

[Chap 3]. Example : the Fokker-Planck equation. (1)

We consider the Fokker-Planck equation.

$$\partial_t f = \mathcal{L}f := \Delta f + \operatorname{div}(xf) \quad \text{in } (0, \infty) \times \mathbb{R}^d.$$

lemma 1 For any $f_0 \in L^2 \exists ! f \in X_{\infty}$ var. sol. $\mathcal{L}$
~~There exists a semigroup.~~ proof JL^2 (chap 1)

lemma 2 $\exists$ SG S_t associated to the eq.

$$S_t \geq 0 \quad \text{and} \quad \mathcal{L}^k 1 = 0$$

$$\mathcal{L}^k \phi = \Delta \phi^k - x \cdot \nabla \phi.$$

lemma 3 For many L^p spaces, $S_t : L^p \rightarrow L^p$, $O(e^{Kt})$.

The L^p spaces are associated to

- $\omega = \langle x \rangle^k$ s.t. $L^p \subset L^1$, i.e. $k > d/p'$
- $\omega = e^{a|x|^2}$ s.t. $\chi \in L^p$, i.e. $a \in (0, 1/2)$.

Proof : After several I by P :

$$\int \mathcal{L}f f^{p-1} \omega^p = - \partial_t \int | \nabla (f \omega)^{p/2} |^2 + \int f^p \omega^p \psi$$

$$\text{with } \psi = a(r) \frac{|\nabla \omega|^2}{\omega^2} + b(r) \frac{\Delta \omega}{\omega} + \frac{d}{p'} - x \cdot \frac{\nabla \omega}{\omega}$$

and we choose ω s.t. $\psi \leq \frac{M}{1+|x|^2}$.

$$\psi \leq K \leq M \mathbf{1}_{B_R} - \nu \quad M, K \geq 0, \nu > 0.$$

As a consequence

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int f^p \omega^p &= \int \mathcal{L}f \cdot f^{p-1} \omega^p \\ &\leq K \int f^p \omega^p \end{aligned}$$

and

$$\|f_t\|_{L^p} \leq e^{Kt} \|f_0\|_{L^p} \quad \text{by Grönwall lemma} \quad \square$$

lemma 4 $S_\varepsilon : L^\infty \rightarrow L^\infty \quad O\left(\frac{e^{-Kt}}{t^{d/4}}\right)$

That Nash argument.

proof 1: $f := S_\varepsilon(H)$ fo.

$$\frac{1}{2} \frac{d}{dt} \int f^2 \omega^2 \leq - \int |\nabla(f\omega)|^2 + K_2 \int f^2 \omega^2$$

$$\frac{d}{dt} \int f \omega \leq K_1 \int f \omega$$

$$U_p(H) := e^{-K_1 t} \|f\omega\|_{L^p}^p$$

$$\frac{d}{dt} U_1(H) \leq 0 \quad \text{if } K \geq K_1.$$

$$\frac{d}{dt} U_2(H) \leq - \int |\nabla(f\omega)|^2 e^{-2Kt} \quad \text{if } K \geq K_2$$

Using Nash inequality

$$\|f\omega\|_{L^2}^{1+2/d} \leq \|f\omega\|_{L^1}^{2/d} \|\nabla(f\omega)\|_{L^2}^2,$$

we get.

$$\frac{d}{dt} U_2 \leq - \frac{\|f\omega\|_{L^2}^{2(1+2/d)} e^{-2(1+2/d)Kt}}{\|f\omega\|_{L^1}^{2/d} e^{-4/d Kt}}$$

$$\leq - \frac{U_2(t)^{1+2/d}}{U_1(t)^{4/d}} \leq - \frac{U_2^{1+2/d}}{U_1(0)^{4/d}}.$$

We obtain

$$U_2(H) \leq \frac{C U_1(0)^2}{t^{d/2}}$$

and the result with $K := \max(K_1, K_2) \quad \square$.

($d \leq 3$).

proof 2. We set $B := \mathcal{L} - M X_R$ $\mathbb{1}_{B_R} \in C^{(2)}(\mathbb{R}^d)$ 2

We now have for $f(A) := S_B A f_0$

$$\frac{1}{2} \frac{d}{dt} \int f^2 \omega \leq - \int |\nabla(f\omega)|^2 - \nu \int f^2 \omega^2$$

$$\frac{d}{dt} \int f \omega \leq - \nu \int f \omega.$$

Forgetting, Throwing up the negative part, we get

$$\frac{1}{2} \frac{d}{dt} \int f^2 \omega \leq - \int |\nabla(f\omega)|^2$$

$$\frac{d}{dt} \int f \omega \leq 0$$

and Nash proof gives

$$\|f\omega\|_2 \leq \frac{1}{t^{d/4}} \|f_0\omega\|_1.$$

Thanks to Duhamel formula

$$S_B f_0 = S_B f_0 + S_B A * S_B f_0.$$

$$\|S_B(A)f_0\|_{L^\infty} \leq \|S_B f_0\|_{L^\infty} + \int_0^t \|S_B(t-s) A S_B f_0\|_{L^\infty} ds$$

$$\leq \frac{C}{t^{d/4}} \|f_0\|_{L^1} + \int_0^t \frac{1}{(t-s)^{d/4}} \|A S_B f_0\|_{L^\infty} ds$$

$$\leq \frac{C}{t^{d/4}} \|f_0\|_{L^1} + \int_0^t \frac{1}{(t-s)^{d/4}} C e^{K(t-s)} \|f_0\|_{L^1} ds$$
$$= \frac{C}{t^{d/4}} \|f_0\|_{L^1} + \frac{e^{Kt} \int_0^t \frac{e^{-K(t-s)}}{(t-s)^{d/4}} ds}{1} \|f_0\|_{L^1}$$

$$\leq \left(\frac{C}{t^{d/4}} + C e^{Kt} \right) \|f_0\|_{L^1} \quad \square$$

Lemma 5 Lyapunov condition ($d \leq 3$).

$$\|S_X(t)f\|_{L^\infty} \leq e^{-\nu t} \|f\|_{L^\infty} + K \|f\|_{L^1}.$$

proof. Repeating the first proof of Lemma 4, we have

$$\|S_B(t)f\|_{L^\infty} \leq e^{-\nu t} \|f\|_{L^\infty} \quad \underline{\underline{\nu > 0}}.$$

$$\|S_B(t)f\|_{L^\infty} \leq e^{-\nu t} \|f\|_{L^\infty}$$

$$\|S_B(t)f\|_{L^\infty} \leq \frac{e^{-\nu t}}{t^{d/4}} \|f\|_{L^1}.$$

We write again the Duhamel formula

$$S_X(t) = S_B + S_B A \times S_X$$

For the second term we have

$$\|S_B A \times S_X f\|_{L^\infty} \leq \int_0^t \|S_B(t-s) A S_X(s)f\|_{L^\infty} ds$$

$$\leq \int_0^t \frac{e^{-\nu(t-s)}}{(t-s)^{d/4}} \|A S_X(s)f\|_{L^\infty} ds$$

$$\leq \int_0^t \frac{e^{-\nu(t-s)}}{(t-s)^{d/4}} \|S_X(s)f\|_{L^1} ds$$

$$\leq \underbrace{\int_0^\infty \frac{e^{-\nu \tau}}{\tau^{d/4}} d\tau}_{=: K} \|f\|_{L^1}$$

□.

Lemma 6. (Doblin-Harris).

$$\forall T_0 \geq R_0, \exists R_1 \geq \gamma_1 \quad S_T f_0 \geq \gamma_1 \mathbb{1}_{B_{R_1}} \int_{B_{R_0}} f_0, \quad \forall f_0 \geq 0 \in L^1_\omega. \quad (3)$$

proof. By the maximum principle, we may assume $\text{supp } f_0 \subset B_{R_0}$ and thus $f_0 \in L^1_\omega$, the weight function. We fix (for instance) $\omega(x) = \langle x \rangle$.

We know that

$$\|f_t\|_{L^1_\omega} \leq C e^{Kt} \|f_0\|_{L^1_\omega}.$$

We fix $T > 0$. For any $t \in (0, T)$, we write

$$\begin{aligned} \int_{B_R} f_t &= \int_{\mathbb{R}^d} f_t - \int_{B_R^c} f_t \\ &\geq \int_{\mathbb{R}^d} f_0 - \frac{1}{\omega(R)} \int f_t \omega \\ &\geq \int_{B_{R_0}} f_0 - \frac{C e^{KT}}{\omega(R)} \int f_0 \omega \\ &\geq \left(1 - C e^{KT} \frac{\omega(R_0)}{\omega(R)}\right) \int_{B_{R_0}} f_0 \\ &\geq \frac{1}{2} \int_{B_{R_0}} f_0 \end{aligned}$$

by choosing $R = R_1$ large enough.

We deduce that f satisfies

$$\|f\|_{L^1(Q_0)} \geq C \int_{B_{R_0}} f_0$$

with $Q_0 := (0, T) \times B_{R_1}$, $T_0 := T/2$.

On the other hand, because $f \in L^2(\mathbb{Q})$ is a var sol on $\mathbb{Q} \supset \mathbb{Q}_0$, we may apply the Harnack inequality on $\mathbb{Q}_1 := \mathbb{Q}_0 + \mathbb{Q}_1$ and we get

$$\inf_{\mathbb{Q}_1} f \geq \sup_{\mathbb{Q}_0} f \geq \inf_{\mathbb{Q}_0} f$$

with $\mathbb{Q}_1 := (T, T+1) \times B_{R_1}$.

We deduce so

$$\begin{aligned} f(T) &\geq \inf_{B_{R_1} \times \mathbb{Q}_1} f \geq \sup_{B_{R_1} \times \mathbb{Q}_0} f \geq \inf_{B_{R_1} \times \mathbb{Q}_0} f \\ &\geq \inf_{B_{R_1} \times \mathbb{Q}_0} f \geq \inf_{B_{R_1} \times \mathbb{Q}_0} f \geq \inf_{B_{R_1} \times \mathbb{Q}_0} f \end{aligned}$$

As a conclusion, $\exists A > 0$.

Theorem $\exists A > 0, C \geq 1$, s.t.

$$\|f - \int_{\mathbb{Q}_0} G(x) \, dx\|_{L^\infty} \leq C e^{-At} \|f_0 - \int_{\mathbb{Q}_0} G(x) \, dx\|_{L^\infty}.$$