Homoclinic orbits near saddle-center fixed points of Hamiltonian systems with two degrees of freedom

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To Jacob Palis for his 60^{th} Birthday

Resumo : Damos exemplos de sistemas Hamiltonianos sobre uma variedade simplética de dimensão 4, com um ponto fixo sela-centro, apresentando a seguinte propriedade: Toda órbita periódica na variedade central do ponto fixo tem uma órbita homoclínica a ela, apesar de o ponto fixo não ter. Além disso, provamos que tais sistemas têm um comportamento caótico na vizinhança do nível de energia do ponto fixo.

Résumé : On étudie une classe de systèmes Hamiltoniens, sur une variété symplectique de dimension 4, admettant un point fixe de type selle-centre. On montre que ces systèmes ont la propriété suivante: Chaque orbite périodique de la variété centrale du point fixe a une orbite homocline, mais le point fixe lui-même n'a pas necessairement d'orbite homocline. On montre de plus que ces systèmes ont un comportement chaotique au voisinage de la surface d'énergie du point fixe.

Abstract : We study a class of Hamiltonian systems on a 4 dimensional symplectic manifold with a saddle-center fixed point. We prove that these systems have the following interesting property: All the periodic orbits in the center manifold of the fixed point have an orbit homoclinic to them, although there may not exist any orbit homoclinic to the fixed point. In addition, we prove that these systems have a chaotic behavior in the neighborhood of the energy shell of the fixed point.

Introduction

A fixed point of a Hamiltonian system with two degrees of freedom is called a Saddle-Center if the linearized vector field has one pair of purely imaginary eigenvalues and one pair of non zero real eigenvalues. A saddle-center fixed point is surrounded by a two-dimensional invariant manifold, the center manifold, filled by closed orbits. A saddle-center fixed point has also a one-dimensional stable manifold and a one-dimensional unstable manifold; the periodic orbits in the center manifold have two dimensional stable and unstable manifolds. If a point belongs to the intersection of the stable and unstable manifold of the fixed point (resp. of one periodic orbit) then its orbit is biasymptotic to the fixed point (resp. the periodic orbit). We call such an orbit homoclinic.

Some consequences of the existence of an orbit homoclinic to the fixed point have been investigated in [5], [9], [7], [8], [11], [16] (specially section 7.2) and other papers. It should be noted, however, that the existence of such a homoclinic is exceptional, in contrast to the case of hyperbolic fixed points. Dimensional considerations show that orbits homoclinic to the periodic motions of the center manifold are more likely to exist. The existence of such homoclinics has been studied in [4], [14] (see also [11], [9], [10], [7], [12]) by perturbation methods, and in [2] by global methods. In these papers, orbits homoclinic to periodic orbits sufficiently far away from the fixed point are found.

In the present work, we study analytic perturbations of an integrable system with a homoclinic loop. We prove the following interesting behavior : Given any periodic orbit sufficiently close to the equilibrium in the center manifold, there exists an orbit homoclinic to it, although in general there does not exist any orbit homoclinic to the fixed point. This illustrates a question asked in [2].

In addition, topological entropy near the energy shell of the fixed point is obtained as a consequence of the presence of these homoclinics. More precisely, we prove that every neighborhood of the energy shell of the fixed point contains an energy shell with chaotic behavior on it. A similar result for reversible Hamiltonian systems is claimed, with no proof, in [14] pg 116. Other results in this direction under the hypothesis of the system being far from integrable can be found in [9], [7], [13].

Our method is semi-global and heavily relies on the low dimension: We first use the perturbative setting to prove the existence of quasiperiodic invariant tori confining the system in a neighborhood of the unperturbed homoclinic loop. We then reduce the problem to an area preservation argument on appropriate Poincaré return maps. It would of course be very interesting to obtain similar results by global methods and in higher dimension, in the spirit of [2], and to understand to what extent the phenomenon described here is general.

This paper emanated from a discussion between the authors after a talk of one of them at the international conference on dynamical systems dedicated to Jacob Palis. The authors would like to thank the organizers of that conference, who made that encounter possible. The first author learned a lot during his numerous conversations with Michel Herman, and was moved a lot by his sudden death.

1 Notations and results

1.1 Let M be a four-dimensional analytic manifold, endowed with a symplectic form Ω, and let

$$\begin{aligned} H : M \times I &\longrightarrow \mathbb{R}, \\ (x,\mu) &\longmapsto H(x,\mu) = H_\mu(x) \end{aligned}$$

be an analytic one-parameter family of Hamiltonians, where I is some interval containing 0 in its interior. In all this paper, we shall assume that the Hamiltonian system H_μ has a saddle-center fixed point r_μ for all $\mu \in I$, and that $H_\mu(r_\mu) = 0$. It is by now classical (see [5], [9], [14],...), that the system H_μ is integrable in the neighborhood of the saddle-center r_μ. More precisely, there exist a neighborhood U of 0 in $\mathbb{R}^4$ and an analytic mapping $\phi : U \times I \longrightarrow M$ such that ϕ_μ is a symplectic embedding for each μ, $\phi_\mu(0) = r_\mu$, and

$$H_\mu \circ \phi_\mu(q_1, p_1, q_2, p_2) = h(I_1, I_2, \mu),$$

where

$$I_1 = p_1 q_1 \,, \quad I_2 = (p_2^2 + q_2^2)/2,$$

and the function h is analytic (one may have to reduce I). Furthermore, one can be reduced via a change in time-scale and a canonical transformation to the case where

$$\partial_{I_1} h(0,0,\mu) = -1 \text{ and } \partial_{I_2} h(0,0,\mu) = \omega(\mu) > 0.$$

The functions I_1 and I_2 are preserved by the flow restricted to the local chart, this flow is determined by the equations

$$\begin{array}{ll} \dot{p}_1 = -\partial_{I_1} h(I_1,I_2,\mu)p_1 & \dot{p}_2 = -\partial_{I_2} h(I_1,I_2,\mu)q_2 \\ \dot{q}_1 = \ \ \partial_{I_1} h(I_1,I_2,\mu)q_1 & \dot{q}_2 = \ \ \partial_{I_2} h(I_1,I_2,\mu)p_2. \end{array}$$

It follows that the center manifold of r_μ has equation $I_1 = 0$, its stable manifold has equation $I_2 = 0$, $p_1 = 0$ and its unstable manifold $I_2 = 0$, $q_1 = 0$. In the following, we will call $P_{E,\mu}$ the periodic orbit of H_μ at energy E, which in local coordinates is the circle $p_1 = q_1 = 0$, $I_2 = E$.

1.2 We shall also suppose that H_0 is integrable (namely, its associated Hamiltonian vector field has an additional real analytic first integral J such that $dH_0(x)$ and $dJ(x)$ are independent for almost every x) and that the vector field associated to H_0 has an orbit homoclinic to r_0 which connects the branch $p_1 > 0$ of the unstable manifold to the branch $q_1 > 0$ of the stable manifold. Integrable systems with a saddle-center and an orbit doubly asymptotic to it have been studied in [9], where it is explained that there exist two different kinds of homoclinics. For comparison, let us mention that we are here in case (A) of [9].

1.3 THEOREM Let us consider an analytic one-parameter family H_μ of Hamiltonian systems satisfying the above hypotheses. There exists a positive number ϵ such that for all $E \in]0,\epsilon[$ and all $\mu \in]-\epsilon,\epsilon[\subset I$, there exists an orbit of H_μ homoclinic to the periodic orbit $P_{E,\mu}$. In fact, there even exist infinitely many geometrically distinct orbits homoclinic to $P_{E,\mu}$.

1.4 THEOREM Let us fix $\mu \in]-\epsilon,\epsilon[$. For each $E \in]0,\epsilon[$, either the stable and unstable manifolds of $P_{E,\mu}$ coincide, or the flow of H_μ on the energy shell $H_\mu = E$ has positive topological entropy.

1.5 THEOREM Let us fix a value of μ satisfying the hypothesis of theorem 1.3. Assume in addition that the stable and unstable manifolds of the fixed point r_μ do not coincide. Then there exists a sequence $E_n > 0$ converging to 0 and such that the stable and unstable manifolds of $P_{E_n,\mu}$ do not coincide. It follows that, for each n, the flow of H_μ restricted to the energy surface $H_\mu = E_n$ has positive topological entropy.

1.6 The main result of the present paper is Theorem 1.3. It is proved in section 3. Theorem 1.4 may be considered classical. However we include a proof in section 4 because we could not find any reference matching precisely our needs. Theorem 1.5 is a simple but, we believe, interesting consequence. It is proved in section 5. The main notations and tools that will be used throughout the paper are introduced in section 2

1.7 REMARK In order to apply Theorem 1.5, one has to be able to decide whether there exists an orbit homoclinic to the fixed point. Let us mention a result in that direction. Under an additional hypothesis of reversibility of the family of Hamiltonian systems H_μ (see [7]) it is possible to prove that the set of values of μ for which a homoclinic orbit to the equilibrium point r_μ occurs is either a whole interval or it is countable ([7], section 6). The same result may hold for the non reversible case considered here but this is an open question.

2 Local sections and invariant curves

We analyze the orbit structure near the homoclinic loop in a rather usual way (see [5], [9], [14],...), via Poincaré sections. More details in these papers. The existence of invariant curves was already obtained in [8].

2.1 Let us define the two Poincaré sections given in local coordinates by

$$\Sigma_1 = \{q_1 = \delta\}\ , \quad \Sigma_2 = \{p_1 = \delta\},$$

where δ is a small positive number. Since $\partial_{I_1} h = -1$, the equation $h(I_1, I_2, \mu) = E$ can be solved in I_1 for sufficiently small I_2, E and μ *i.e.* there exists an analytic function v defined in a neighborhood of 0 in $\mathbb{R}^3$ and such that

$$h(I_1, I_2, \mu) = E \iff I_1 = v(I_2, E, \mu).$$

As a consequence, for sufficiently small E and μ, the intersection $\Sigma_i(E,\mu)$ of Σ_i with the energy shell $H_\mu = E$ is a graph over the (p_2, q_2)-plane. More precisely, the analytic mappings $\sigma_i^{E,\mu} : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ given by

$$\sigma_1^{E,\mu}(p_2, q_2) = \sigma_1(p_2, q_2, E, \mu) = \big(v\big(I_2(p_2, q_2), E, \mu\big)/\delta,\ \delta,\ p_2,\ q_2\big),$$
$$\sigma_2^{E,\mu}(p_2, q_2) = \sigma_2(p_2, q_2, E, \mu) = \big(\delta,\ v\big(I_2(p_2, q_2), E, \mu\big)/\delta,\ p_2,\ q_2\big)$$

are symplectic charts of $\Sigma_i(E,\mu)$. In the following, we note $y = (p_2, q_2)$ and take it as coordinates of $\Sigma_i(E,\mu)$.

2.2 The intersection between the stable manifold of $P_{E,\mu}$ and Σ_1, as well as the intersection between the unstable manifold and Σ_2, are the circles $I_2(y) = I^c(E,\mu)$ in coordinates, where $I^c(E,\mu)$ is the solution of the equation

$$h(0, I^c(E,\mu), \mu) = E \iff v(I^c(E,\mu), E, \mu) = 0.$$

The orbits starting in $\Sigma_1(E,\mu)$ outside of this circle hit $\Sigma_2(E,\mu)$ after a time

$$T(y, E, \mu) = t(I_2(y), E, \mu) = \frac{1}{\partial_{I_1} h\big(v(I_2(y), E, \mu), I_2(y), \mu\big)} \log \frac{v(I_2(y), E, \mu)}{\delta^2}.$$

Notice in the previous expression that $v(I_2(y), E, \mu)$ is positive if and only if y lies outside of the stable circle. The local transition map $l_{E,\mu} : \Sigma_1(E,\mu) \longrightarrow \Sigma_2(E,\mu)$ is defined outside of the stable circle and can be computed in local coordinates

$$l_{E,\mu}(y) = l(y, E, \mu) = R\big(\theta(I_2(y), E, \mu)\big)\, y$$

where $R(\theta)$ is the matrix of the rotation of angle θ, and

$$\theta(I_2, E, \mu) = t(I_2)\, \partial_{I_2} h\big(v(I_2, E, \mu), I_2, \mu\big).$$

The outer transition map $g_{E,\mu} : \Sigma_2(E,\mu) \longrightarrow \Sigma_1(E,\mu)$ is defined by following the flow along the homoclinic loop.

2.3 The following estimate will be useful (see [7]):

$$\theta(I_2, E, \mu) = -\omega(\mu) \log |I_2 - I^c(E, \mu)| + \Lambda_{E,\mu}(I_2), \quad I_2 > I^c(E, \mu)$$

where

$$I^c(E, \mu) = \frac{E}{\omega(\mu)} + O(E^2)$$

and where the function $I_2 \longmapsto \Lambda_{E,\mu}(I_2)$ is analytic around $I^c(E, \mu)$ for each E and μ. To see this, just write $v(I_2, E, \mu) = (I_2 - I_c(E, \mu))w(I_2, E, \mu)$, where w is analytic and $w(I_c, E, \mu) \neq 0$.

2.4 The local transition maps $l_{E,\mu}$ seen in coordinates as mappings of $\mathbb{R}^2$ preserve the circles centered at the origin. Since the unperturbed Hamiltonian H_0 is assumed to be integrable, the outer transition map $g_{0,0}$ also preserves these circles, hence this symplectic map can be written

$$g_{0,0}(y) = R(\psi(I_2(y)))\, y,$$

where ψ is a real map analytic in a neighborhood of 0. Let us now define the mapping

$$F_{E,\mu} = g_{E,\mu} \circ l_{E,\mu},$$

we have

$$F_{0,0} = R\big(\psi \circ I_2 + \theta \circ I_2\big).$$

In view of the estimates of 2.3, it is possible to choose positive numbers $I^- < I^+$ such that $F_{0,0}$ is an integrable analytic twist area preserving diffeomorphism of the annulus $\mathcal{A} = \big\{y \text{ s.t. } I^- \leqslant I_2(y) \leqslant I^+\big\}$. For sufficiently small E and μ, $F_{E,\mu}$ is a two-parameter analytic family of exact area preserving diffeomorphisms between $\mathcal{A}$ and its image in $\mathbb{R}^2$. Here exact means that there exists a rotational Jordan curve C in the annulus $\mathcal{A}$ with the following property : The image $F_{E,\mu}(C)$ is also a rotational Jordan curve in $\mathcal{A}$ and the area of the domain between $\{I_2 = I^-\}$ and $F_{E,\mu}(C)$ is equal to the area of the domain between $\{I_2 = I^-\}$ and C. A direct application of KAM theorem now proves the following proposition.

2.5 Proposition There exist positive numbers ϵ and I such that, for all $E \in]0, \epsilon[$ and $\mu \in]-\epsilon, \epsilon[$, there exists an analytic rotational Jordan curve $C(E, \mu)$ contained in $\{y \text{ s.t. } I/2 \leqslant I_2(y) \leqslant I\}$ and invariant under $F_{E,\mu}$. Let us denote $C'(E, \mu) = l_{E,\mu}(C(E, \mu)) = g_{E,\mu}^{-1}(C(E, \mu))$.

3 Homoclinic orbits and multiplicity

We now prove Theorem 1.3. We have to study the dynamics of the flow of H_μ on the energy surface $\{H_\mu = E\}$, where E and μ satisfy the hypotheses of Proposition 2.5. We will not mention any more the parameters E and μ.

3.1 The map $F = g \circ l$ has an invariant circle C. Let S be the intersection between the stable manifold and Σ_1, and S' be the intersection between the unstable manifold and Σ_2. Both S and S' are the circle $\{I_2(y) = I^c\}$ in coordinates. The local transition map l is defined in the open annulus A in Σ_1 enclosed between S and C, and takes values in the annulus A' of Σ_2 enclosed between S' and C'. The outer transition map g is defined and analytic in D', the open disk enclosed in C', and takes values in D, the open disk enclosed in C. We call B the closed disk bounded by S and B' the closed disk bounded by S'. Both l and g preserve area (see figure 1).

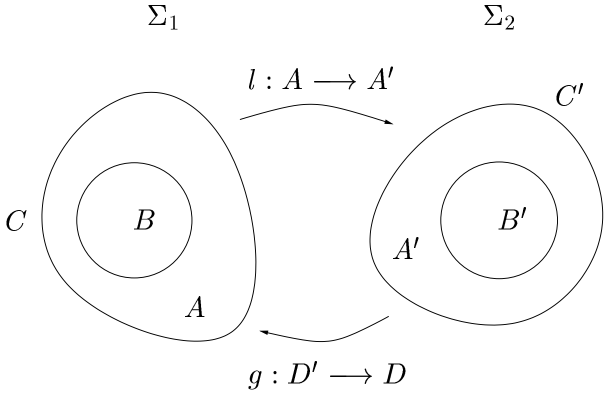

Figure 1: The mappings

3.2 The existence of a homoclinic is a consequence of the facts recalled above, as we shall see now. If $g(S')$ intersects S, then these intersection points are homoclinic points, since S is contained in the stable manifold on one hand, and S', hence $g(S')$, are contained in the unstable manifold on the other hand. We call such intersections 1-bump homoclinic points. If $g(S')$ does not intersect S, then $g(B')$ $(g^{-1}(B))$ is contained in A (A') since it can't be contained in B (B'), by area preservation. It follows that there exists a neighborhood U of B' such that $F \circ g$ is well defined in U.

3.3 Lemma Suppose that for each $n \leqslant N-2$ the map $F^n \circ g$ is well defined in a neighborhood of B' and satisfies $F^n \circ g(S') \cap S = \emptyset$. Then $F^i \circ g(B') \cap F^j \circ g(B') = \emptyset$, for all $0 \leqslant i < j \leqslant N-2$.

Proof The hypothesis $F^n \circ g(S') \cap S = \emptyset$ and the area preservation property of F and g imply that $F^n \circ g(B') \cap B = \emptyset$ for all $n \leqslant N-2$. We also have that $F^n \circ g(B') \cap g(B') = \emptyset$ for all $1 \leqslant n \leqslant N-2$. To prove this, we observe that the image $\mathrm{Im}(F)$ of the map F is $g(\mathrm{Im}(l)) = g(A')$. Since A' is disjoint from B', the image of F is disjoint from $g(B')$. Let us now take $0 \leqslant i < j \leqslant N-2$, we have $F^i \circ g(B') \cap F^j \circ g(B') = F^i\big(g(B') \cap F^{j-i} \circ g(B')\big) = F^i(\emptyset) = \emptyset$. □

3.4 Proposition There exists an integer $N \geqslant 1$ satisfying the hypotheses of Lemma 3.3 and such that

$$F^{N-1} \circ g(S') \cap S \neq \emptyset.$$

The intersection points seen as points of Σ_1, are homoclinic points, we call them N-bump homoclinic points. We have the following alternative: Either $F^{N-1} \circ g(S') = S$ and there are infinitely many N-bumps homoclinics, or $F^{N-1} \circ g(S') \not\subset S$ and there are infinitely many $2N$-bumps homoclinics.

3.5 Proof Since the annulus A has bounded area, and since all the domains $F^n \circ g(B')$ have the same positive area, only finitely many of them can be disjoint, hence the existence of N. It is quite clear that there exist infinitely many N-bumps homoclinic orbits in the case where $F^{N-1} \circ g(S') = S$. We shall now see that there exist infinitely many $2N$-bumps homoclinic points in the second case, *i.e.* if

$$F^{N-1} \circ g(S') \neq S.$$

Definition (see [1]) Let Δ be a compact topological disk in $\mathbb{R}^2$. We say that a continuous curve $\delta \subset \mathbb{R}^2 - \Delta$ has the obstruction property with respect to Δ if any continuous curve γ containing a point in Δ and a point outside Δ intersects the curve δ. It follows that any such curve γ must intersect δ infinitely many times.

Let us note $G = F^{N-1} \circ g$. In view of the estimates 2.3, and since $G(S')$ is not contained in B, the curve

$$\delta = l\big(G(S') \cap A\big)$$

has the obstruction property with respect to B. It follows that the curve $G(\delta \cap \mathrm{dom}(G))$ has the obstruction property with respect to $G(B')$, where $\mathrm{dom}(G)$ is the domain of definition of G. We have supposed that $G(S')$ intersects S (and thus B), and that $G(S')$ is not S, hence is not contained in B, by area preservation. It follows from the obstruction property that $G(\delta)$ has to intersect S infinitely many times. We have proved that the set $G \circ l \circ G(S') = F^{2N-1} \circ g(S')$ has infinitely many points of intersection with S. These points clearly represent geometrically distinct $2N$-bumps homoclinics. $\square$

4 Bernoulli shift

In order to prove Theorem 1.4, we are now going to build a Bernoulli shift. Our construction is quite similar to the one described in [15], chapter III, for the Sitnikov map. However, we only look for a semiconjugacy, instead of a conjugacy in [15]. This avoids many calculations and allows weaker hypotheses.

4.1 We use the notations of 3.1. The mapping $G = F^{N-1} \circ g$ is defined in a neighborhood of B'. We suppose that $G(S')$ and S are neither disjoint nor equal *i.e.* that there exists an N-bump homoclinic to the periodic orbit under interest, but that its stable and unstable manifolds do not coincide. The local transition map l is defined outside of B and satisfies the estimate of 2.3.

4.2 Under the hypotheses recalled in 4.1, the mapping $F^N = G \circ l$ has the Bernoulli shift as a topological factor. As a consequence, the mapping F has positive topological entropy, and there exist infinitely many kN-bump homoclinic orbits for all $k \geqslant 2$.

In order to be more explicit, let us consider the set $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, endowed with the following topology : A subset $U \subset \bar{\mathbb{N}}$ is open if and only if either it does not contain ∞, or it contains the subset $\{n \in \mathbb{N}, \text{ s.t. } n \geqslant N\}$ for some $N \geqslant 1$. This is the classical compactification of $\mathbb{N}$. Let us consider the set $\bar{\Lambda}$ of the sequences $s \in \bar{\mathbb{N}}^{\mathbb{Z}}$ of the form

$$\ldots, \infty, \infty, s_{-m}, \ldots, s_0, \ldots, s_n, \infty, \infty, \ldots$$

with $\infty \geqslant m \geqslant -1$, $\infty \geqslant n \geqslant 0$, and $s_i < \infty$ for all $-m \leqslant i \leqslant n$. It has to be understood that $m = -1$ and $n = 0$ in the above expression stand for the sequence $\ldots, \infty, \infty, \ldots$. The set $\bar{\Lambda}$ is a compact subset of $\bar{\mathbb{N}}^{\mathbb{Z}}$ containing

$$\Lambda = \mathbb{N}^{\mathbb{Z}}.$$

In addition, Λ is dense in $\bar{\Lambda}$, which justifies the notations. The map $\lambda : \Lambda \longrightarrow \Lambda$ is defined by $\lambda(s)_i = s_{i-1}$. Note that the continuous extension $\bar{\lambda}$ of λ to $\bar{\mathbb{N}}^{\mathbb{Z}}$ does not preserve $\bar{\Lambda}$.

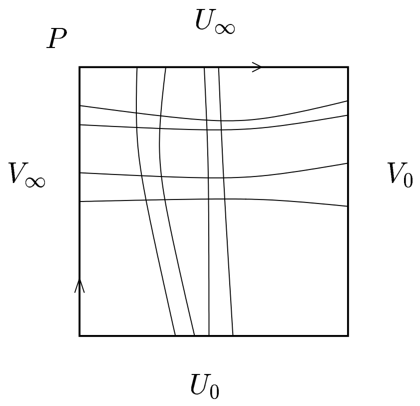

Figure 2: The square

We shall prove that there exist a compact set $\bar{X} \subset \bar{A}$, an invariant set X contained in $\bar{X}$ and dense in it, and a surjective continuous mapping $\bar{\tau} : \bar{X} \longrightarrow \bar{\Lambda}$ satisfying $\bar{\tau}(X) = \Lambda$ and such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{F^N} & X \\ \tau \downarrow & & \downarrow \tau \\ \Lambda & \xrightarrow{\lambda} & \Lambda \end{array}$$

commutes, where τ is the restriction of $\bar{\tau}$ to X. In addition, the points of $\bar{\tau}^{-1}(s)$ are kN-bump homoclinic points when s is a sequence

$$\ldots, \infty, \infty, s_{-m}, \ldots, s_0, \ldots s_n, \infty, \infty, \ldots$$

with $\infty > m \geqslant 0$, $\infty > n \geqslant 0$, $k = m + n + 2$ and $s_i < \infty$ for all $-m \leqslant i \leqslant n$. To finish this description, the preimage $\bar{\tau}^{-1}(\ldots, \infty, \infty, \ldots)$ consists of N-bump homoclinic points.

4.3 In order to prove the statements of 4.2, we shall introduce the notion of vertical and horizontal strips, following [15] for the main lines. However, as we already mentioned, we work under weaker hypotheses, and we will need more topological notions, in the spirit of works of Conley, Easton and McGehee, see for example [6]. See also [3] for related work. Let us consider the square Q as drawn in figure 2, where V_0 is the right edge, U_0 is the lower edge, V_∞ and U_∞ are the left and upper edges, and P is the vertex $V_\infty \cap U_\infty$. We shall also note Q any domain of the plane homeomorphic to this square, and define the following distinguished subsets:

Definition A vertical strip is a compact subset V of Q such that $V \cup U_0 \cup U_\infty$ is connected. A horizontal strip is a compact subset U of Q such that $U \cup V_0 \cup V_\infty$ is connected.

Lemma If V_i is a decreasing sequence of vertical strips, the intersection $\cap_i V_i$ is a vertical strip. The same holds for horizontal strips. A vertical strip and a horizontal strip have non empty intersection.

Proposition There exists a square Q in Σ_1 such that $U_\infty \subset G(S')$, $V_\infty \subset S$ (hence $P \subset G(S') \cap S$) and $\mathrm{int}(Q) \cap (G(B') \cup B) = \emptyset$, where $\mathrm{int}(Q)$ is the interior of Q. In this square Q, there exists a sequence U_i, $i \in \mathbb{N}$ of disjoint horizontal strips, and a sequence V_i, $i \in \mathbb{N}$ of disjoint vertical strips such that

$$F^N(V_i) = U_i.$$

The strip U_{i+1} is above U_i in Q and U_i is converging to U_∞ for the Hausdorff metric. Seemingly, the strips V_i are ordered from the right to the left and converge to V_∞. In addition, we have the following property:
If V is a vertical strip, then each of the sets $F^{-N}(V) \cap V_i$ contains a vertical strip. If U is a horizontal strip, then each $F^N(U) \cap U_i$ contains a horizontal strip.

4.4 The structure described in 4.3 implies the existence of a Bernoulli shift as defined in 4.2. We shall prove this fact now, and delay the proof of Proposition 4.3 up to 4.5. We closely follow the presentation of [15], which may be consulted for more details. Let us consider a sequence $s_i \in \bar{\Lambda}$, and define the sets

$$V_{s_0 s_{-1} \cdots s_{-n}} = \bigcap_{i=0}^{j} F^{-iN}(V_{s_{-i}}),$$

where $j = n$ if $s_{-n} < \infty$, and $j = \min\{k \leqslant n, \text{ s.t. } s_{-k} = \infty\}$ otherwise. These sets are vertical strips, as can be proved by induction using Proposition 4.3 and noticing that

$$V_{s_0 s_{-1} \cdots s_{-n}} = V_{s_0 s_{-1} \cdots s_{-j}} = V_{s_0} \cap F^{-N}(V_{s_{-1} \cdots s_{-j}}).$$

In the same way, we define the horizontal strips

$$U_{s_1 \cdots s_n} = \bigcap_{i=1}^{j} F^{iN}(U_{s_i}),$$

where $j = n$ if $s_n < \infty$, and $j = \min\{k \leqslant n, \text{ s.t. } s_k = \infty\}$ otherwise. It follows from Lemma 4.3 that

$$V(s) = \bigcap_{n=0}^{\infty} V_{s_0 s_{-1} \cdots s_{-n}}$$

is a vertical strip, and that

$$U(s) = \bigcap_{n=1}^{\infty} U_{s_1 \cdots s_n}$$

is a horizontal strip. The set $V(s) \cap U(s)$ is thus a non empty compact set. If $s \in \Lambda$, we have

$$V(s) \cap U(s) = \{p \in Q \text{ s.t. } F^{-iN}(p) \in V_{s_i}\}.$$

We can now define the invariant set

$$X = \bigcup_{s \in \Lambda} V(s) \cap U(s),$$

the compact set

$$\bar{X} = \bigcup_{s \in \bar{\Lambda}} V(s) \cap U(s)$$

and the mapping $\bar{\tau}$ which, to each point of $V(s) \cap U(s)$, associates the sequence $s \in \bar{\Lambda}$. This mapping is well defined since the sets $V(s) \cap U(s)$ and $V(s') \cap U(s')$ are obviously disjoint for different sequences s and s'. It is straightforward with these definitions to check the statements of 4.2.

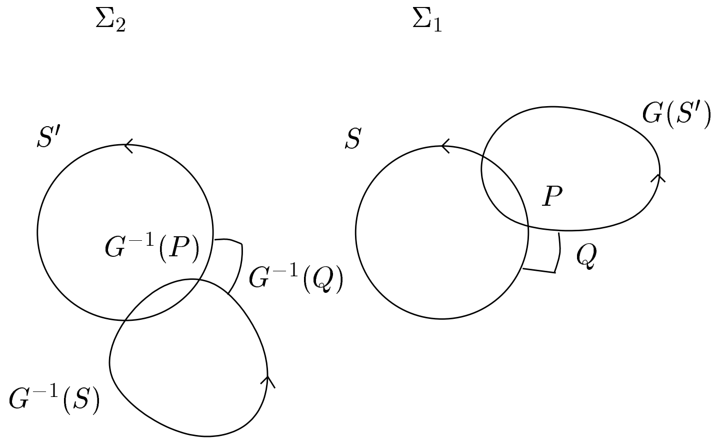

Figure 3: Construction of Q

4.5 In order to prove Proposition 4.3, we shall first build the square Q. Let us choose a point P of $G(S') \cap S$. There are two cases.

i. The curves $G(S')$ and S are outer tangent, *i.e.* $G(B') \cap B \subset S$ and we can take any point $P \in G(S') \cap S$.

ii. The curves $G(S')$ and S are crossing each other. In this case, we choose P such that the curves $G(S')$ and S locally cross each other at P.

In both cases, P is isolated in $G(S') \cap S$ since both curves are analytic. Let us consider the action-angle coordinates (I_2, θ) on Σ_2, defined by the relations

$$p_2 = \sqrt{2I_2}\cos\theta\,, \quad q_2 = \sqrt{2I_2}\sin\theta.$$

There exists a positive integer a, a positive real number δ and an analytic function $h : [I_c, I_c+\delta] \longrightarrow \mathbb{R}$ such that the curve $(I_2, h((I_2-I_c)^{1/a}))$, $I_2 \in [I_c, I_c+\delta]$ is contained in $G^{-1}(S) \cap A$. Recall that the circle S' has the equation $I_2 = I_c$. In the case where P is a point of transverse intersection, we can take $a = 1$. It is possible to choose P, δ and h in such a way that the open set $\{I_c < I_2 < I_c+\delta, h((I_2-I_c)^{1/a}) < \theta < h((I_2-I_c)^{1/a}) + \delta\}$ is disjoint from $G^{-1}(S)$. We then set

$$Q = G\big(\{I_c \leqslant I_2 \leqslant I_c+\delta, h((I_2-I_c)^{1/a}) \leqslant \theta \leqslant h((I_2-I_c)^{1/a}) + \delta\}\big).$$

We orient the curves S, S', $G(S')$ and $G^{-1}(S)$ positively, and give U_0 and U_∞ the induced orientation. In order to prove that Proposition 4.3 holds with this square Q, it is enough to prove the following proposition.

4.6 Proposition Any sufficiently small neighborhood of U_∞ in Q contains a horizontal strip U which is the image by F^N of a vertical strip V of Q, and satisfies the following property : If $\tilde{V}$ is a vertical strip of Q, then $F^{-N}(\tilde{V} \cap U) \subset V$ contains a vertical strip of Q, and if $\tilde{U}$ is a horizontal strip of Q, then $F^N(\tilde{U} \cap V) \subset U$ contains a horizontal strip.

Proof We need a Lemma.

Lemma Let $c : [0,1] \to \bar{A}$, be an analytic curve such that $c(]0,1]) \subset A$ and $c(0) \in S$. Then for ϵ small enough, the curve $F^N \circ c :]0,\epsilon] \to A$ is an analytic spiral that accumulates

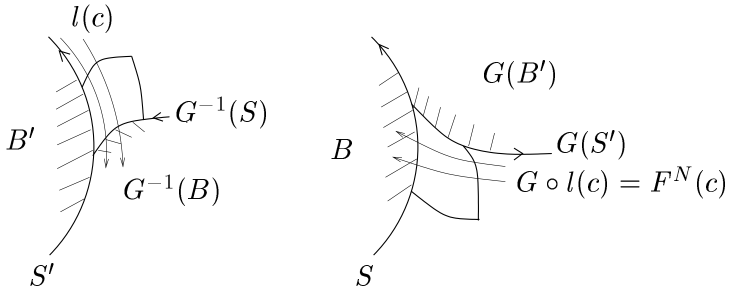

Figure 4: Spirals

on $G(S')$ and that crosses Q infinitely many times. Moreover, every connected component of $F^N \circ c(]0,\epsilon]) \cap Q$ crosses Q from V_0 to V_∞ (the orientation of $F^N \circ c$ is that defined by the parameterization).

To prove this lemma we first write $F^N \circ c$ as $G \circ l \circ c$. Then using estimate 2.3 and recalling that l is explicitly given by (see 2.2)

$$l_{E,\mu}(y) = l(y, E, \mu) = R\big(\theta(I_2(y), E, \mu)\big)\, y$$

we conclude that $l \circ c$ is an infinite spiral turning monotonically around S and accumulating on S. In addition, easy explicit estimates show that, when ϵ is small enough, each connected component of $l(c(]0,\epsilon])) \cap G^{-1}(Q)$ is crossing $G^{-1}(Q)$ from $G^{-1}(V_0)$ to $G^{-1}(V_\infty)$. The lemma follows from the fact that G is a local diffeomorphism in a neighborhood of P (see figure 4).

This lemma implies that the set $F^N(Q) \cap Q$ has infinitely many connected components which are horizontal strips accumulating on U_∞. Each of these strips is bounded by two horizontal arcs, a lower and an upper one, which are contained in $F^N(U_\infty)$ and $F^N(U_0)$, respectively, and two small sub-arcs of V_0 and V_∞. Let U be one of these horizontal strips, sufficiently close to U_∞. Using that: $F^{-N}(U)$ is connected, $F^{-N}(U) \cap U_\infty \neq \emptyset$ and $F^{-N}(U) \cap U_0 \neq \emptyset$, we conclude that $F^{-N}(U) = V$ is a vertical strip. In addition, we see that the vertical strip V is a topological square bounded on one side by a connected component of $F^{-N}(V_0) \cap Q$ crossing Q, and on the other side by a connected component of $F^{-N}(V_\infty) \cap Q$. To finish the proof of the proposition we need another lemma

LEMMA Let us consider a compact curve γ in Q connecting U_0 and U_∞. There exist connected components of $\gamma \cap U$ intersecting both $F^N(U_0)$ and $F^N(U_\infty)$.

To prove this fact, let us orient γ from U_∞ to U_0, and consider the last point of intersection of γ with $F^N(U_0) \cap U$. Just after this last intersection, γ lies inside U, hence has to leave U through $F^N(U_\infty)$. This proves the lemma □

Let $\tilde{V}$ be a vertical strip. It intersects U, by lemma 4.3. We are going to prove that $F^{-N}(\tilde{V} \cap U)$ is a vertical strip. Assume that this is not true. In this case, the compact set $\tilde{V} \cap U$ is disconnected, and is the union of two disjoint compact sets K_1 and K_2, where K_1 is the union of the connected components of $\tilde{V} \cap U$ which intersect $F^N(U_0)$, and K_2 the union of those which intersect $F^N(U_\infty)$. We can find two disjoint open sets of Q, Ω_1 and Ω_2, containing respectively K_1 and K_2. In addition, since $F^N(U_0) \cap U$ and $F^N(U_\infty) \cap U$ are compact, we can choose Ω_1 and Ω_2 such that Ω_1 does not intersect $F^N(U_\infty) \cap U$ and Ω_2 does not intersect $F^N(U_0) \cap U$. The sets $U - (\Omega_1 \cup \Omega_2)$ and $\tilde{V}$ are compact and disjoint. It follows that one can find a connected open neighborhood Ω of $\tilde{V}$ such that $\Omega \cap U \subset \Omega_1 \cup \Omega_2$. The open set Ω contains a curve γ connecting U_0 and U_∞. Each connected component of $\gamma \cap U$ is contained either in Ω_1 or in Ω_2, which is in contradiction with the conclusion of the lemma. The intersection between V and horizontal strips can be studied exactly in the same way. □

5 Chaos near the energy shell of the fixed point

5.1 In this section, we fix a value of the parameter μ and work with a fixed Hamiltonian H. We suppose that the conditions of existence of invariant curves (see Proposition 2.5) is satisfied, hence there exists a critical energy $\eta > 0$ such that, for all $E \in]0, \eta[$, there exists a homoclinic orbit to the periodic orbit P_E of energy E contained in the center manifold. We also suppose that the stable and unstable manifolds of the fixed point do not coincide.

5.2 Theorem Under the hypotheses recalled above, there exists a sequence $E_n \longrightarrow 0$ of positive numbers such that, for each n, the stable manifold of P_{E_n} and its unstable manifold do not coincide.

5.3 In order to prove this theorem, let us define the function $N(E)$ which, to each value of energy $E \in]0, \eta[$, associates the minimal number of bumps of an orbit homoclinic to P_E

$$N(E) = \min\{n \in \mathbb{N} \text{ s.t. } F_E^{n-1} \circ g_E(S_E) \cap S_E \neq \emptyset\},$$

which is finite in view of Theorem 1.3. See 3.1 for the definition of S_E.

Lemma The function $]0, \eta[\ni E \longmapsto N(E)$ is lower semi-continuous and continuous at each point E_0 such that $F_{E_0}^{N(E_0)-1} \circ g_{E_0}(S_{E_0}) \cap S_{E_0} \neq \emptyset$. In addition, $\lim_{E \longrightarrow 0} N(E) = \infty$.

This lemma implies the desired result. Assume by contradiction that the stable and unstable manifolds of P_E coincide for all energies E in an interval $]0, \epsilon[$. By the lemma the function N would be continuous, hence constant on this interval, and N would have a finite limit in 0, which is in contradiction with the last part of the lemma. There remains to prove the lemma:

Proof of the lemma Let us fix a value E_0 of the energy, and consider a sequence $E_n \longrightarrow E_0$ such that $N(E_n) = N$ is constant. We have

$$F_{E_n}^{N-1} \circ g_{E_n}(S_{E_n}) \cap S_{E_n} \neq \emptyset.$$

for each n. This clearly implies that

$$F_{E_0}^{N-1} \circ g_{E_0}(S_{E_0}) \cap S_{E_0} \neq \emptyset.$$

hence $N(E_0) \leqslant N$. This proves lower semi-continuity of N. If the stable and unstable manifolds of P_{E_0} coincide, there holds

$$F_{E_0}^{N(E_0)-1} \circ g_{E_0}(S_{E_0}) = S_{E_0}.$$

It is then clear, by area preservation, that

$$F_E^{N(E_0)-1} \circ g_E(S_E) \cap S_E \neq \emptyset$$

for E sufficiently close to E_0, hence $N(E) \leqslant N(E_0)$. As a consequence, E_0 is a point of upper semi-continuity of N, hence a point of continuity. To end the proof, we note that if there existed a sequence $E_n \longrightarrow 0$ with $N(E_n)$ bounded, there would exist a homoclinic orbit to the fixed point. This can be checked by a compactness argument similar to the proof of lower semi-continuity above. □

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