COMPLEMENTS TO LECTURE 4: THE MCKEAN-VLASOV EQUATION

We establish the existence and uniqueness of solutions to the McKean-Vlasov equation.

1. The McKean-Vlasov equation

1.1. **The well-posedness.** In this section, we consider the nonlinear McKean-Vlasov equation

$$\partial_t f = \mathcal{L}_f f := \Delta f + \mathrm{div}(a_f f), \quad f(0) = f_0, \tag{1.1}$$

with

$$a_f := a * f, \quad a \in L^\infty(\mathbb{R}^d)^d, \tag{1.2}$$

and we aim to prove the following existence and uniqueness result. We define

$$H = L^2_k := \big\{ f \in L^2(\mathbb{R}^d);\ \|f\|^2_{L^2_k} := \int f^2 \langle x\rangle^{2k}\, dx < \infty \big\},$$

with $\langle x\rangle^2 := 1 + |x|^2$, and

$$V = H^1_k := \big\{ f \in L^2_k(\mathbb{R}^d);\ \nabla f \in L^2_k(\mathbb{R}^d) \big\}.$$

Theorem 1.1. *For any $0 \le f_0 \in H := L^2_k$, $k > d/2$, there exists a unique global variational solution f to the McKean-Vlasov equation* (1.1)*, and more precisely $f \in X_T$, for any $T > 0$, where X_T is defined thanks to*

$$f \in X_T := C([0,T]; H) \cap L^2(0,T;V) \cap H^1(0,T;V') \tag{1.3}$$

with the choices $H := L^2_k$ and $V := H^1_k$.

Proof of Theorem 1.1. Step 1. A priori estimates. Integrating in the x variable a nice solution f to the McKean-Vlasov equation (1.1) and using the Green-Ostrogradski divergence formula, we have

$$\frac{d}{dt}\int f dx = \int \mathrm{div}(\dots)\, dx = 0,$$

so that the mass (the integral) is conserved. On the other hand, multiplying the equation by f_+ and integrating in the x variable, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f_+^2 &= -\int |\nabla f_+|^2 - \int \nabla f_+ \cdot a_f f_+ \\ &\le \frac{1}{4}\|a_f\|^2_{L^\infty}\int f_+^2, \end{aligned}$$

by using the Young inequality and assuming $a_f \in L^\infty$. Thanks to the Gronwall Lemma, we deduce that $f(t)_+ = 0$ if $f_{0+} = 0$, what is equivalent to the fact that the equation is positivity preserving: $f(t) \geq 0$ if $f_0 \geq 0$. These two previous properties together imply

$$\|f(t)\|_{L^1} \leq \|f_0\|_{L^1}, \quad \forall\, t \geq 0, \tag{1.4}$$

with in fact equality. We finally multiply the equation by $f\langle x\rangle^{2k}$ and we integrate in the x variable, in order to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f^2\langle x\rangle^{2k} &= -\int |\nabla f|^2\langle x\rangle^{2k} + \frac{1}{2}\int f^2\Delta\langle x\rangle^{2k} \\ &\quad -\int f(a_f\cdot\nabla f)\langle x\rangle^{2k} - \int f^2 a_f\cdot\nabla\langle x\rangle^{2k}, \end{aligned}$$

by performing several integration by parts. Using the Young inequality in order to get ride of the third term, we get

$$\frac{d}{dt}\int f^2\langle x\rangle^{2k} \leq -\int |\nabla f|^2\langle x\rangle^{2k} + \int f^2(\Delta\langle x\rangle^{2k} + |a_f|^2\langle x\rangle^{2k} - 2a_f\cdot\nabla\langle x\rangle^{2k}).$$

From (1.2) and (1.4), we have

$$\|a_f\|_{L^\infty} \leq \|a\|_{L^\infty}\|f_0\|_{L^1},$$

and therefore

$$\Delta\langle x\rangle^{2k} + |a_f|^2\langle x\rangle^{2k} - 2a_f\cdot\nabla\langle x\rangle^{2k} \leq C_0\langle x\rangle^{2k},$$

for a constant $C_0 := C_0(k, \|a\|_{L^\infty}\|f_0\|_{L^1})$. Together with the Gronwall lemma, we deduce

$$\|f(t)\|^2_{L^2_k} + \int_0^t \|\nabla f\|^2_{L^2_k}\,ds \leq e^{C_0 t}\,\|f_0\|^2_{L^2_k} \quad \forall\, t \geq 0. \tag{1.5}$$

That last estimate is strong enough for defining variational solutions as in the case of a linear parabolic equation at least when $L^2_k \subset L^1$, which means $k > d/2$.

Step 2. We observe that for $g \in L^2_k$ and $f \in V = H^1_k$, the formula

$$\langle \mathcal{L}_g f, h\rangle := -\int_{\mathbb{R}^d} (\nabla f + a_g f)\cdot\nabla(h\langle x\rangle^{2k})\,dx, \quad \forall\, h \in V,$$

defines a linear form on V. Repeating the same computations as for the proof of (1.5), we have

$$\langle \mathcal{L}_g f, f\rangle \leq -\|\nabla f\|^2_{L^2_k} + C_0\|f\|^2_{L^2_k}, \quad \forall\, f \in H^1_k,$$

with $C_0 := C_0(k, \|a\|_{L^\infty}\|g\|_{L^1})$. We consider now

$$g \in \mathscr{C} := \big\{h \in C([0,T]; L^2_k);\ \|h(t)\|_{L^1} \leq \|f_0\|_{L^1}\big\}$$

and the linear time depending problem

$$\partial_t f = \mathcal{L}_g f := \Delta f + \mathrm{div}(a_g f), \quad f(0) = f_0. \tag{1.6}$$

It is worth emphasizing that $a_g \in L^\infty((0,T)\times\mathbb{R}^d)$. We apply J.-L. Lions version of the Lax-Milgram Theorem which implies that there exists a unique variational solution $f \in X_T$, and more precisely

$$(f(t),\varphi(t))_H = (f_0,\varphi(0))_H + \int_0^t \big\{\langle \mathcal{L}_{g(s)} f(s), \varphi(s)\rangle + \langle \varphi'(s), f(s)\rangle\big\}\,ds,$$

for any $\varphi \in X_T$ and any $0 \le t \le T$. Choosing $\varphi := \chi_M \langle x\rangle^{-2k}$ as a test function in the above variational formulation, with $\chi_M(x) := \chi(x/M)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B(0,1)} \le \chi \le \mathbf{1}_{B(0,2)}$, we deduce

$$\int_{\mathbb{R}^d} f(t)\chi_M = \int_{\mathbb{R}^d} f_0\chi_M - \int_{\mathcal{U}} (\nabla f + a_g f)\cdot\nabla\chi_M.$$

Using that $f(t), f_0 \in L^2_k \subset L^1$, $0 \le \chi_M \nearrow 1$, $f, \nabla f \in L^2(0,T;L^2_k) \subset L^1_{tx}$ and $\|\nabla\chi_M\|_{L^\infty} \to 0$, we may pass to the limit $M \to \infty$, and we (rigorously) obtain the same mass conservation (1.4) for the solution to this linear equation. Because $f_0 \ge 0$, we have $f(t) \ge 0$ thanks to the weak maximum principle, and thus $f \in \mathscr{C}$.

Step 3. From the previous step, we have built a mapping $\mathscr{C} \to \mathscr{C}$, $g \mapsto f$. For $g_1, g_2 \in \mathscr{C}$, we consider the associated solutions $f_1, f_2 \in \mathscr{C} \cap X_T$ and we define $f := f_2 - f_1$, $g := g_2 - g_1$. We observe that

$$\partial_t f = \Delta f + \mathrm{div}(a_{g_1} f) + \mathrm{div}(a_g f_2), \quad f(0) = 0.$$

Adapting the L^2_k estimate established in Step 1, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int f^2\langle x\rangle^{2k} &= -\int |\nabla f|^2\langle x\rangle^{2k} + \frac{1}{2}\int f^2\Delta\langle x\rangle^{2k} - \int f(a_{g_1}\cdot\nabla f)\langle x\rangle^{2k} \\
&\quad - \int f^2 a_{g_1}\cdot\nabla\langle x\rangle^{2k} - \int f_2 a_g\cdot(\nabla f\langle x\rangle^{2k} + f\nabla\langle x\rangle^{2k}) \\
&\le \frac{1}{2}\int f^2\Delta\langle x\rangle^{2k} + \frac{1}{2}\int f^2|a_{g_1}|^2\langle x\rangle^{2k} - \int f^2 a_{g_1}\cdot\nabla\langle x\rangle^{2k} \\
&\quad + \frac{1}{2}\int f_2^2|a_g|^2\langle x\rangle^{2k} + \int (f_2^2|a_g|^2\langle x\rangle^{2k} + f^2|\nabla\langle x\rangle^k|^2),
\end{aligned}$$

where we have used three times the Young inequality. Using (1.5) and the fact that $g_1 \in \mathscr{C}$, we deduce

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int f^2\langle x\rangle^{2k} &\lesssim (1 + \|a\|^2_{L^\infty}\|g_1\|^2_{L^1})\int f^2\langle x\rangle^{2k} + \|a\|^2_{L^\infty}\|g\|^2_{L^1}\int f_2^2\langle x\rangle^{2k} \\
&\lesssim (1 + \|a\|^2_{L^\infty}\|f_0\|^2_{L^1})\int f^2\langle x\rangle^{2k} + \|a\|^2_{L^\infty}\|g\|^2_{L^1} e^{C_0 t}\|f_0\|^2_{L^2_k} \\
&= \frac{C_1}{2}\int f^2\langle x\rangle^{2k} + \|g\|^2_{L^1}\frac{C_2}{2}e^{C_0 t},
\end{aligned}$$

with $C_i := C_i(k, \|a\|_{L^\infty}, \|f_0\|_{L^1})$. Thanks to the Gronwall lemma, we finally obtain

$$\begin{aligned}
\sup_{[0,T]}\|f\|^2_{L^2_k} &\le \int_0^T \|g\|^2_{L^1} C_2 e^{C_0 s + C_1(t-s)}\,ds \\
&\le C_2 e^{(C_0+C_1)T}\,T\sup_{[0,T]}\|g\|^2_{L^1},
\end{aligned}$$

and, because $L^2_k \subset L^1$,

$$\sup_{[0,T]}\|f\|^2_{L^2_k} \le \frac{1}{2}\sup_{[0,T]}\|g\|^2_{L^2_k},$$

for $T > 0$ small enough. The Banach-Picard contraction mapping theorem tells us that there exists a unique $f \in \mathscr{C}\cap X_T$ variational solution to the nonlinear McKean-Vlasov equation. Iterating the above process we get a unique global solution. $\square$

1.2. **Aubin-Lions Lemma and application.** We present first a simple but typical version of Aubin-Lions Lemma.

Lemma 1.2. *Consider a sequence (f_n) of functions satisfying $f_n \in C([0,T]; L^2\mathbb{R}^d))$ and*

$$\partial_t f_n - \Delta f_n = F_n + \mathrm{div}(G_n) \quad in \quad \mathcal{D}'((0,T) \times \mathbb{R}^d),$$

with

$$(f_n) \text{ is bounded in } Y_T := L^\infty(0,T; L^2_k(\mathbb{R}^d)) \cap L^2(0,T; H^1(\mathbb{R}^d)), \;\; k > 0;$$
$$(F_n),\, (G_n) \text{ are bounded in } L^2((0,T) \times B_R),\, \forall\, R > 0.$$

Then, there exists $f \in Y_T$ and a subsequence (f_{n_ℓ}) such that

$$f_{n_\ell} \to f \text{ strongly in } L^2((0,T) \times \mathbb{R}^d)) \text{ and weakly in } Y_T.$$

Proof of Lemma 1.2. Step 1. We introduce a sequence of mollifiers (ρ_ε), that is $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x)$ with $0 \le \rho \in \mathcal{D}(\mathbb{R}^2)$, $\|\rho\|_{L^1} = 1$. We observe that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d} f_n(t,y)\,\rho_\varepsilon(x-y)\,dy = \int_{\mathbb{R}^d}(f_n\,\Delta\rho_\varepsilon + F_n\rho_\varepsilon - G_n\cdot\nabla\rho_\varepsilon)\,dy,$$

where the RHS term is bounded in $L^2((0,T) \times B_R)$ uniformly in n for any fixed $\varepsilon > 0$. We also clearly have

$$\nabla_x \int_{\mathbb{R}^d} f_n(t,y)\,\rho_\varepsilon(x-y)\,dy = -\int_{\mathbb{R}^d} f_n \nabla_y \rho_\varepsilon(x-y)\,dy,$$

where again the RHS term is bounded in $L^2((0,T) \times B_R)$ uniformly in n for any fixed $\varepsilon > 0$. In other words, $f_n * \rho_\varepsilon$ is bounded in $H^1((0,T) \times B_R)$. We finally observe that

$$\begin{aligned}
&\sup_{[0,T]} \int (f_{n_\ell} * \rho_\varepsilon)^2 \langle x\rangle^k dx \\
&\quad\le \sup_{[0,T]} \int\!\!\int (f_{n_\ell}(x))^2 \rho_\varepsilon(y)\langle x-y\rangle^k dxdy \\
&\quad\le \sup_n \sup_{[0,T]} \int (f_n(x))^2\langle x\rangle^k dx \int \rho(y)\langle y\rangle^k dy < \infty.
\end{aligned}$$

Thanks to the Rellich-Kondrachov Theorem, we get that (up to the extraction of a subsequence) $(f_n * \rho_\varepsilon)_n$ is strongly convergent in $L^2((0,T) \times \mathbb{R}^d)$. Thanks to the boundedness assumption on (f_n), we may extract a second subsequence (f_{n_ℓ}) such that $f_{n_\ell} \rightharpoonup f$ weakly in Y_T for some $f \in Y_T$. We then have $f_{n_\ell} * \rho_\varepsilon \to f * \rho_\varepsilon$ weakly in Y_T. Coming back to the previous strong compactness result, we thus also have

$$f_{n_\ell} * \rho_\varepsilon \to f * \rho_\varepsilon \text{ strongly in } L^2((0,T) \times \mathbb{R}^d) \text{ as } k \to \infty.$$

Step 2. Now, we observe that

$$\begin{aligned}
\int_{(0,T)\times\mathbb{R}^d} |g - g*\rho_\varepsilon|^2\,dxdt &= \int_{(0,T)\times\mathbb{R}^2}\Big|\int_{\mathbb{R}^d}(g(t,x) - g(t,x-y))\rho_\varepsilon(y)\,dy\Big|\,dxdt \\
&= \int_{(0,T)\times\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\int_0^1 \nabla_x g(t,z_s)\cdot y\rho_\varepsilon(y)\,dsdy\Big|^2\,dxdt,
\end{aligned}$$

with $z_s := x + sy$ thanks to a Taylor expansion. As a consequence, we have

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}^d} |g - g * \rho_\varepsilon|^2 \, dxdt &\le \int_{(0,T)\times\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |\nabla_x g(t, z_s)|^2 |y|^2 \rho_\varepsilon(y) \, dsdydxdt \\ &\le \varepsilon^2 \int_{(0,T)\times\mathbb{R}^d} |\nabla_x g(t,z)|^2 dtdz \int_{\mathbb{R}^d} |\zeta|^2 \rho(\zeta)\, d\zeta,\end{aligned}$$

where we have used the Jensen inequality and two changes of variables. We conclude that $f_{n_\ell} \to f$ in $L^2((0,T)\times\mathbb{R}^d)$ by writing

$$f_{n_\ell} - f = (f_{n_\ell} - f_{n_\ell} * \rho_\varepsilon) + (f_{n_\ell} * \rho - f * \rho) + (f * \rho_\varepsilon - f)$$

and using the previous convergence and estimates. □

We give now a typical example of application of the Aubin-Lions lemma that we illustrate on the McKean-Vlasov equation. It is worth emphasizing that more than the result in itself, it is the strategy which is interesting because, for example, a small variation around the same ideas allows one to establish the existence of Leray solution to the Navier-Stokes equation in any dimension $d \ge 2$.

Let us then consider $a_n \in L^\infty(\mathbb{R}^d)$ and $f_{0,n} \in L^2_k(\mathbb{R}^d)$, $k > d/2$, and the associated unique solution $f_n \in X_T$ to the McKean-Vlasov equation

$$\partial_t f_n = \Delta f_n + \mathrm{div}((a_n * f_n) f_n), \quad f_n(0) = f_{0,n}, \tag{1.7}$$

as built in Theorem 1.1.

Proposition 1.3. *Assume that $f_{0,n} \rightharpoonup f_0$ weakly in L^2_k and $a_n \rightharpoonup a_n$ weakly in L^∞. Then $f_n \to f$ in $L^2((0,T)\times\mathbb{R}^d)$ (for instance), where $f \in X_T$ is the unique solution to the McKean-Vlasov equation* (1.7) *associated to the interaction kernel a and to the initial datum f_0.*

Proof of Proposition 1.3. Because of Theorem 1.1, we know that

$$\begin{aligned}&(f_n) \text{ is bounded in } Y_T := L^\infty(0,T; L^2_k(\mathbb{R}^d)) \cap L^2(0,T; H^1(\mathbb{R}^d)),\\ &((a_n * f_n) f_n) \text{ is bounded in } L^2((0,T)\times\mathbb{R}^d).\end{aligned}$$

We may thus apply the Aubin-Lions Lemma 1.2 and we deduce that there exists $f \in Y_T$ and a subsequence (f_{n_ℓ}) such that

$$f_{n_\ell} \to f \text{ strongly in } L^2((0,T)\times\mathbb{R}^d)) \text{ and weakly in } Y_T.$$

For $\varphi \in C^1_c([0,T)\times\mathbb{R}^d)$, the weak formulation of (1.7) writes

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} f_{n_\ell} (\partial_t \varphi + \Delta\varphi) dxdt + \int_{\mathbb{R}^d} f_{0,n_\ell} \varphi(0,\cdot) dxdt \\ &\quad = \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_{n_\ell}(t,x) f_{n_\ell}(t,y) a_{n_\ell}(x-y)\cdot\nabla\varphi(y) dxdydt.\end{aligned}$$

We observe that

$$f_{n_\ell}(t,x) f_{n_\ell}(t,y) \to f(t,x) f(t,y) \text{ strongly in } L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$$

and

$$a_{n_\ell}(x-y)\cdot\nabla\varphi(y) \rightharpoonup a(x-y)\cdot\nabla\varphi(y) \text{ weakly in } L^\infty(\mathbb{R}^d\times\mathbb{R}^d).$$

Using the weak convergence in the LHS term and a standard weak-strong convergence trick in order to deal with the RHS term, we immediately deduce that we may pass to the limit $\ell \to \infty$ in the above formulation and thus that f is a weak solution to the McKean-Vlasov equation (1.7) associated to the interaction kernel a and to the initial datum f_0. Because weak solutions are variational solutions and Theorem 1.1, f is in fact the unique variational solution and by uniqueness of the limit it is the whole sequence (f_n) which converges toward f. $\square$

We recall and accept the following classical result.

Theorem 1.4 (Brouwer-Schauder-Tychonoff)**.** *Consider a locally convex topological vector space X, a convex set $\mathcal{Z} \subset X$ which is metrizable and closed for the induced topology and a function $\varphi : \mathcal{Z} \to \mathcal{Z}$. We assume further that one of the two following conditions holds*
(1) $\mathcal{Z}$ is compact and φ is continuous;
(2) φ is compact and continuous.
In both cases, φ has at least one fixed point.

We now recover the existence part of Theorem 1.1 as a consequence of the two last results.

Corollary 1.5. *For any $f_0 \in L^2_k$, $k > d/2$, there exists at least one solution to the McKean-Vlasov equation* (1.1)

First proof of Corollary 1.5. We will use Theorem 1.4 with $X := X_T$ endowed with the **strong** topology of $L^2(\mathcal{U})$, with $\mathcal{Z} := \mathscr{C}$ which is convex and closed for the strong topology of $L^2(\mathcal{U})$ and with $\varphi : \mathscr{C} \to \mathscr{C}$ the mapping $\varphi(g) := f$, where f is the unique variational solution of equation (1.6). Consider a sequence (g_n) of $\mathscr{C}$ and assume that $g_n \rightharpoonup g$ weakly in $L^2(\mathcal{U})$. We proceed exactly as during the proof of Proposition 1.3. From the Aubin-Lions Lemma 1.2, we know that there exists $f \in L^2(\mathcal{U})$ and a subsequence (f_{n_ℓ}) such that $f_{n_\ell} \to f$ strongly in $L^2(\mathcal{U})$. We may pass to the limit in the weak formulation of equation (1.6) written for f_{n_ℓ} and we deduce that f is a weak solution (and thus a variational solution) to equation (1.6) associated to g. By uniqueness of the solution, we have $f = \varphi(g)$ and by uniqueness of the limit, we have $\varphi(g_n) \to \varphi(g)$ strongly in $L^2(\mathcal{U})$. We immediately deduce that φ is both compact and continuous. We may apply the second version of Theorem 1.4 and conclude. $\square$

Second proof of Corollary 1.5. We will use Theorem 1.4 with $X := X_T$ endowed with the **weak** topology of $L^2(\mathcal{U})$, with same definitions of $\mathcal{Z}$ and φ. The set $\mathcal{Z}$ is clearly weakly compact for the weak topology of $L^2(\mathcal{U})$ and the function φ has been proved to be weakly continuous. We may apply the first version of Theorem 1.4 and conclude. $\square$