theta_{L^{q_0} L^{r_0}} \|g\|^{1-\theta}_{L^{q_1} L^{r_1}}, \tag{1.10}$$

where

$$\frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}, \quad \frac{1}{r_\theta} = \frac{\theta}{r_0} + \frac{1-\theta}{r_1}, \quad \theta \in [0, 1],$$

which proof is left as an exercise (using twice the Holder inequality). Using this interpolation inequality with θ such that

$$\frac{1}{p} := \frac{1-\theta}{2} = \frac{\theta}{2} + \frac{1-\theta}{2^*},$$

we deduce

$$\|f\|^2_{L^p(I_1;L^p)} \le \frac{C}{t_1 - t_0} \|f\|^2_{L^2(I_0;L^2)}, \quad p := 2(1 + 2/d). \tag{1.11}$$

• Before continuing, let us make two observations. First, for any smooth function $\beta : \mathbb{R} \to \mathbb{R}$, we have

$$\partial_t \beta(f) = \beta'(f) \Delta f = \Delta \beta(f) - \beta''(f) |\nabla f|^2,$$

so that when β is convex, the function $\beta(f)$ is a subsolution, namely it satisfies

$$\partial_t \beta(f) \le \Delta \beta(f). \tag{1.12}$$

Next, for any subsolution $g \ge 0$ to the heat equation, we may repeat all the above argument, and we get in the very same manner

$$\|g\|^2_{L^p(\mathcal{U}_{k+1})} \le C \frac{1}{t_{k+1} - t_k} \|g\|^2_{L^2(\mathcal{U}_k)}, \tag{1.13}$$

with $\mathcal{U}_k := I_k \times \mathbb{R}^d$, $I_k := (t_k, T]$ and $0 \le t_k < t_{k+1} < T$.

• We consider now a solution $f \ge 0$ to the heat equation and we define

$$t_k := \frac{T}{2} - \frac{T}{2^k}, \quad k \ge 1, \quad p_{k+1} := (1 + 2/d) p_k, \quad k \ge 1, \quad p_1 := 2.$$

Because $p_k/2 \geq 1$, and thus $s \mapsto |s|^{p_k/2}$ is convex, and because of (1.12), the function $g := f^{p_k/2}$ is a subsolution. Applying (1.13) to this function g, we obtain

$$\begin{aligned} \|f\|_{L^{p_{k+1}}(\mathcal{U}_{k+1})} &= \|f^{p_k/2}\|^{2/p_k}_{L^p(\mathcal{U}_{k+1})} \\ &\leq \big(C\frac{2^k}{T}\|f^{p_k/2}\|^2_{L^2(\mathcal{U}_k)}\big)^{1/p_k} = \big(C\frac{2^k}{T}\big)^{1/p_k}\|f\|_{L^{p_k}(\mathcal{U}_k)}. \end{aligned}$$

Observing that

$$\sum_{k=1}^\infty \frac{1}{p_k} = \frac{1}{2}\sum_{j=0}^\infty \frac{1}{(1+2/d)^j} = \frac{1}{2}+\frac{d}{4},$$

we deduce that

$$\prod_{k=1}^\infty \big(C\frac{2^k}{T}\big)^{1/p_k} \lesssim T^{-1/2-d/4}.$$

As a consequence, we have

$$\begin{aligned} \|f\|_{L^\infty(\mathcal{U}_\infty)} &\leq \liminf_{k\to\infty} \|f\|_{L^{p_k}(\mathcal{U}_k)} \\ &\leq \liminf_{k\to\infty} \prod_{j=1}^k \big(C\frac{2^j}{T}\big)^{1/p_j}\|f\|_{L^{p_1}(\mathcal{U}_1)} \end{aligned}$$

and thus

$$\|f\|_{L^\infty(\mathcal{U}_\infty)} \lesssim T^{-1/2-d/4}\|f\|_{L^2(\mathcal{U}_1)}. \tag{1.14}$$

Finally, together with the decay of the L^2 norm (1.6) which implies

$$\|f\|_{L^2(\mathcal{U}_1)} \leq T^{1/2}\|f_0\|_{L^2},$$

we have thus established

$$\|f_T\|_{L^\infty} \lesssim \frac{1}{T^{d/4}}\|f_0\|_{L^2}. \tag{1.15}$$

Estimate (1.15) is the dual estimate of (1.3) with $p=2$ and $q=1$ obtained as a first step using Nash method. We may thus end the proof of the full range estimate (1.3) by arguing by duality and interpolation exactly as in Nash proof.

1.2. **De Giorgi approach.** For a fixed $K>0$, we rather use the convex function $\beta(s) := (s-K)^2_+$ in (1.4), which gives

$$\frac{1}{2}\frac{d}{dt}\int (f-K)^2_+ = -\int |\nabla(f-K)_+|^2.$$

Repeating the first step in Moser's proof and using the notation $f_K := (f-K)_+$, we straightforwardly obtain

$$\|f_K\|^2_{L^p(\mathcal{U}')} \leq \frac{C}{\tau'-\tau}\|f_K\|^2_{L^2(\mathcal{U})}, \tag{1.16}$$

with $\mathcal{U} := (\tau,T)$, $\mathcal{U}' := (\tau',T)$, $0\leq\tau<\tau'<T$ and $p := 2(1+2/d) > 2$.

Now, for $K' > K > 0$, we first observe that thanks to the Holder inequality

$$\|f_{K'}\|_{L^2(\mathcal{U}')} \leq \|f_{K'}\|_{L^p(\mathcal{U}')}|\{f_{K'}>0\}\cap\mathcal{U}'|^{1/2-1/p}. \tag{1.17}$$

On the other hand, because $f_{K'}>0$ is equivalent to $f_K > K'-K$, the Tchebychev inequality implies

$$|\{f_{K'}>0\}\cap\mathcal{U}'| \leq \frac{1}{(K'-K)^2}\|f_K\|^2_{L^2(\mathcal{U})}. \tag{1.18}$$

Putting together the above three estimates and using the elementary inequality $f_{K'} \le f_K$, we have key inequality

$$\|f_{K'}\|^2_{L^2(\mathcal{U}')} \quad \le \quad \frac{C}{\tau' - \tau} \frac{1}{(K'-K)^{2(1-2/p)}} \|f_K\|^{2(2-2/p)}_{L^2(\mathcal{U})}.$$

Let us introduce an iterative argument. We define the sequence of (increasing) time, (decreasing) cylinder and (increasing) truncation barrier

$$T_n := T - \tfrac{T}{2}(1+2^{-n}), \quad \mathcal{U}_n := (T_n, T) \times \mathbb{R}^d, \quad K_n := \tfrac{1}{2}(1-2^{-n}),$$

and the sequence of energy

$$\mathcal{E}_n := \int_{\mathcal{U}_n} (f-K_n)^2_+ \, dxdt.$$

Using the above key inequality with $\tau = T_n$, $\tau' = T_{n+1}$, $K = K_n$, $K' = K_{n+1}$ and observing that $T_{n+1} - T_n = T2^{-(n+2)}$, $K_{n+1} - K_n = 2^{-(n+2)}$, $2 - 2/p = 1 + \alpha$ with $\alpha := d/(d+2) \in (0,1)$, we get

$$\mathcal{E}_{n+1} \le M^n \mathcal{E}_n^{1+\alpha}, \quad \forall\, n \ge 1,$$

for some constant $M > 1$. We use now the following elementary result.

Lemma 1.1. *If a sequence $(v_n)_{n\ge0}$ satisfies $0 \le v_{n+1} \le M^n v_n^{1+\alpha}$ for any $n \ge 1$ and $v_0 < M^{-\frac{1}{\alpha^2}}$ for some $M > 1$ and $\alpha > 0$, then $v_n \to 0$ as $n \to \infty$.*

Proof of Lemma 1.1. Let us define $u_n := u_0 \varrho^n$. We observe that

$$M^n u_n^{1+\alpha} = M^n \varrho^{n\alpha-1} u_0^\alpha u_{n+1} = u_{n+1},$$

provided that $M\varrho^\alpha = 1$ and $\varrho^{-1} u_0^\alpha = 1$. With this choice, we have $\varrho := M^{-1/\alpha}$ and $u_0 := \varrho^{1/\alpha} = M^{-1/\alpha^2}$. For $v_0 \le u_0$, we straighforwardly establish recursively that $v_n \le u_n$ and thus $v_n \to 0$. □

Assuming $\mathcal{E}_0 < \delta := M^{-\frac{1}{\alpha^2}}$, we deduce from Lemma 1.1 that $\mathcal{E}_n \to 0$ as $n \to \infty$, and in particular

$$\mathcal{E}_\infty := \int_{\mathcal{U}_\infty} (f-K_\infty)^2_+ dxdt = 0,$$

with $\mathcal{U}_\infty = (T/2, T) \times \mathbb{R}^d$ and $K_\infty = 1/2$. That precisely means that

$$f \le 1/2 \quad \text{on} \quad \mathcal{U}_\infty. \tag{1.19}$$

In other words, we have established (first De Giorgi Lemma)

$$\|f_+\|_{L^\infty(\mathcal{U}_\infty)} \le \frac{1}{2\delta} \|f\|_{L^2(\mathcal{U}_0)}.$$

It is not difficult to recover again (1.15) from that last estimate.

2. General relative entropy

We consider the parabolic equation

$$\partial_t f = \operatorname{div}(A\nabla f + af) + b\cdot\nabla f + cf. \tag{2.20}$$

If $g > 0$ is a solution to the same equation

$$\partial_t g = \operatorname{div}(A\nabla g) + b\cdot\nabla g + cg$$

and if $\phi \ge 0$ is a solution to the dual evolution problem

$$-\partial_t \phi = \operatorname{div}(A^\top \nabla\phi) - \operatorname{div}(b\phi) + c\phi,$$

we can exhibit a family of entropies associated to the evolution PDE (2.20). More precisely, we establish the following result (and in fact a bit more accurate formulation of it).

Theorem 2.2. *For any real values convex function H, the generalized entropy functional*

$$f \mapsto \mathcal{H}(f) := \int_{\mathbb{R}^d} H(f/g)\, g\, \phi, \tag{2.21}$$

is an Lyapunov function for the evolution PDE (2.20) *(meaning that is is decaying function of time along the flows of the evolution PDE).*

Step 1. First order PDE. We assume that

$$\begin{aligned} \partial_t f &= b \cdot \nabla f + cf \\ \partial_t g &= b \cdot \nabla g + cg \\ -\partial_t \phi &= -\mathrm{div}(b\,\phi) + c\,\phi, \end{aligned}$$

and we show that

$$\partial_t (H(X) g\phi) - \mathrm{div}(bH(X)g\phi) = 0, \quad X = f/g.$$

We compute

$$\begin{aligned} &\partial_t (H(X) g\phi) - \mathrm{div}(bH(X)g\phi) \\ &= H'(X) g\phi\, [\partial_t X - b \cdot \nabla X] + H(X)\,[\partial_t (g\phi) - \mathrm{div}(bg\phi)]. \end{aligned}$$

The first term vanishes because

$$\partial_t X - b \cdot \nabla X = \frac{1}{g}\,(\partial_t f - b \cdot \nabla f) - \frac{f}{g^2}\,(\partial_t g - b \cdot \nabla g) = \frac{1}{g}\,(cf) - \frac{f}{g^2}\,(cg) = 0.$$

The second term also vanishes because

$$\partial_t (g\phi) - \mathrm{div}(bg\phi) = \phi\,[\partial_t g - b \cdot \nabla g] + g\,[\partial_t \phi - \mathrm{div}(b\phi)] = \phi\,[-cg] + g\,[c\phi] = 0.$$

Step 2. Second order PDE. We assume that

$$\begin{aligned} \partial_t f &= \mathrm{div}(A\nabla f) + cf \\ \partial_t g &= \Delta g + cg \\ -\partial_t \phi &= \Delta \phi + c\,\phi, \end{aligned}$$

and we show

$$\partial_t (H(X) g\phi) - \mathrm{div}(\phi \nabla (H(X) g)) + \mathrm{div}(gH(X)\nabla\phi) = -H''(X) g\phi |\nabla X|^2.$$

We first observe that

$$\begin{aligned} \Delta X &= \mathrm{div}\Big(\frac{\nabla f}{g} - f\,\frac{1}{g^2}\,\nabla g\Big) \\ &= \frac{\Delta f}{g} - 2\nabla f\,\frac{\nabla g}{g^2} + 2\,f\,\frac{|\nabla g|^2}{g^3} - \frac{f}{g^2}\,\Delta g \\ &= \frac{\Delta f}{g} - \frac{f\,\Delta g}{g^2} - 2\,\frac{\nabla g}{g} \cdot \nabla X, \end{aligned}$$

which in turn implies

$$\partial_t X - \Delta X = 2\,\frac{\nabla g}{g} \cdot \nabla X.$$

We then compute

$$\begin{aligned}
&\partial_t(H(X)g\phi) - \operatorname{div}(\phi\nabla(H(X)g)) + \operatorname{div}(gH(X)\nabla\phi) = \\
&= (\partial_t H(X))\, g\phi + H(X)\,\partial_t(g\phi) - \phi\operatorname{div}[gH'(X)\nabla X + H(X)\nabla g] + gH(X)\Delta\phi \\
&= H'(X)g\phi\left\{\partial_t X - \Delta X - 2\frac{\nabla g}{g}\cdot\nabla X\right\} - g\phi\, H''(X)\,|\nabla X|^2 + H(X)\,[\partial_t(g\phi) - \phi\Delta g + g\Delta\phi] \\
&= -g\phi\, H''(X)\,|\nabla X|^2,
\end{aligned}$$

since the first term and the last term independently vanish.

Step 3. Conclusion. For any solutions (f, g, ϕ) to the system of equations (with $A = I$ and $a \equiv 0$), we have summing up the three computations

$$\begin{aligned}
&\partial_t(g\phi H(X)) - \operatorname{div}(bH(X)g\phi) - \operatorname{div}(\phi\nabla(H(X)g)) + \operatorname{div}(gH(X)\nabla\phi) \\
&= -g\phi\, H''(X)\,|\nabla X|^2
\end{aligned}$$

Since when we integrate in the x variable the term on the second line vanishes, we find out

$$\frac{d}{dt}\mathcal{H}(f) = -D_{\mathcal{H}}(f),$$

with

$$D_{\mathcal{H}}(f) := \int g\phi\, H''(X)\,|\nabla X|^2 \geq 0,$$

so that (2.21) is proved. □

Exercise 2.3. *(1) Complete the proof for a general diffusion matrix A and for a given vector field a.*

(2) Generalize the result to the case where we add a integral term

$$\int b\, f_* := \int b(x, x_*) f(x_*)\, dx_*, \quad b \geq 0,$$

to the RHS of the equation.

Exercise 2.4. *We consider a semigroup $S_t = e^{tL}$ of linear and bounded operators on L^1 and we assume that*
(i) $S_t \geq 0$;
(ii) $\exists\, g > 0$ such that $Lg = 0$, or equivalently $S_t g = g$ for any $t \geq 0$;
(iii) $\exists\, \phi \geq 0$ such that $L^*\phi = 0$, or equivalently $\langle S_t h, \phi\rangle = \langle h, \phi\rangle$ for any $h \in L^1$ and $t \geq 0$.

Our aim is to generalize to that a bit more general (and abstract) framework the general relative entropy principle we have presented for the evolution PDE (2.20).

(a) Prove that for any real affine function ℓ, there holds $\ell[(S_t f)/g]g = S_t[\ell(f/g)g]$.

(b) Prove that for any convex function H and any f, there holds $H[(S_t f)/g]g \leq S_t[H(f/g)g]$. (Hint. Use the fact that $H = \sup_{\ell \leq H} \ell$).

(c) Deduce that

$$\int H[(S_t f)/g]g\phi \leq \int H[f/g]g\phi, \quad \forall\, t \geq 0.$$

3. Doblin-Harris Theorem in a Banach lattice

We formulate a general abstract constructive Doblin-Harris theorem.

We consider a Banach lattice X, which means that X is a Banach space endowed with a closed positive cone X_+ (we write $f \ge 0$ if $f \in X_+$ and we recall that $f = f_+ - f_-$ with $f_\pm \in X_+$ for any $f \in X$. We also denote $|f| := f_+ + f_-$). We assume that X is in duality with another Banach lattice Y, with closed positive cone Y_+, so that the bracket $\langle \phi, f\rangle$ is well defined for any $f \in X$, $\phi \in Y$, and that $f \in X_+$ (resp. $\phi \ge 0$) iff $\langle \psi, f\rangle \ge 0$ for any $\psi \in Y_+$ (resp. iff $\langle \phi, g\rangle \ge 0$ for any $g \in X_+$), typically $X = Y'$ or $Y = X'$. We write $\psi \in Y_{++}$ if $\psi \in Y$ satisfies $\langle \psi, f\rangle > 0$ for any $f \in X_+\backslash\{0\}$.

Example 3.5. *The typical case (and unique example) we have in mind is $X := L^p_\omega$, for $p \in [1,\infty]$ and a weight function $\omega : \mathbb{R}^d \to \mathbb{R}$, where*

$$L^p_\omega := \{f \in L^1_{\rm loc}(\mathbb{R}^d);\ \|f\|_{L^p_\omega} := \|f\omega\|_{L^p} < \infty\},$$

and $Y := L^{p'}_{\omega^{-1}}$.

We consider a positive and conservative (or stochastic) semigroup $S = (S_t) = (S(t))$ on X, that means that (S_t) is a semigroup on X such that

- $S_t : X_+ \to X_+$ for any $t \ge 0$, in particular, $\exists\, b \in \mathbb{R}$, $\exists\, M \ge 1$,

$$\|S_t\|_{\mathscr{B}(X)} \le M\, e^{bt} \quad \forall\, t \ge 0; \tag{3.22}$$

- there exist $\phi_1 \in Y_{++}$, $\|\phi_1\| = 1$, and a dual semigroup $S^* = S^*_t = S^*(t)$ on Y such that $S^*_t\phi_1 = \phi_1$ for any $t \ge 0$. More precisely, we assume that S^*_t is a bounded linear mapping on Y such that $\langle S_t f, \phi\rangle = \langle f, S^*_t\phi\rangle$, for any $f \in X$, $\phi \in Y$ and $t \ge 0$, and thus in particular $S^*_t : Y_+ \to Y_+$ for any $t \ge 0$.

Example 3.6. *For the linear McKean equation associated to the partial differential operator $\mathcal{L}f := \Delta f + {\rm div}(af)$ defined on (a subspace of) $X := L^p_\omega \subset L^1$, the function $\phi_1 := 1 \in L^\infty \subset Y$ fulfills the second condition (conservative property) and the operator $\mathcal{L}$ generates a positive semigroup.*

We denote by $\mathcal{L}$ the generator of S with domain $D(\mathcal{L})$. For $\psi \in Y_+$, we define the seminorm

$$[f]_\psi := \langle |f|, \psi\rangle,\ \forall\, f \in X.$$

Proposition 3.7. *A positive and conservative semigroup S on a Banach lattice X is a semigroup of contraction for the seminorm associated to the conservation ϕ_1, in other words*

$$[S(t)f]_{\phi_1} \le [f]_{\phi_1}, \quad \forall\, t \ge 0,\ \forall\, f \in X. \tag{3.23}$$

Proof of Proposition 3.7. For $f \in X$, we may write $f = f_+ - f_-$, $f_\pm \in X_+$, and then compute

$$\begin{array}{rcl} |S_t f| & \le & |S_t f_+| + |S_t f_-| \\ & = & S_t f_+ + S_t f_- = S_t|f|, \end{array}$$

where we have used the positivity property of S_t in the second line. We deduce

$$[S_t f]_{\phi_1} \le \langle S_t|f|, \phi_1\rangle = \langle |f|, S^*_t\phi_1\rangle$$

and thus (3.23), because of the stationarity property of ϕ_1. $\square$

In order to obtain a very accurate and constructive description of the longtime asymptotic behaviour of the semigroup S, we introduce additional assumptions.

• Lyapunov condition. We first make the strong dissipativity assumption

$$\|S(t)f\| \quad \leq \quad C_0 e^{\lambda t}\|f\| + C_1 \int_0^t e^{\lambda(t-s)} [S(s)f]_{\phi_1} ds, \tag{3.24}$$

for any $f \in X$ and $t \geq 0$, where $\lambda < 0$ and $C_i \in (0, \infty)$.

• Doblin-Harris condition. Next, we make the conditional positivity assumption: for any $T \geq T_1 > 0$ and $A > 0$, there exists $g_{T,A} \in X_+ \backslash \{0\}$ such that

$$S_T f \geq g_{T,A}[f]_{\phi_1}, \quad \forall\, f \in X_+,\ \|f\| \leq A[f]_{\phi_1}. \tag{3.25}$$

Example 3.8. *The above Lyapunov condition is satisfied by the Fokker-Planck semigroup, but not by the heat semigroup (where the same estimate is however true with $\lambda = 0$). The above Doblin-Harris condition is true for the partial differential operator $\mathcal{L}f := \Delta f + \mathrm{div}(af)$.*

Theorem 3.9. *Consider a semigroup S on a Banach lattice X which satisfies the above conditions. Then, there exists a unique normalized positive stationary state $f_1 \in D(\mathcal{L})$, that is*

$$\mathcal{L}f_1 = 0, \quad f_1 \geq 0, \quad \langle \phi_1, f_1 \rangle = 1.$$

Furthermore, there exist some constructive constants $C \geq 1$ and $\lambda_2 < 0$ such that

$$\|S(t)f - \langle f, \phi_1\rangle f_1\| \leq C e^{\lambda_2 t}\|f - \langle f, \phi_1\rangle f_1\| \tag{3.26}$$

for any $f \in X$ and $t \geq 0$.

Sketch of the proof of Theorem 3.9.

Step 1. The Lyapunov condition. From (3.24) and (3.23), we have

$$\|S_t f\| \leq C_0 e^{\lambda t}\|f\| + C_1 \int_0^t e^{\lambda(t-s)}[f]_{\phi_1} ds,$$

and we may thus choose $T \geq T_1$ large enough in such a way that

$$\|S_T f\| \leq \gamma_L \|f\| + K[f]_{\phi_1}, \tag{3.27}$$

with

$$\gamma_L := C_0 e^{\lambda T} \in (0,1), \quad K := C_1/\lambda.$$

For further references, we denote by A a positive real number such that

$$\gamma_L + K/A < 1. \tag{3.28}$$

Step 2. The conditional coupling property. We may now improve the non-expensive estimate (3.23) on the set $\mathcal{N} := \{f \in X; \langle \phi_1, f\rangle = 0\}$. Take indeed $f \in \mathcal{N}$ such that $\|f\| \leq A[f]_{\phi_1}$. Observing that $[f_\pm]_{\phi_1} = [f]_{\phi_1}/2$ and thus

$$\|f_\pm\| \leq \|f\| \leq 2A[f_\pm]_{\phi_1},$$

the previous estimate tells us that

$$S_T f_\pm \geq \varrho g_{T,A} \quad \varrho := c_{2A}[f]_{\phi_1}.$$

Slightly modifying the arguments of Proposition 3.7, we compute now

$$\begin{aligned} |S_T f| \quad &\leq \quad |S_T f_+ - \varrho g_{T,A}| + |S_T f_- - \varrho g_{T,A}| \\ &= \quad S_T|f| - 2\varrho g_{T,A}. \end{aligned}$$

We deduce

$$[S_T f]_{\phi_1} \le \langle |f|, \phi_1\rangle - 2\varrho\,\langle \phi_1, g_{T,A}\rangle,$$

and thus conclude to the *conditional coupling estimate*

$$[S_T f]_{\phi_1} \le \gamma_H [f]_{\phi_1}, \tag{3.29}$$

with $\gamma_H := 1 - 2c_{2A}\langle \phi_1, g_{T,A}\rangle \in (0,1)$.

Step 3. We introduce a new equivalent norm $|||\cdot|||$ on X defined by

$$|||f||| := [f]_{\phi_1} + \beta\|f\|. \tag{3.30}$$

Using the three properties (3.23), (3.27) and (3.29), we may prove that there exist $\beta > 0$ small enough and $\gamma \in (0,1)$ such that

$$|||S_T f_0||| \le \gamma |||f_0|||, \quad \text{for any } f_0 \in \mathcal{N}. \tag{3.31}$$

We observe that we have the alternative

$$A\,[f_0]_{\phi_1} \ge \|f_0\| \quad \text{or} \quad A\,[f_0]_{\phi_1} < \|f_0\|.$$

In the first case of the alternative, using the Lyapunov estimate (3.27) and the coupling estimate (3.29), we have

$$\begin{aligned} |||S_T f_0||| &= [S_T f_0]_{\phi_1} + \beta\|S_T f_0\| \\ &\le (\gamma_H + \beta K)[f_0]_{\phi_1} + \beta\gamma_L\|f_0\| \\ &\le \gamma_1 |||f_0|||, \end{aligned}$$

with $\gamma_1 := \max(\gamma_H + \beta K, \gamma_L) < 1$, by fixing from now on $\beta > 0$ small enough. In the second case of the alternative, using the Lyapunov estimate (3.27) and the non expansion estimate (3.23), we have

$$\begin{aligned} |||S_T f_0||| &= [S_T f_0]_{\phi_1} + \beta\|S_T f_0\| \\ &\le (1 + \beta K - \beta\delta)[f_0]_{\phi_1} + \beta(\gamma_L + \delta/A)\|f_0\| \\ &\le \gamma_2 |||f_0|||, \end{aligned}$$

with $\gamma_2 := \max(1 + \beta K - \beta\delta, \gamma_L + \delta/A)$ for any $0 < \beta\delta < 1 + \beta K$. We take $\delta := K + \vartheta$, $\vartheta > 0$, so that we get

$$\gamma_2 = \max(1 - \beta\vartheta, (\gamma_L + K/A) + \vartheta/A) < 1,$$

by choosing $\vartheta > 0$ small enough and using the very definition (3.28) of A. In any cases, we have thus established (3.31) with $\gamma := \max(\gamma_1, \gamma_2) < 1$.

Step 4. In order to prove the existence and uniqueness of the stationary state $f_1 \in X_1$, we fix $g_0 \in \mathcal{M} := \{g \in X_1, g \ge 0, \langle g, \phi_1\rangle = 1\}$, and we define recursively $g_k := S_T g_{k-1}$ for any $k \ge 1$. Thanks to (3.31), we get

$$\sum_{k=1}^{\infty} |||g_k - g_{k-1}||| \le \sum_{k=0}^{\infty} \gamma^k |||g_1 - g_0||| < \infty,$$

so that (g_k) is a Cauchy sequence in $\mathcal{M}$. We set $f_1 := \lim g_k \in \mathcal{M}$ which is a stationary state for the mapping S_T, as seen by passing to the limit in the recursive equations defining (g_k). From (3.31) again, this is the unique stationary state for this mapping in $\mathcal{M}$. From the semigroup property, we have $S_t f_1 = S_t S_T f_1 = S_T(S_t f_1)$ for any $t > 0$, so that $S_t f_1$ is also a stationary state in $\mathcal{M}$, and thus $S_t f_1 = f_1$ for any $t > 0$, by uniqueness.

Step 5. For $f \in X$, we see that $h := f - \langle f, \phi_1 \rangle \phi_1 \in \mathcal{N}$, and using recursively (3.31), we deduce

$$|||S_{nT} h||| \le \gamma^n |||h|||, \quad \forall\, n \ge 0.$$

The estimate (3.26) then follows by using the equivalence of the norms $\|\cdot\|$ and $|||\cdot|||$, the semigroup property and the growth estimate (3.22) for dealing with intermediate times $t \in (nT, (n+1)T)$. For $t \ge 0$ and $t = nT + \tau$, $n \ge 0$, $\tau \in [0, T)$, we may indeed write

$$\begin{aligned} \|S(t)h\| &\le \beta^{-1} |||S(T)^n S(\tau) h||| \\ &\le \beta^{-1} \gamma^n |||S(\tau) h||| \\ &\le \beta^{-1} e^{n \log \gamma} (1+\beta) \|S(\tau) h\| \\ &\le e^{(t/T - \tau/T) \log \gamma} (1 + \beta^{-1}) M e^{bT} \|h\|, \end{aligned}$$

where in the last line we have used (3.22) with $b \ge 0$ (what we may always assume and it is also imposed in our case by the conservation hypothesis), from what we conclude to (3.26) with $\lambda_2 := T^{-1} \log \gamma < 0$ and $C := \gamma^{-1}(1 + \beta^{-1}) M e^{bT}$. ☐

4. Semigroups factorization technique

In this section, we establish the following weighted L^1 decay through a semigroups factorization technique and the already known weighted L^2 decay (consequence and equivalent to the Poincaré inequality).

Theorem 4.1. *For any $a \in (-\lambda_P, 0)$ and for any $k > k^* := \lambda_P$ there exists $C_{k,a}$ such that for any $\varphi \in L^1_k$, the associated solution f to the Fokker-Planck equation satisfies*

$$\|f - \langle \varphi \rangle\, G\|_{L^1_k} \le C_{k,a}\, e^{a\,t}\, \|\varphi - \langle \varphi \rangle\, G\|_{L^1_k}. \tag{4.1}$$

A refined version of the proof below shows that the same estimate holds with $a := -\lambda_P$.

Proof of Theorem 4.1. In order to simplify a bit the presentation, we only present the proof in the case of the dimension $d \le 3$, but the same arguments can be generalized to any dimension $d \ge 1$.

Step 1. The splitting. We introduce the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{B} f := \Delta f + \nabla \cdot (f\, x) - M\, f\, \chi_R, \quad \mathcal{A} f := M\, f\, \chi_R,$$

where $\chi_R(x) = \chi(x/R)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B_1} \le \chi \le \mathbf{1}_{B_2}$, and where $R, M > 0$ are two real constants to be chosen later. We define, in any Banach space $\mathcal{X}$ such that $G \in \mathcal{X} \subset L^1$, the projection operator

$$\Pi f := \langle f \rangle\, G,$$

which thus satisfies $\Pi^2 = \Pi$ and $\Pi \in \mathcal{B}(\mathcal{X})$. When $S_\mathcal{L}$ is well defined as a semigroup in $\mathcal{X}$, we have

$$S_\mathcal{L}(I - \Pi) = (I - \Pi)\, S_\mathcal{L} = (I - \Pi)\, S_\mathcal{L}(I - \Pi) \tag{4.2}$$

as a consequence of the projection property $(I - \Pi)^2 = (I - \Pi)^2$, of the facts that G is a stationary solution to the Fokker-Planck equation and that the mass is preserved by the associated flow. Now, iteraration the Duhamel formula

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{L} * \mathcal{A} S_\mathcal{B},$$

we have

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}}) + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}) * (\mathcal{A}S_{\mathcal{B}}). \tag{4.3}$$

The two identities (4.2) and (4.3) together and using the shorthand $\Pi^\perp = I - \Pi$, we have

$$S_{\mathcal{L}}\Pi^\perp \quad = \quad \Pi^\perp S_{\mathcal{B}}\Pi^\perp + \Pi^\perp S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})\Pi^\perp + S_{\mathcal{L}}\Pi^\perp * (\mathcal{A}S_{\mathcal{B}}) * (\mathcal{A}S_{\mathcal{B}}\Pi^\perp) =: \sum_{i=1}^{3} \mathcal{T}_i(t).$$

In order to get (4.1), we will establish that

$$S_{\mathcal{B}}(t) : L^1_k \to L^1_k, \text{ with bound } \mathcal{O}(e^{a't}), \ \forall t \geq 0, \ \forall a' > a^*, \ \forall k > k^*, \tag{4.4}$$

$$S_{\mathcal{B}}(t) : L^1_K \to L^2_K, \text{ with bound } \mathcal{O}(\frac{e^{a't}}{t^{3/4}}), \ \forall t > 0, \ \forall a' > a^*, \ \forall K > K^*, \tag{4.5}$$

$$\mathcal{A} : L^1_k \to L^1_K, \ \mathcal{A} : L^2_K \to L^2(G^{-1}), \ \forall K > k^*, \ \forall k > k^*, \tag{4.6}$$

with $K^* := \lambda_P + d/2$. We also recall that

$$S_{\mathcal{L}}(t)\Pi^\perp : L^2(G^{-1}) \to L^2(G^{-1}), \text{ with bound } \mathcal{O}(e^{-\lambda_P t}), \ \forall t \geq 0, \tag{4.7}$$

which is nothing but the convergence obtained thanks to the Poincaré inequality. We finally observe that

$$u * w(t) = \mathcal{O}(e^{at}) \ \text{ and } u * v * w(t) = \mathcal{O}(e^{at}), \ \ \forall t \geq 0, \ \forall a > a^*, \tag{4.8}$$

if

$$u(t) = \mathcal{O}(e^{a't}), \ \ v(t) = \mathcal{O}(\frac{e^{a't}}{t^{3/4}}), \ \ w(t) = \mathcal{O}(e^{a't}), \ \ \forall t > 0, \ \forall a' > a^*. \tag{4.9}$$

The first estimate in (4.8) is obtained by writing

$$u * w(t) = \int_0^t u(s)w(t-s)\,ds \lesssim \int_0^t e^{a's}e^{a'(t-s)}\,ds \lesssim te^{a't} \lesssim e^{at},$$

for any $t \geq 0$ and any $a > a' > a^*$. For the second estimate in (4.8), we first write

$$v * w(t) = \int_0^t v(s)w(t-s)\,ds \lesssim \int_0^t \frac{e^{a''s}}{s^{3/4}}e^{a''(t-s)}\,ds \lesssim t^{1/4}e^{a''t} \lesssim e^{a't},$$

for any $t \geq 0$ and any $a' > a'' > a^*$, and we conclude by combining that estimate with the first estimate in (4.8).

Step 2. The conclusion. With the help of the estimates stated in step 1, we are in position to prove (4.1) or equivalently that

$$\|\mathcal{T}_i(t)\|_{L^1_k \to L^1_k} \lesssim e^{at}, \ \forall t \geq 0, \ \forall a > a^*, \ \forall k > k^*, \tag{4.10}$$

for any $i = 1, 2, 3$. For $i = 1$, (4.10) is nothing but (4.4) together with $\Pi^\perp \in \mathcal{B}(L^1_k)$. For proving (4.10) when $i = 2$, we use the first estimate in (4.8) with

$$u(t) := \|\Pi^\perp S_{\mathcal{B}}(t)\|_{L^1_k \to L^1_k}, \quad w(t) := \|\mathcal{A}S_{\mathcal{B}}(t)\Pi^\perp\|_{L^1_k \to L^1_k},$$

where both functions satisfy the hypothesizes of (4.9) because of $\Pi^\perp \in \mathcal{B}(L^1_k)$, of the first estimate on $\mathcal{A}$ with $K = k$ in (4.6) and of the estimate (4.5) on $S_{\mathcal{B}}(t)$ in L^1_k. For proving (4.10) when $i = 3$, we use the second estimate in (4.8) with

$$u(t) := \|S_{\mathcal{L}}(t)\Pi^\perp\|_{L^2(G^{-1}) \to L^1_k}, \quad v(t) := \|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1_K \to L^2(G^{-1})},$$
$$w(t) := \|\mathcal{A}S_{\mathcal{B}}(t)\Pi^\perp\|_{L^1_k \to L^1_K},$$

where the three functions satisfy the hypothesizes of (4.9). To check the estimate on u, we use (4.7) and $L^2(G^{-1}) \subset L^1_k$. For the estimate on v, we use (4.5) and the second estimate on $\mathcal{A}$ in (4.6). Finally, to check the estimate on w, we use the first estimate on $\mathcal{A}$ in (4.6), the estimate (4.4) on $S_{\mathcal{B}}(t)$ in L^1_k and $\Pi^\perp \in \mathcal{B}(L^1_k)$.

In order to conclude the proof of Theorem 4.1, we thus need to establish (4.4), (4.5) and (4.6). That is done in the three following steps.

Step 3. Proof of (4.6). The operator $\mathcal{A}$ is clearly bounded in any Lebesgue space and more precisely

$$\|\mathcal{A}\, f\|_{L^p(m)} \le C_{R,M}\, \|f\|_{L^p_\ell}, \quad \forall\, f \in L^p_\ell,\ \forall\, p = 1, 2,$$

for $m := \langle x \rangle^K$ or $m := G^{-1}$ and with

$$C_{R,M} := M \| \frac{m}{\langle \cdot \rangle^{p\ell}} \|^{1/p}_{L^\infty(B_{2R})}.$$

Step 4. Proof of (4.4). For any $k, \varepsilon > 0$ and for any $M, R > 0$ large enough (which may depend on k and ε) the operator $\mathcal{B}$ is dissipative in L^1_k in the sense that

$$\forall\, f \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} (\mathcal{B} f)\, (\mathrm{sign} f)\, \langle x \rangle^k \le (\varepsilon - k)\, \|f\|_{L^1_k}. \tag{4.11}$$

We immediately deduce (4.4) from (4.11) and the Gronwall lemma. In oder to establish (4.11), we set $\beta(s) = |s|$ (and more rigorously we must take a smooth version of that function) and $m = \langle x \rangle^k$, and we compute

$$\begin{aligned} \int (\mathcal{L}\, f)\, \beta'(f)\, m &= \int (\Delta f + d\, f + x \cdot \nabla f)\, \beta'(f)\, m \\ &= \int \{ -\nabla f\, \nabla(\beta'(f) m) + d\, |f|\, m + m\, x \cdot \nabla |f| \} \\ &= - \int |\nabla f|^2\, \beta''(f)\, m + \int |f|\, \{ \Delta m + d - \nabla(x\, m) \} \\ &\le \int |f|\, \{ \Delta m - x \cdot \nabla m \}, \end{aligned}$$

where we have used that β is a convex function. Defining

$$\begin{aligned} \psi &:= \Delta m - x \cdot \nabla m - M \chi_R\, m \\ &= (k^2\, |x|^2\, \langle x \rangle^{-4} - k\, |x|^2\, \langle x \rangle^{-2} - M\, \chi_R)\, m \end{aligned}$$

we easily see that we can choose $M, R > 0$ large enough such that $\psi \le (\varepsilon - k)\, m$ and then (4.11) follows.

Step 5. Proof of (4.5). Fix now $K > K^*$ and $a > -\lambda_P$. There holds

$$\|S_{\mathcal{B}}(t) \varphi\|_{L^2_K} \le \frac{C_{a,K}}{t^{d/4}}\, e^{a\, t}\, \|\varphi\|_{L^1_K}, \quad \forall\, \varphi \in L^1_K, \tag{4.12}$$

which immediately implies (4.5) since we are restricted to the case of a dimension $d \le 3$. We set $m = \langle x \rangle^K$. A similar computation as in step 4 gives

$$\begin{aligned} \int (\mathcal{B}\, f)\, f\, m^2 &= - \int |\nabla (f\, m)|^2 + \int |f|^2\, \Big\{ \frac{|\nabla m|^2}{m^2} + \frac{d}{2} - x \cdot \nabla m - M\, \chi_R \Big\} m^2 \\ &= - \int |\nabla (f\, m)|^2 + (\frac{d}{2} + \varepsilon - K) \int |f|^2\, m^2, \end{aligned}$$

for $M, R > 0$ chosen large enough. Denoting by $f(t) = S_{\mathcal{B}}(t)\varphi$ the solution to the evolution PDE

$$\partial_t f = \mathcal{B} f, \qquad f(0) = \varphi,$$

we (formally) have

$$\frac{1}{2}\frac{d}{dt}\int f^2\, m^2 = \int (\mathcal{B}\, f)\, f\, m^2 \le -\int |\nabla (f\, m)|^2 + a\int |f|^2\, m^2.$$

On the one hand, throwing away the last (negative) term at the RHS of the above differential inequality and using the Nash trick, we get

$$\|f(t)\, m\|_{L^2} \le \frac{C}{t^{d/4}}\, \|f(0)\, m\|_{L^1}, \quad \forall\, t > 0. \tag{4.13}$$

On the other hand, throwing away the first (negative) term at the RHS of the above differential inequality and using the Gronwall lemma exactly as in step 4, we get

$$\|f(t)\, m\|_{L^2} \le C e^{a(t-t_0)}\, \|f(t_0)\, m\|_{L^2}, \quad \forall\, t \ge t_0 \ge 0. \tag{4.14}$$

Using (4.13) for $t \in (0, 1]$ and (4.14) for $t \ge 1$, we deduce (4.12). $\square$