A GENERIC PROPERTY OF FAMILIES OF LAGRANGIAN SYSTEMS

PATRICK BERNARD AND GONZALO CONTRERAS

ABSTRACT. We prove that a generic lagrangian has finitely many minimizing measures for every cohomology class.

1. INTRODUCTION

Let M be a compact boundaryless smooth manifold.

Let $\mathbb{T}$ be either the group $(\mathbb{R}/\mathbb{Z}, +)$ or the trivial group $(\{0\}, +)$.

A *Tonelli lagrangian* is a C^2 function $L : \mathbb{T} \times TM \to \mathbb{R}$ such that

- The restriction to each fiber of $\mathbb{T} \times TM \to \mathbb{T} \times M$ is a *convex* function.
- It is fiberwise *superlinear*:
$$\lim_{|\theta|\to+\infty} L(t,\theta)/|\theta| = +\infty, \qquad (t,\theta) \in \mathbb{T} \times TM.$$
- The Euler-Lagrange equation
$$\tfrac{d}{dt} L_v = L_x$$
defines a *complete* flow $\varphi : \mathbb{R} \times (\mathbb{T} \times TM) \longrightarrow \mathbb{T} \times TM$.

When $\mathbb{T} = \{0\}$ we say that the lagrangian is *autonomous*.

Let $\mathcal{P}(L)$ be the set of Borel probability measures on $\mathbb{T} \times TM$ which are invariant under the Euler-Lagrange flow φ. The action functional $A_L : \mathcal{P}(L) \to \mathbb{R} \cup \{+\infty\}$ is defined as
$$A_L(\mu) := \langle L, \mu \rangle := \int_{\mathbb{T}\times TM} L \, d\mu.$$
The functional A_L is lower semi-continuous and the minimizers of A_L on $\mathcal{P}(L)$ are called *minimizing measures*. The ergodic components of a minimizing measure are also minimizing, so that the set $\mathfrak{M}(L)$ of minimizing measures is a simplex whose extremal points are the ergodic minimizing measures.

In general, the simplex $\mathfrak{M}(L)$ may be of infinite dimension. The goal of the present paper is to prove that this is a very exeptional phenomenon. The first results in that direction were obtained by Mañé in [3]. His paper has been very influential to our work.

We say that a property is *generic* in the sense of Mañé if, for each Tonelli Lagrangian L, there exists a residual subset $\mathcal{O} \subset C^\infty(\mathbb{T} \times M, \mathbb{R})$ such that the property holds for each of the Lagrangians $L - u, u \in \mathcal{O}$. A set is called residual if it is a countable intersection

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of open and dense sets. We recall which topology is used on $C^\infty(\mathbb{T}\times M,\mathbb{R})$. Denoting by $\|u\|_k$ the C^k-norm of a function $u:\mathbb{T}\times M\longrightarrow\mathbb{R}$, define

$$\|u\|_\infty := \sum_{k\in\mathbb{N}} \frac{\arctan(\|u\|_k)}{2^k}.$$

Note that $\|.\|_\infty$ is not a norm. Endow the space $C^\infty(\mathbb{T}\times M,\mathbb{R})$ with the translation-invariant metric $\|u-v\|_\infty$. This metric is complete, hence $C^\infty(\mathbb{T}\times M,\mathbb{R})$ is a Frechet space. Consequently the Baire property holds: any residual subset of $C^\infty(\mathbb{T}\times M,\mathbb{R})$ is dense.

Theorem 1. *Let A be a finite dimensional convex family of Lagrangians. Then there exists a residual subset $\mathcal{O}$ of $C^\infty(\mathbb{T}\times M,\mathbb{R})$ such that,*

$$u\in\mathcal{O},\quad L\in A \quad\Longrightarrow\quad dim\,\mathfrak{M}(L-u)\leqslant dim\,A.$$

In other words, there exist at most $1+dim\,A$ ergodic minimizing measures of $L-u$.

The main result of Mañé in [3] is that having a unique minimizing measure is a generic property. This corresponds to the case where A is a point in our statement. Our generalization of Mañé's result is motivated by the following construction due to John Mather:

We can view a 1-form on M as a function on TM which is linear on the fibers. One can see that the Euler-Lagrange equation of the Lagrangian $L-\lambda$ is the same as that of L. However, the minimizing measures of $L-\lambda$, are not the same as the minimizing measures of L. Mather proves in [4] that the set $\mathfrak{M}(L-\lambda)$ of minimizing measures of the lagrangian $L-\lambda$ depends only on the cohomology class c of λ. If $c\in H^1(M,\mathbb{R})$ we write $\mathfrak{M}(L-c):=\mathfrak{M}(L-\lambda)$, where λ is a closed form of cohomology c. We call

$$\mathfrak{N}(L) := \bigcup_{c\in H^1(M,\mathbb{R})} \mathfrak{M}(L-c)$$

the set of all *Mather minimizing measures* of L.

It truns out that important applications of Mather theory, such as the existence of orbits wandering in phase space, require understanding not only of the set $\mathfrak{M}(L)$ of minimizing measures for a fixed or generic cohomology classes but of the set $\mathfrak{N}(L)$ of all Mather minimizing measures for every $c\in H^1(M,L)$.

The following corollaries are crutial for these applications.

Corollary 2. *The following property is generic in the sense of Mañé:*

For all $c\in H^1(M,\mathbb{R})$, there are at most $1+dim\,H^1(M,\mathbb{R})$ ergodic minimizing measures of $L-c$.

We say that a property is of infinite codimension if, for each finite dimensional convex family A of Lagrangians, there exists a residual subset $\mathcal{O}$ in $C^\infty(\mathbb{T}\times M,\mathbb{R})$ such that the property is satisfied by all the Lagrangians $L-u$, $L\in A$, $u\in\mathcal{O}$.

Corollary 3. *The following property is of infinite codimension:*

There exists $c \in H^1(M,\mathbb{R})$, such that $L - c$ has infinitely many ergodic minimizing measures.

Another important issue concerning variational methods for Arnold diffusion questions is the total disconectedness of the quotient Aubry set. John Mather proves in [6, § 3] that the quotient Aubry set $\overline{\mathcal{A}}$ of any Tonelli lagrangian on $\mathbb{T} \times TM$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\dim M \leqslant 2$ or with $\mathbb{T} = \{0\}$ and $\dim M \leqslant 3$. See [6] for its definition.

The elements of the quotient Aubry set are called *static classes.* They are disjoint subsets of $\mathbb{T} \times TM$ and each static class contains at least one ergodic minimizing measure. We then get

Corollary 4. *The following property is generic in the sense of Mañé:*

For all $c \in H^1(M,\mathbb{R})$ the quotient Aubry set $\overline{\mathcal{A}}_c$ of $L - c$ has at most $1 + \dim H^1(M,\mathbb{R})$ elements.

2. Abstract Results

Assume that we are given

- Three topological vector spaces E, F, G.
- A continuous linear map $\pi : F \to G$.
- A bilinear pairing $\langle u, \nu\rangle : E \times G \to \mathbb{R}$.
- Two metrizable convex compact subsets $H \subset F$ and $K \subset G$ such that $\pi(H) \subset K$.

Suppose that

(i) The map

$$E \times K \ni (u, \nu) \longmapsto \langle u, \nu\rangle$$

is continuous.

We will also denote by $\langle u, \pi(\mu)\rangle$ by $\langle u, \mu\rangle$ when $\mu \in H$. Observe that each element $u \in E$ gives rise to a linear functional on F

$$F \ni \mu \longmapsto \langle u, \mu\rangle$$

which is continuous on H. We shall denote by H^* the set of linear functionals on F which are continuous on H and use the same symbol u for an element of E and for the element $u \longmapsto \langle u, \mu\rangle$ of H^* which is associated to it.

(ii) The compact K is separated by E. This means that, if η and ν are two different points of K, then there exists a point u in E such that $\langle u, \eta - \nu\rangle \neq 0$.

Note that the topology on K is then the weak topology associated to E. In other words, a sequence η_n of elements of K converges to η if and only if we have $\langle u, \eta_n\rangle \longrightarrow \langle u, \eta\rangle$ for each $u \in E$. We shall, for notational conveniences, fix once and for all a metric d on K.

(iii) E is a Frechet space. It means that E is a topological vector space whose topology is defined by a translation-invariant metric, and that E is complete for this metric.

Note then that E has the Baire property. We say that a subset is residual if it is a countable intersection of open and dense sets. The Baire property says that any residual subset of E is dense.

For each $L \in H^*$ denote by

$$M_H(L) := \arg\min_H L$$

the set of points $\mu \in H$ which minimize $L|_H$, and by $M_K(L)$ the image $\pi(M_H(L))$. These are compact convex subsets of H and K.

Our main abstract result is:

Theorem 5. *For every finite dimensional affine subspace A of H^*, there exists a residual subset $\mathcal{O}(A) \subset E$ such that, for all $u \in \mathcal{O}(A)$ and all $L \in A$, we have*

$$dim\, M_K(L - u) \leq dim\, A. \tag{1}$$

Proof: We define the ε-neighborhood V_ε of any subset V of K as the union of all the open balls in K which have radius ε and are centered in the subset. Given a subset $D \subset A$, a positive number ε, and a positive integer k, denote by $\mathcal{O}(D, \varepsilon, k) \subset E$ the set of points $u \in E$ such that, for each $L \in D$, the convex set $M_K(L-u)$ is contained in the ε-neighborhood of some k-dimensional convex subset of K.

We shall prove that the theorem holds with

$$\mathcal{O}(A) = \bigcap_{\varepsilon>0} \mathcal{O}(A, \varepsilon, \dim A).$$

If u belongs to $\mathcal{O}(A)$, then (1) holds for every $L \in A$. Else, for some $L \in A$, the convex set $M_K(L-u)$ would contain a ball of dimension $\dim A + 1$, and, if ε is small enough, such a ball is not contained in the ε-neighborhood of any convex set of dimension $\dim A$.

So we have to prove that $\mathcal{O}(A)$ is residual. In view of the Baire property, it is enough to check that, for any compact subset $D \subset A$ and any positive ε, the set $\mathcal{O}(D, \varepsilon, \dim A)$ is open and dense. We shall prove in 2.1 that it is open, and in 2.2 that it is dense.

□

2.1. **Open.**

We prove that, for any $k \in \mathbb{Z}^+$, $\varepsilon > 0$ and any compact $D \subset A$, the set $\mathcal{O}(D, \varepsilon, k) \subset E$ is open. We need a Lemma.

Lemma 6. *The set-valued map $(L, u) \longmapsto M_H(L-u)$ is upper semi-continuous on $A \times E$. This means that for any open subset U of H, the set*

$$\{(L, u) \in A \times E \ : \ M_H(L-u) \subset U\} \subset A \times E$$

is open in $A \times E$. Consequently, the set-valued map $(L, u) \longmapsto M_K(L-u)$ is also upper semi-continuous.

Proof: This is a standard consequence of the continuity of the map

$$A \times E \times H \ni (L, u, \mu) \longmapsto (L-u)(\mu) = L(\mu) - \langle u, \mu \rangle.$$

□

Now let u_0 be a point of $\mathcal{O}(D, \varepsilon, k)$. For each $L \in D$, there exists a k-dimensional convex set $V \subset K$ such that $M_K(L-u_0) \subset V_\varepsilon$. In other words, the open sets of the form

$$\{(L, u) \in D \times E \ : \ M_H(L-u) \subset V_\varepsilon\} \subset D \times E,$$

where V is some k-dimensional convex subset of K, cover the compact set $D \times \{u_0\}$. So there exists a finite subcovering of $D \times \{u_0\}$ by open sets of the form $\Omega_i \times U_i$, where Ω_i is an open set in A and $U_i \subset \mathcal{O}(\Omega_i, \varepsilon, k)$ is an open set in E containing u_0. We conclude that the open set $\cap U_i$ is contained in $\mathcal{O}(D, \varepsilon, k)$, and contains u_0. This ends the proof.

□

2.2. Dense.

We prove the density of $\mathcal{O}(A, \varepsilon, \dim A)$ in E for $\varepsilon > 0$. Let w be a point in E. We want to prove that w is in the closure of $\mathcal{O}(A, \varepsilon, \dim A)$.

Lemma 7. *There exists an integer m and a continuous map*

$$T_m = (w_1, \dots, w_m) : K \longrightarrow \mathbb{R}^m,$$

with $w_i \in E$ such that

$$\forall\, x \in \mathbb{R}^m \qquad \operatorname{diam} T_m^{-1}(x) < \varepsilon, \tag{2}$$

where the diameter is taken for the distance d on K.

Proof: In $K \times K$, to each element $w \in E$ we associate the open set

$$U_w = \{\, (\eta, \mu) \in K \times K \ : \ \langle w, \eta - \mu \rangle \neq 0 \,\}.$$

Since E separates K, the open sets $U_w, w \in E$ cover the complement of the diagonal in $K \times K$. Since this complement is open in the metrizable set $K \times K$, we can extract a countable subcovering from this covering. So we have a sequence U_{w_k}, with $w_k \in E$, which covers the complement of the diagonal in $K \times K$. This amounts to say that the sequence w_k seperates K. Defining $T_m = (w_1, \dots, w_m)$, we have to prove that (2) holds for m large enough. Otherwise, we would have two sequences η_m and μ_m in K such that

$$T_m(\mu_m) = T_m(\eta_m) \quad \text{and} \quad d(\mu_m, \eta_m) \geqslant \varepsilon.$$

By extracting a subsequence, we can assume that the sequences μ_m and η_m have different limits μ and η, which satisfy $d(\eta, \mu) \geqslant \varepsilon$. Take m large enough, so that $T_m(\eta) \neq T_m(\mu)$. Such a value of m exists because the linear forms w_k separate K. We have that

$$T_m(\mu_k) = T_m(\eta_k) \quad \text{for} \quad k \geq m.$$

Hence at the limit $T_m(\eta) = T_m(\mu)$. This is a contradiction.

□

Define the function $F_m : A \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ as

$$F_m(L, x) := \min_{\substack{\mu \in H \\ T_m \circ \pi(\mu) = x}} (L - w)(\mu),$$

when $x \in T_m(\pi(H))$ and $F_m(L, x) = +\infty$ if $x \in \mathbb{R}^m \setminus T_m(\pi(H))$. For $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, let

$$M_m(L, y) := \operatorname*{arg\,min}_{x \in \mathbb{R}^m} \big[F_m(L, x) - y \cdot x \big] \subset \mathbb{R}^m$$

be the set of points which minimize the function $x \longmapsto F_m(L, x) - y \cdot x$. We have that

$$M_K\big(L - w - y_1 w_1 - \cdots - y_m w_m\big) \subset T_m^{-1}(M_m(L, y)).$$

Let

$$\mathcal{O}_m(A, \dim A) := \{\ y \in \mathbb{R}^m \mid \forall\, L \in A \ : \ \dim M_m(L, y) \leqslant \dim A\ \}.$$

From Lemma 7 it follows that

$$y \in \mathcal{O}_m(A, \dim A) \quad \Longrightarrow \quad w + y_1 w_1 + \cdots + y_m w_m \in \mathcal{O}(A, \varepsilon, \dim A).$$

Therefore, in order to prove that w is in the closure of $\mathcal{O}(A, \varepsilon, \dim A)$, it is enough to prove that 0 is in the closure of $\mathcal{O}_m(A, \dim A)$, which follows from the next proposition.

Proposition 8. *The set $\mathcal{O}_m(A, dim\, A)$ is dense in $\mathbb{R}^m$.*

Proof: Consider the Legendre transform of F_m with respect to the second variable,

$$\begin{aligned} G_m(L, y) &= \max_{x \in \mathbb{R}^m} y \cdot x - F_m(L, x) \\ &= \max_{\mu \in H} \langle w + y_1 w_1 + \cdots + y_m w_m, \mu\rangle - L(\mu). \end{aligned}$$

It follows from this second expression that the function G_m is convex and finite-valued, hence continuous on $A \times \mathbb{R}^m$.

Consider the set $\tilde{\Sigma} \subset A \times \mathbb{R}^m$ of points (L, y) such that $\dim \partial G_m(L, y) \geq \dim A + 1$, where ∂G_m is the subdifferential of G_m. It is known, see [1], that this set has Hausdorff dimension at most

$$(m + \dim A) - (\dim A + 1) = m - 1.$$

Consequently, the projection Σ of the set $\tilde{\Sigma}$ on the second factor $\mathbb{R}^m$ also has Hausdorff dimension at most $m - 1$. Therefore, the complement of Σ is dense in $\mathbb{R}^m$. So it is enough to prove that

$$y \notin \Sigma \quad \Longrightarrow \quad \forall\, L \in A\ : \quad \dim M_m(L, y) \leq \dim A.$$

Since we know by definition of Σ that $\dim \partial G_m(L, y) \leq \dim A$, it is enough to observe that

$$\dim M_m(L, y) \leq \dim \partial G_m(L, y).$$

The last inequality follows from the fact that the set $M_m(L, y)$ is the subdifferential of the convex function

$$\mathbb{R}^m \ni z \longmapsto G_m(L, z)$$

at the point y.

$\square$

3. Application to Lagrangian dynamics

Let C be the set of continuous functions $f : \mathbb{T} \times TM \to \mathbb{R}$ with linear growth, i.e.

$$\|f\|_\ell := \sup_{(t,\theta)\in\mathbb{T}\times TM} \frac{|f(t,\theta)|}{1+|\theta|} < +\infty,$$

endowed with the norm $\|\cdot\|_\ell$.

We apply Theorem 5 to the following setting

- $F = C^*$ is the vector space of continuous linear functionals $\mu : C \to \mathbb{R}$ with the topology given by

$$\lim_n \mu_n = \mu \quad \Longleftrightarrow \quad \lim_n \mu_n(f) = \mu(f), \quad \forall\, f \in C.$$

- $E = C^\infty(\mathbb{T} \times M, \mathbb{R})$.
- G is the vector space of finite Borel signed measures on $\mathbb{T} \times M$.
- The pairing $E \times G \to \mathbb{R}$ is given by integration:

$$\langle u, \nu\rangle = \int_{\mathbb{T}\times M} u \; d\nu.$$

- The compact $K \subset G$ is the set of Borel probability measures on $\mathbb{T} \times M$, endowed with the weak-$*$ topology. Observe that K is separated by E.
- The compact $H_n \subset F$ is the set of holonomic probability measures which are supported on

$$B_n := \{(t,\theta) \in \mathbb{T} \times TM \mid |\theta| \leqslant n\,\}.$$

Holonomic probabilities are defined as follows: Given a C^1 curve $\gamma : \mathbb{R} \to M$ of period $T \in \mathbb{N}$ define the probability measure μ_γ on $\mathbb{T} \times TM$ by

$$\int_{\mathbb{T}\times TM} f \; d\mu_\gamma = \frac{1}{T}\int_0^T f(s,\gamma(s),\dot\gamma(s))\; ds$$

for each $f \in C^0(\mathbb{T} \times TM, \mathbb{R})$. Let

$$\Gamma := \{\, \mu_\gamma \mid \gamma \in C^1(\mathbb{R}, M) \text{ is periodic of integral period }\}.$$

The set $\mathcal{H}$ of holonomic probabilities is the closure of Γ in F. One can see that $\mathcal{H}$ is convex (cf. Mañé [3, prop. 1.1(a)]).

Let $\mathfrak{M}(L)$ be the set of minimizing measures for L and let $\operatorname{supp}\mathfrak{M}(L)$ be the union of their supports.

Lemma 9. *If L is a Tonelli lagrangian then there is $n \in \mathbb{N}$ such that*

$$dim\, \pi(M_{H_n}(L)) = dim\, \mathfrak{M}(L).$$

Proof:

Birkhoff theorem implies that $\mathfrak{M}(L) \subset \mathcal{H}$ (cf. Mañé [3, prop. 1.1.(b)]).

In [3, §1] Mañé proves that minimizing measures are also all the minimizers of the action functional $A_L : \mathcal{H} \to \mathbb{R}$,

$$A_L(\mu) = \int L \, d\mu \,,$$

on holonomic measures.

In [4, Prop. 4, p. 185] Mather proves that $\operatorname{supp} \mathfrak{M}(L)$ is compact, therefore

$$\mathfrak{M}(L) \subset H_n \qquad \text{for some } n \in \mathbb{N}.$$

In [4, Th. 2, p. 186] Mather proves that the restriction $\operatorname{supp} \mathfrak{M}(L) \to M$ of the projection $TM \to M$ is injective. Therefore the linear map $\pi : \mathfrak{M}(L) \to G$ is injective.

□

Proof of Theorem 1.

Given $n \in \mathbb{N}$ apply Theorem 5 and obtain a residual subset $\mathcal{O}_n(A) \subset E$ such that

$$\forall \, L \in A, \quad \forall \, u \in \mathcal{O}_n(A) \;\; : \qquad \dim \pi(M_{H_n}(L)) \leqslant \dim A.$$

Let $\mathcal{O}(A) = \cap_n \mathcal{O}_n(A)$. By the Baire property $\mathcal{O}(A)$ is residual. We have that

$$\forall \, L \in A, \quad \forall \, u \in \mathcal{O}(A), \quad \forall \, n \in \mathbb{N} \;\; : \qquad \dim \pi(M_{H_n}(L)) \leqslant \dim A.$$

Then by Lemma 9, $\dim \mathfrak{M}(L) \leqslant \dim A$ for all $L \in A$.

□

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CEREMADE, Université de Paris Dauphine, Pl. du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France,

E-mail address: patrick.bernard@ceremade.dauphine.fr

CIMAT, P.O. Box 402, 36.000 Guanajuato, GTO, México.

E-mail address: gonzalo@cimat.mx