SYMPLECTIC ASPECTS OF MATHER THEORY

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Abstract
We prove that the Aubry and Mañé sets introduced by Mather in Lagrangian dynamics are symplectic invariants. In order to do so, we introduce a barrier on phase space. This is also an occasion to suggest an Aubry-Mather theory for nonconvex Hamiltonians.

Résumé
On montre que les ensembles d'Aubry et de Mañé introduits par Mather en dynamique Lagrangienne sont des invariants symplectiques. On introduit pour ceci une barriere dans l'espace des phases. Ceci est aussi l'occasion d'ébaucher une théorie d'Aubry-Mather pour des Hamiltoniens non convexes.

In Lagrangian dynamics, John Mather has defined several invariant sets, now called the Mather set, the Aubry set, and the Mañé set. These invariant sets provide obstructions to the existence of orbits wandering in phase space. Conversely, the existence of interesting orbits has been proved under some assumptions on the topology of these sets. Such results were first obtained by John Mather in [12] and then in several papers (see [1], [3], [4], [5], [17], [18] as well as recent unpublished works of John Mather).

In order to apply these results to examples, one has to understand the topology of the Aubry and Mañé sets, which is a very difficult task. In many perturbative situations, averaging methods appear as a promising tool in that direction. In order to use these methods, one has to understand how the averaging transformations modify the Aubry-Mather sets. In the present article, we answer this question and prove that the Mather set, the Aubry set, and the Mañé set are symplectic invariants.

In order to do so, we define a barrier on phase space, which is some symplectic analogue of the function called the Peierls's barrier by Mather in [12]. We then propose definitions of Aubry and Mañé sets for general Hamiltonian systems. We hope that these definitions may also serve as the starting point of an Aubry-Mather theory for some classes of nonconvex Hamiltonians. We develop the first steps of such a theory.

Several earlier works gave hints towards the symplectic nature of Aubry-Mather theory (see [2], [14], [15], [16], [11], for example). These works prove the symplectic invariance of the α function of Mather, and one may consider that the symplectic

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invariance of the Aubry set is not a surprising result after them. However, it was not clear to us beforehand that the Mañé set is a symplectic invariant, although a similar result has been proved by Mather in dimension 2 (see [11, Proposition 3.1]). It is possible that the geometric methods introduced in [14] may also be used to obtain symplectic definitions of the Aubry and Mañé sets.

1. Mather theory in Lagrangian dynamics

We recall the basics of Mather theory and state our main result, Theorem 1. The original references for most of the material presented in this section are Mather's papers [10] and [12]. The central object is the Peierls's barrier, introduced by Mather in [12]. Our presentation is also influenced by the work of Fathi [9].

1.1

In this section, we consider a C^2 Hamiltonian function $H : T^*M \times \mathbb{T} \longrightarrow \mathbb{R}$, where M is a compact connected manifold without boundary and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We denote by $P = (q, p)$ the points of T^*M. The cotangent bundle is endowed with its canonical one-form $\eta = pdq$ and with its canonical symplectic form $\omega = -d\eta$. Following a very standard device, we reduce our nonautonomous Hamiltonian function H to an autonomous one by considering the extended phase space $T^*(M \times \mathbb{T}) = T^*M \times T^*\mathbb{T}$. We denote by (P, t, E), $P \in T^*M$, $(t, E) \in T^*\mathbb{T}$ the points of this space. We consider the canonical one-form $\lambda = pdq + Edt$ and the associated symplectic form $\Omega = -d\lambda$. We define the new Hamiltonian $G : T^*(M \times \mathbb{T}) \longrightarrow \mathbb{R}$ to be the expression

$$G(P, t, E) = E + H(P, t).$$

We denote by $V_G(P, t, E)$ the Hamiltonian vector field of G, which is defined by the relation

$$\Omega_{(P,t,E)}(V_G, \cdot) = dG_{(P,t,E)}.$$

We fix once and for all a Riemannian metric on M and use it to define norms of tangent vectors and tangent covectors of M. We denote this norm indifferently by $|P|$ or by $|p|$ when $P = (q, p) \in T_q^*M$. We denote by π the canonical projections $T^*M \longrightarrow M$ or $T^*(M \times \mathbb{T}) \longrightarrow M \times \mathbb{T}$. The theory of Mather relies on the following standard set of hypotheses.

(1) *Completeness.* The Hamiltonian vector field V_G on $T^*(M \times \mathbb{T})$ generates a complete flow, denoted by Φ_t. The flow Φ_t preserves the level sets of G.

(2) *Convexity.* For each $(q, t) \in M \times \mathbb{T}$, the function $p \longmapsto H(q, p, t)$ is convex on T_q^*M, with positive definite Hessian. In short, $\partial_p^2 H > 0$.

(3) *Superlinearity.* For each $(q, t) \in M \times \mathbb{T}$, the function $p \longmapsto H(q, p, t)$ is superlinear, which means that $\lim_{|p| \longrightarrow \infty} H(t, x, p)/|p| = \infty$.

1.2

We associate to the Hamiltonian H a Lagrangian function $L : TM \times \mathbb{T} \longrightarrow \mathbb{R}$ defined by

$$L(t, q, v) = \sup_{p \in T_q^* M} p(v) - H(t, q, p).$$

The Lagrangian satisfies the following.

(1) *Convexity.* For each $(q, t) \in M \times \mathbb{T}$, the function $v \longmapsto L(q, v, t)$ is a convex function on $T_q M$, with positive definite Hessian. In short, $\partial_v^2 L > 0$.

(2) *Superlinearity.* For each $(q, t) \in M \times \mathbb{T}$, the function $v \longmapsto L(q, v, t)$ is superlinear on $T_q M$.

Let $X(t) = (P(t), s + t, E(t))$ be a Hamiltonian orbit of G, and let $q(t) = \pi(P(t))$. Then we have the identities

$$\lambda_{X(t)}\big(\dot{X}(t)\big) - G\big(X(t)\big) = \eta_{P(t)}\big(\dot{P}(t)\big) - H\big(P(t), s + t\big) = L\big(q(t), \dot{q}(t), s + t\big).$$

1.3

Following John Mather, we define the function $F : M \times \mathbb{T} \times M \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$F(q_0, t; q_1, s) = \min_{\gamma} \int_0^s L\big(\gamma(\sigma), \dot{\gamma}(\sigma), t + \sigma\big)\, d\sigma,$$

where the minimum is taken on the set of absolutely continuous curves $\gamma : [0, s] \longrightarrow M$, which satisfy $\gamma(0) = q_0$ and $\gamma(1) = q_1$. We also define the Peierls's barrier $h : M \times \mathbb{T} \times M \times \mathbb{T} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ by

$$h(q_0, t_0; q_1, t_1) := \liminf_{n \in \mathbb{N}} F(q_0, t_0; q_1, s_1 + n),$$

where $t_0 + s_1 \bmod 1 = t_1$. This barrier is the central object in Mather's study of globally minimizing orbits.

1.4

Let us set $m(H) = \inf_{(q,t) \in M \times \mathbb{T}} h(q, t; q, t)$. It follows from [10] (see also [12] and [13]), that $m(H) \in \{-\infty, 0, +\infty\}$. In addition, for each Hamiltonian H satisfying the hypotheses in Section 1.1, there exists one and only one real number $\alpha(H)$ such that $m(H - \alpha(H)) = 0$. As a consequence, there is no loss of generality in assuming that $m(H) = 0$ or, equivalently, that $\alpha(H) = 0$. We make this assumption from now on in this section. Let us mention the terminology of Mañé, who called the Hamiltonians H satisfying $m(H) = +\infty$ supercritical, the Hamiltonians satisfying $m(H) = -\infty$ subcritical, and the Hamiltonians satisfying $m(H) = 0$ critical.

1.5

If $m(H) = 0$, then the function h is a real-valued Lipschitz function on $M \times \mathbb{T} \times M \times \mathbb{T}$, which satisfies the triangle inequality

$$h(q_0, t_0; q_2, t_2) \leqslant h(q_0, t_0; q_1, t_1) + h(q_1, t_1; q_2, t_2)$$

for all (q_0, t_0), (q_1, t_1) and (q_2, t_2) in $M \times \mathbb{T}$. In addition, for each $(q, t) \in M \times \mathbb{T}$, the function $h(q, t; \cdot, \cdot)$ is a weak Kolmogorov-Arnold-Moser (KAM) solution in the sense of Fathi, which means that, for $\tau \geqslant \theta$ in $\mathbb{R}$ and $x \in M$, we have

$$h(q, t; x, \tau \bmod 1) = \min \Big(h\big(q, t; q(\theta), \theta \bmod 1\big) + \int_\theta^\tau L\big(q(s), \dot{q}(s), s\big)\, ds \Big),$$

where the minimum is taken on the set of absolutely continuous curves $q(s) : [\theta, \tau] \longrightarrow M$ such that $q(\tau) = x$. Similarly, we have, for $\tau \geqslant \theta$ in $\mathbb{R}$ and $x \in M$,

$$h(x, \theta \bmod 1; q, t) = \min \Big(h\big(q(\tau), \tau \bmod 1; q, t\big) + \int_\theta^\tau L\big(q(s), \dot{q}(s), s\big)\, ds \Big),$$

where the minimum is taken on the set of absolutely continuous curves $q(s) : [\theta, \tau] \longrightarrow M$ such that $q(\theta) = x$.

1.6

The projected Aubry set $\mathscr{A}(H)$ is the set of points $(q, t) \in M \times \mathbb{T}$ such that $h(q, t; q, t) = 0$. Fathi proved that, for each point $(q, t) \in \mathscr{A}(H)$, the function $h(q, t; \cdot, \cdot)$ is differentiable at (q, t). Let us denote by $X(q, t)$ the differential $\partial_3 h(q, t; q, t) \in T_q^* M$ of the function $h(q, t; \cdot, t)$ at point q. The Aubry set $\tilde{\mathscr{A}}(H)$ is defined as

$$\tilde{\mathscr{A}}(H) = \big\{ \big(X(q, t), t, -H(X(q, t), t) \big); (q, t) \in \mathscr{A}(H) \big\} \subset T^*(M \times \mathbb{T}).$$

The Aubry set is compact and Φ-invariant, and it is a Lipschitz graph over the projected Aubry set $\mathscr{A}(H)$. These are Mather's results (see [12]). In our presentation, which follows Fathi, this amounts to say that the function $(q, t) \longmapsto X(q, t)$ is Lipschitz on $\mathscr{A}(H)$.

1.7

The Mather set $\tilde{\mathscr{M}}(H)$ is defined as the union of the supports of all Φ-invariant probability measures on $T^*(M \times \mathbb{T})$ concentrated on $\tilde{\mathscr{A}}(H)$. This set was first defined by Mather, but our definition is due to Mañé.

1.8

The projected Mañé set $\mathcal{N}(H)$ is the set of points $(q,t)\in M\times\mathbb{T}$ such that there exist points (q_0,t_0) and (q_1,t_1) in $\mathcal{A}(H)$, satisfying

$$h(q_0,t_0;q_1,t_1)=h(q_0,t_0;q,t)+h(q,t;q_1,t_1).$$

Let us denote by $\mathcal{I}(q_0,t_0;q_1,t_1)$ the set of points $(q,t)\in M\times\mathbb{T}$, which satisfy this relation. If $(q_0,t_0)\in\mathcal{A}(H)$ and $(q_1,t_1)\in\mathcal{A}(H)$ are given and if $(q,t)\in\mathcal{I}(q_0,t_0;q_1,t_1)$, then the function $h(q_0,t_0;\cdot,t)$ is differentiable at q, as well as the function $h(\cdot,t;q_1,t_1)$, and $\partial_3 h(q_0,t_0,q,t)+\partial_1 h(q,t;q_1,t_1)=0$. This is proved in [3] following ideas of Fathi. We define

$$\begin{aligned}&\tilde{\mathcal{I}}(q_0,t_0;q_1,t_1)\\ &:=\big\{\big(\partial_3 h(q_0,t_0,q,t),t,-H(\partial_3 h(q_0,t_0,q,t),t)\big),(q,t)\in\mathcal{I}(q_0,t_0;q_1,t_1)\big\}.\end{aligned}$$

The set $\tilde{\mathcal{I}}(q_0,t_0;q_1,t_1)$ is a compact Φ-invariant subset of $T^*(M\times\mathbb{T})$, and it is a Lipschitz graph. The Mañé set $\tilde{\mathcal{N}}(H)$ is the set

$$\tilde{\mathcal{N}}(H)=\bigcup_{(q_0,t_0),(q_1,t_1)\in\mathcal{A}(H)}\tilde{\mathcal{I}}(q_0,t_0;q_1,t_1)\subset T^*(M\times\mathbb{T}).$$

The Mañé set was first introduced by Mather in [12]; it is compact and Φ-invariant, and it contains the Aubry set. In other words, we have the inclusions

$$\tilde{\mathcal{M}}(H)\subset\tilde{\mathcal{A}}(H)\subset\tilde{\mathcal{N}}(H).$$

The Mañé set is usually not a graph. However, it satisfies

$$\tilde{\mathcal{N}}(H)\cap\pi^{-1}\big(\mathcal{A}(H)\big)=\tilde{\mathcal{A}}(H).$$

This follows from the fact, proved by Fathi, that, for each $(x,\theta)\in M\times\mathbb{T}$ and each $(q,t)\in\mathcal{A}(H)$, the function $h(x,\theta;\cdot,t)$ is differentiable at q and satisfies $\partial_3 h(x,\theta;q,t)=X(q,t)$.

1.9

Mather introduced the function $d(q,t;q',t')=h(q,t;q',t')+h(q',t';q,t)$ on $M\times\mathbb{T}$. When restricted to $\mathcal{A}(H)\times\mathcal{A}(H)$, it is a pseudometric. This means that this function is symmetric and nonnegative, satisfies the triangle inequality, and $d(q,t;q,t)=0$ for $(q,t)\in\mathcal{A}(H)$. We also denote by d the pseudometric $d(P,t,-H(P,t);P',t',-H(P',t'))=d(\pi(P),t;\pi(P'),t')$ on $\tilde{A}(H)$. The relation $d(P,t,E;P',t',E')=0$ is an equivalence relation on $\tilde{\mathcal{A}}(H)$. The classes of equivalence are called the static classes. Let us denote by $\dot{\mathcal{A}}(H)$ the set of static classes. The pseudometric d gives rise to a metric $\dot{d}$ on $\dot{\mathcal{A}}(H)$. The compact metric space $(\dot{\mathcal{A}}(H),\dot{d})$ is called the quotient Aubry set. It was introduced by Mather.

1.10
The diffeomorphism $\Psi : T^*(M \times \mathbb{T}) \longrightarrow T^*(M \times \mathbb{T})$ is called exact if the form $\Psi^*\lambda - \lambda$ is exact.

THEOREM
Let H be a Hamiltonian satisfying the hypotheses in Section 1.1, and let $\Psi : T^*(M \times \mathbb{T}) \longrightarrow T^*(M \times \mathbb{T})$ be an exact diffeomorphism such that the Hamiltonian

$$\Psi^* H := G \circ \Psi(P, t, E) - E$$

is independent of E and satisfies the hypotheses in Section 1.1 when considered as a function on $T^*M \times \mathbb{T}$. Then $m(\Psi^* H) = m(H)$, and hence, $\alpha(H) = \alpha(\Psi^* H)$. If $m(H) = 0$, then we have

$$\Psi\big(\tilde{\mathscr{M}}(\Psi^* H)\big) = \tilde{\mathscr{M}}(H), \qquad \Psi\big(\tilde{\mathscr{A}}(\Psi^* H)\big) = \tilde{\mathscr{A}}(H), \qquad \Psi\big(\tilde{\mathscr{N}}(\Psi^* H)\big) = \tilde{\mathscr{N}}(H).$$

In addition, Ψ sends the static classes of $\Psi^* H$ onto the static classes of H, and the induced mapping

$$\dot{\Psi} : \dot{\mathscr{A}}(\Psi^* H) \longrightarrow \dot{\mathscr{A}}(H)$$

is an isometry for the quotient metrics.

1.11
We prove this result in the following. In Section 2, we set the basis of a symplectic Aubry-Mather theory for general Hamiltonian systems. We prove that the analogue of Theorem 1.10 holds in this general setting. We also continue the theory a bit further than would be necessary to prove Theorem 1.10. In Section 3, we prove that, under the hypotheses of Theorem 1.10, the symplectic Aubry-Mather sets coincide with the standard Aubry-Mather sets, which ends the proof of Theorem 1.10.

2. A barrier in phase space

We propose general definitions for a Mather theory of Hamiltonian systems. Of course, the definitions given here provide relevant objects only for some specific Hamiltonian systems. It would certainly be interesting to give natural conditions on H implying the nontriviality of the theory developed in this section. We only check, in Section 3, that our definitions coincide with the standard ones in the convex case, obtaining nontriviality in this special case. Let us mention once again that it might be possible and interesting to find more geometric definitions using the methods of [14].

2.1
In Section 2, we work in a very general setting. We consider a manifold N, not necessarily compact, and an autonomous Hamiltonian function $G : T^*N \longrightarrow \mathbb{R}$. We

assume that G generates a complete Hamiltonian flow Φ_t. We make no convexity assumption. We denote by λ the canonical one-form of T^*N, and we denote by $V_G(P)$ the Hamiltonian vector field of G. Let $D(P, P')$ be a distance on T^*N induced from a Riemannian metric. We identify N with the zero section of T^*N, so that D is also a distance on N. We assume that $D(\pi(X), \pi(X')) \leqslant D(X, X')$ for X and X' in T^*N.

2.2

Let X_0 and X_1 be two points of T^*N. A preorbit between X_0 and X_1 is the data of a sequence $\underline{Y} = (Y_n)$ of curves $Y_n(s) : [0, T_n] \longrightarrow T^*N$ such that we have the following.

(1) For each n, the curve Y_n has a finite number N_n of discontinuity points $T_n^i \in]0, T_n[$, $1 \leqslant i \leqslant N_n$, such that $T_n^{i+1} > T_n^i$. We also often use the notations $T_n^0 = 0$ and $T_n^{N_n+1} = T_n$.

(2) The curve Y_n satisfies $Y_n(T_n^i + s) = \Phi_s(Y_n(T_n^i))$ for each $s \in [0, T_n^{i+1} - T_n^i[$. We denote by $Y_n(T_n^i-)$ the point $\Phi_{T_n^i - T_n^{i-1}}(Y(T_n^{i-1}))$ and impose that $Y_n(T_n) = Y_n(T_n-)$.

(3) We have $T_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

(4) We have $Y_n(0) \longrightarrow X_0$ and $Y_n(T_n) \longrightarrow X_1$. In addition, we have $\lim_{n\longrightarrow\infty} \Delta(Y_n) = 0$, where we denote by $\Delta(Y_n)$ the sum $\sum_{i=1}^{N_n} D(Y_n(T_n^i-), Y_n(T_n^i))$.

(5) There exists a compact subset $K \subset T^*N$ which contains the images of all the curves Y_n.

The preorbits do not depend on the metric, which has been used to define the distance D. In a standard way, we call action of the curve $Y_n(t)$ the value

$$A(Y_n) = \int_0^{T_n} \lambda_{Y_n(t)}\big(\dot{Y}_n(t)\big) - G\big(Y_n(t)\big)\, dt.$$

The action of the preorbit $\underline{Y}$ is

$$A(\underline{Y}) := \liminf_{n\longrightarrow\infty} A(Y_n).$$

2.3

LEMMA

If there exists a preorbit between X_0 and X_1, then $G(X_0) = G(X_1)$.

Proof

This follows easily from the fact that the Hamiltonian flow Φ preserves the Hamiltonian function G. □

2.4

We define the barrier $\tilde{h} : T^*N \times T^*N \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ by the expression

$$\tilde{h}(X_0, X_1) = \inf_{\underline{Y}} A(\underline{Y}),$$

where the infimum is taken on the set of preorbits between X_0 and X_1. As usual, we set $\tilde{h}(X_0, X_1) = +\infty$ if there does not exist any preorbit between X_0 and X_1. If $\tilde{h}(X_0, X_1) < +\infty$, then the forward orbit of X_0 and the backward orbit of X_1 are bounded. As a consequence, if $\tilde{h}(X, X) < +\infty$, then the orbit of X is bounded.

2.5

PROPERTY

For each $t > 0$, we have the equality

$$\tilde{h}(X_0, X_1) = \tilde{h}\big(\Phi_t(X_0), X_1\big) + \int_0^t \lambda_{\Phi_s(X_0)}\big(V_G(\Phi_s(X_0))\big) - G\big(\Phi_s(X_0)\big)\, ds$$

and

$$\tilde{h}\big(X_0, \Phi_t(X_1)\big) = \tilde{h}(X_0, X_1) + \int_0^t \lambda_{\Phi_s(X_1)}\big(V_G(\Phi_s(X_1))\big) - G\big(\Phi_s(X_1)\big)\, ds.$$

Proof

We prove the first equality; the proof of the second one is similar. To each preorbit $\underline{Y}$ between X_0 and X_1, we associate the preorbit $\underline{Z}$ between $\Phi_t(X_0)$ and X_1 defined by $Z_n(s) : [0, T_n - t] \ni s \longmapsto Y_n(s + t)$. We have

$$A(\underline{Y}) = A(\underline{Z}) + \int_0^t \lambda_{\Phi_s(X_0)}\big(V_G(\Phi_s(X_0))\big) - G\big(\Phi_s(X_0)\big)\, ds.$$

This implies that

$$\tilde{h}\big(\Phi_t(X_0), X_1\big) \leqslant \tilde{h}(X_0, X_1) - \int_0^t \lambda_{\Phi_s(X_0)}\big(V_G(\Phi_s(X_0))\big) - G\big(\Phi_s(X_0)\big)\, ds.$$

In a similar way, we associate to each preorbit $\underline{Z} = Z_n(s) : [0, T_n] \longrightarrow T^*N$ between $\Phi_t(X_0)$ and X_1 the preorbits $\underline{Y} : [0, T_n + t] \longrightarrow T^*N$ between X_0 and X_1 defined by $Y_n(s) = \Phi_{s-t}(Z_n(0))$ for $s \in [0, t]$ and $Y_n(s) = Z_n(s - t)$ for $s \in [t, T_n + t]$. We have

$$A(\underline{Y}) = A(\underline{Z}) + \int_0^t \lambda_{\Phi_s(X_0)}\big(V_G(\Phi_s(X_0))\big) - G\big(\Phi_s(X_0)\big)\, ds.$$

This implies that

$$\tilde{h}(X_0, X_1) \leqslant \tilde{h}\big(\Phi_t(X_0), X_1\big) + \int_0^t \lambda_{\Phi_s(X_0)}\big(V_G(\Phi_s(X_0))\big) - G\big(\Phi_s(X_0)\big)\, ds. \qquad \square$$

2.6

PROPERTY

The function $\tilde{h}$ satisfies the triangle inequality. More precisely, the relation

$$\tilde{h}(X_1, X_3) \leqslant \tilde{h}(X_1, X_2) + \tilde{h}(X_2, X_3)$$

holds for each point X_1, X_2, and X_3 such that the right-hand side has a meaning.

Proof

If one of the values $\tilde{h}(X_1, X_2)$ or $\tilde{h}(X_2, X_3)$ is $+\infty$, then there is nothing to prove. If they are both different from $+\infty$, then, for each $\epsilon > 0$, there exists a preorbit $\underline{Y} = Y_n : [0, T_n] \longrightarrow T^*N$ between X_1 and X_2 such that $A(\underline{Y}) \leqslant \tilde{h}(X_1, X_2) + \epsilon$ (resp., $A(\underline{Y}) \leqslant -1/\epsilon$ in the case where $\tilde{h}(X_1, X_2) = -\infty$) and a preorbit $\underline{Y}' = Y_n' :$ $[0, S_n] \longrightarrow T^*N$ between X_2 and X_3 such that $A(\underline{Y}') \leqslant \tilde{h}(X_2, X_3)+\epsilon$ (resp., $A(\underline{Y}') \leqslant$ $-1/\epsilon$ in the case where $\tilde{h}(X_1, X_2) = -\infty$). Let us consider the sequence of curves $Z_n(t) : [0, T_n + S_n] \longrightarrow T^*N$ such that $Z_n = X_n$ on $[0, T_n[$ and $Z_n(t + T_n) = Y_n(t)$ for $t \in [0, S_n]$. It is clear that the sequence $\underline{Z} = Z_n$ is a preorbit between X_1 and X_3 and that its action satisfies

$$A(\underline{Z}) = A(\underline{X}) + A(\underline{Y}) \leqslant \tilde{h}(X_1, X_2) + \tilde{h}(X_2, X_3) + 2\epsilon.$$

As a consequence, for all $\epsilon > 0$, we have $\tilde{h}(X_1, X_3) \leqslant \tilde{h}(X_1, X_2) + \tilde{h}(X_2, X_3) + 2\epsilon$, and hence, the triangle inequality holds. □

2.7

PROPERTY

Let $\Psi : T^*N \longrightarrow T^*N$ be an exact diffeomorphism. We have the equality

$$\tilde{h}_{G\circ\Psi}(X_0, X_1) = \tilde{h}_G\big(\Psi(X_0), \Psi(X_1)\big) + S(X_0) - S(X_1),$$

where $S : T^*N \longrightarrow \mathbb{R}$ is a function such that $\Psi^*\lambda - \lambda = dS$.

Proof

Observe first that $\underline{Y} = Y_n$ is a preorbit for the Hamiltonian $G \circ \Psi$ between points X_0 and X_1 if and only if $\Psi(\underline{Y}) = \Psi(Y_n)$ is a preorbit for the Hamiltonian G between $\Psi(X_0)$ and $\Psi(X_1)$. As a consequence, it is enough to prove that

$$A_{G\circ\Psi}(\underline{Y}) = A_G\big(\Psi(\underline{Y})\big) + S(X_0) - S(X_1).$$

Let us denote by $\underline{Z} = Z_n$ the preorbit $\Psi(Y_n)$. Setting $T_n^0 = 0$ and $T_n^{N_n+1} = T_n$, we have

$$\begin{aligned}
A_G(Z_n) &= \sum_{i=0}^{N_n} \int_{T_n^i}^{T_n^{i+1}} \lambda_{Z_n(t)}\big(\dot{Z}_n(t)\big) - G\big(Z_n(t)\big)\,dt \\
&= \sum_{i=0}^{N_n} \int_{T_n^i}^{T_n^{i+1}} (\Psi^*\lambda)_{Y_n(t)}\big(\dot{Y}_n(t)\big) - G\circ\Psi\big(Y_n(t)\big)\,dt \\
&= \sum_{i=0}^{N_n} \Big(\int_{T_n^i}^{T_n^{i+1}} \lambda_{Y_n(t)}\big(\dot{Y}_n(t)\big) - G\circ\Psi\big(Y_n(t)\big)\,dt \\
&\quad + S\big(Y_n(T_n^{i+1}-)\big) - S\big(Y_n(T_n^i)\big)\Big) \\
&= A_{G\circ\Psi}(Y_n) - S\big(Y_n(0)\big) + S\big(Y_n(T_n)\big) \\
&\quad + \sum_{i=1}^{N_n} \big(S(Y_n(T_n^i-)) - S\big(Y_n(T_n^i)\big)\big).
\end{aligned}$$

Since the function S is Lipschitz on the compact set K that contains the image of the curves Y_n, we obtain at the limit

$$A_G(\underline{Z}) = A_{G\circ\Psi}(\underline{Y}) - S(X_0) + S(X_1). \qquad \square$$

2.8

PROPOSITION

Let us set $\tilde{m}(G) := \inf_{X\in T^*N} \tilde{h}(X, X)$. *We have* $\tilde{m}(G) \in \{-\infty, 0, +\infty\}$. *In addition, if* $\tilde{m}(G) = 0$, *then there exists a point* X *in* T^*N *such that* $\tilde{h}(X, X) = 0$.

Proof

It follows from the triangle inequality that, for each $X \in T^*N$, $\tilde{h}(X, X) \geqslant 0$ or $\tilde{h}(X, X) = -\infty$. As a consequence, $\tilde{m}(G) \geqslant 0$ or $\tilde{m}(G) = -\infty$. Let us assume that $\tilde{m}(G) \in [0, \infty[$. Then there exists a point $X_0 \in T^*N$ and a preorbit $\underline{Y} = Y_n : [0, T_n] \longrightarrow T^*N$ between X_0 and X_0 such that $A(\underline{Y}) \in [0, \infty[$. Let K be a compact subset of T^*N which contains the image of all the curves Y_n. Let S_n be a sequence of integers such that $T_n/S_n \longrightarrow \infty$ and $S_n \longrightarrow \infty$. Let b_n be the integer part of T_n/S_n. Note that $b_n \longrightarrow \infty$. Let d_n be a sequence of integers such that $d_n \longrightarrow \infty$ and $d_n/b_n \longrightarrow 0$. Since the set K is compact, there exists a sequence $\epsilon_n \longrightarrow 0$ such that whenever b_n points are given in K, then at least d_n of them lie in the same ball of radius ϵ_n. So there exists a point $X_n \in K$ such that at least d_n of the points $Y_n(S_n), Y_n(2S_n), \dots, Y_n(b_nS_n)$ lie in the ball of radius ϵ_n and center X_n. Let us denote by $Y_n(t_n^1), Y_n(t_n^2), \dots, Y_n(t_n^{d_n})$ these points, where $t_n^{i+1} \geqslant t_n^i + S_n$. Taking a subsequence, we can assume that the sequence X_n has a limit X in K. It is not hard to see that $\underline{Y}^i = Y_{n\,|\,[t_n^i, t_n^{i+1}]}$ is a preorbit between X and X. On the other hand, for each $k \in \mathbb{N}$, we define the sequence of curves $Z_n^k : [0, T_n + t_n^1 - t_n^k] \longrightarrow T^*N$ by $Z_n^k(t) = Y_n(t)$ for $t \in [0, t_n^1[$ and $Z_n^k(t) = Y_n(t + t_n^k - t_n^1)$ for $t \in [t_n^1, T_n + t_n^1 - t_n^k]$. For each k, the sequence Z_n^k is a preorbit between X_0 and X_0. We have

$$A(Y_n) = A(Z_n^k) + \sum_{i=1}^{k-1} A(Y_n^i),$$

and hence,

$$A(\underline{Y}) \geqslant \tilde{h}(X_0, X_0) + (k-1)\tilde{h}(X, X).$$

Since $A(\underline{Y})$ is a real number and since this inequality holds for all $k \in \mathbb{N}$, this implies that $\tilde{h}(X, X) = 0$. $\square$

2.9

Let us define the symplectic Aubry set of G as the set

$$\tilde{\mathscr{A}}_s(G) := \left\{ X \in T^*N \text{ such that } \tilde{h}(X, X) = 0 \text{ and } G(X) = 0 \right\} \subset T^*N.$$

The symplectic Mather set $\tilde{\mathscr{M}}_s(G)$ of G is the union of the supports of the compactly supported Φ-invariant probability measures concentrated on $\tilde{\mathscr{A}}_s(G)$. Note that, in general, it is not clear that the symplectic Aubry set should be closed. The symplectic Mather set, then, may not be contained in the symplectic Aubry set but only in its closure. The Mather set and the Aubry set are Φ-invariant, as follows directly from Property 2.5. If $\tilde{m}(G) = 0$, then the symplectic Aubry set is not empty, and all its orbits are bounded; hence, the symplectic Mather set $\tilde{\mathscr{M}}_s(G)$ is not empty.

2.10

For each pair X_0, X_1 of points in $\tilde{\mathscr{A}}_s(G)$, we define the set $\tilde{\mathscr{I}}_s(X_0, X_1)$ of points $P \in T^*N$ such that

$$\tilde{h}(X_0, X_1) = \tilde{h}(X_0, X) + \tilde{h}(X, X_1)$$

if $\tilde{h}(X_0, X_1) \in \mathbb{R}$, and $\tilde{\mathscr{I}}_s(X_0, X_1) = \emptyset$ otherwise. Note that the sets $\tilde{\mathscr{I}}_s(X_0, X_1)$ are all contained in the level $\{G = 0\}$. Indeed, the finiteness of $\tilde{h}(X_0, X)$ implies that $G(X_0) = G(X)$, while $G(X_0) = 0$ by definition of $\tilde{\mathscr{A}}_s(G)$. It follows from Property 2.5 that the set $\tilde{\mathscr{I}}_s(X_0, X_1)$ is Φ-invariant. We now define the symplectic Mañé set as

$$\tilde{\mathscr{N}}_s(G) := \bigcup_{X_0, X_1 \in \tilde{\mathscr{A}}_s(G)} \tilde{\mathscr{I}}_s(X_0, X_1).$$

The Mañé set is Φ-invariant; all its orbits are bounded. We have the inclusion

$$\tilde{\mathscr{A}}_s(G) \subset \tilde{\mathscr{N}}_s(G).$$

In order to prove this inclusion, just observe that $X_0 \in \tilde{\mathscr{I}}(X_0, X_0)$ for each $X_0 \in \tilde{\mathscr{A}}_s(G)$.

2.11
If $\Psi : T^*N \longrightarrow T^*N$ is an exact diffeomorphism, then we have

$$\Psi\big(\tilde{\mathscr{M}}_s(G\circ\Psi)\big) = \tilde{\mathscr{M}}_s(G), \qquad \Psi\big(\tilde{\mathscr{A}}_s(G\circ\Psi)\big) = \tilde{\mathscr{A}}_s(G), \qquad \Psi\big(\tilde{\mathscr{N}}_s(G\circ\Psi)\big) = \tilde{\mathscr{N}}_s(G).$$

This obviously follows from Property 2.7 and from the fact that Ψ conjugates the Hamiltonian flow of G and the Hamiltonian flow of $G \circ \Psi$.

2.12
Let us assume that $\tilde{m}(G) = 0$, and set

$$\tilde{d}(X, X') = \tilde{h}(X, X') + \tilde{h}(X', X).$$

We have $\tilde{d}(X, X') \geqslant 0$, and the function $\tilde{d}$ satisfies the triangle inequality and is symmetric. In addition, we obviously have $\tilde{d}(X, X) = 0$ if and only if $X \in \tilde{\mathscr{A}}_s(G)$. The restriction of the function $\tilde{d}$ to the set $\tilde{\mathscr{A}}_s(G)$ is a pseudometric with $+\infty$ as a possible value. We define an equivalence relation on $\tilde{\mathscr{A}}_s(G)$ by saying that the points X and X' are equivalent if and only if $\tilde{d}(X, X') = 0$. The equivalence classes of this relation are called the static classes. Let us denote by $(\dot{\mathscr{A}}_s(G), \dot{d}_s)$ the metric space obtained from $\tilde{\mathscr{A}}_s$ by identifying points X and X' when $\tilde{d}(X, X') = 0$. In other words, the set $\dot{\mathscr{A}}_s(G)$ is the set of static classes of H. We call $(\dot{\mathscr{A}}_s(G), \dot{d}_s)$ the quotient Aubry set. Note that the metric $\dot{d}_s$ can take the value $+\infty$. The quotient Aubry set is also well behaved under exact diffeomorphisms. More precisely, if Ψ is an exact diffeomorphism of T^*N, then the image of a static class of $G \circ \Psi$ is a static class of G. This defines a map

$$\dot{\Psi} : \dot{\mathscr{A}}_s(G \circ \Psi) \longrightarrow \dot{\mathscr{A}}_s(G),$$

which is an isometry for the quotient metrics.

2.13
PROPOSITION
Assume that $\tilde{m}(G) = 0$, and in addition assume that the function $\tilde{h}$ is bounded from below. Then the orbits of $\tilde{\mathscr{N}}_s(G)$ are biasymptotic to $\tilde{\mathscr{A}}_s(G)$. In addition, for each

orbit $X(s)$ in $\tilde{\mathcal{N}}_s(G)$, there exists a static class $S-$ in $\tilde{\mathcal{A}}_s(G)$ and a static class $S+$ such that the orbit $X(s)$ is α-asymptotic to $S-$ and ω-asymptotic to $S+$.

Proof
Let ω and ω' be two points in the ω-limit of the orbits $X(t) = \Phi_t(X)$. We have to prove that ω and ω' belong to the symplectic Aubry set and to the same static class. It is enough to prove that $\tilde{d}(\omega, \omega') = 0$. In order to do so, we consider two increasing sequences t_n and s_n, such that $t_n - s_n \longrightarrow \infty$, $s_n - t_{n-1} \longrightarrow \infty$, $X(t_n) \longrightarrow \omega$, and $X(s_n) \longrightarrow \omega'$. Let $\underline{Y} = Y_n : [0, t_n - s_n] \longrightarrow T^*N$ be the preorbit between ω' and ω defined by $Y_n(t) = X(t - s_n)$. Similarly, we consider the preorbit $\underline{Z} = Z_n$: $[0, s_{n+1} - t_n] \longrightarrow T^*N$ between ω and ω' defined by $Z_n(t) = X(t - t_n)$. Since X belongs to $\tilde{\mathcal{N}}_s(G)$, there exist points X_0 and X_1 in $\tilde{\mathcal{A}}_s(G)$ such that $X \in \tilde{\mathcal{I}}(X_0, X_1)$. In view of Property 2.5, we have

$$\tilde{h}\big(X(t_n), X_1\big) = \tilde{h}\big(X(t_m), X_1\big) + \int_{t_n}^{t_m} \lambda_{X(t)}\big(\dot{X}(t)\big) - G\big(X(t)\big)\, dt$$

for all $m \geqslant n$. Since the function $\tilde{h}$ is bounded from below, we conclude that the double sequence $\int_{t_n}^{t_m} \lambda_{X(t)}(\dot{X}(t)) - G(X(t))\, dt$, $m \geqslant n$, is bounded from above, so that

$$\liminf \int_{t_n}^{t_{n+1}} \lambda_{X(t)}\big(\dot{X}(t)\big) - G\big(X(t)\big)\, dt \leqslant 0.$$

As a consequence, we have $\liminf A(Y_{n+1}) + A(Z_n) \leqslant 0$, and hence, $A(\underline{Y}) + A(\underline{Z}) = 0$ and $\tilde{d}(\omega, \omega') = 0$. The proof is similar for the α-limit. □

It is useful to finish the section with a technical remark.

2.14
LEMMA
Let $\underline{Y} = Y_n : [0, T_n] \longrightarrow T^*N$ be a preorbit between X_0 and X_1. There exists a preorbit $\underline{Z}$ between X_0 and X_1 that has the same action as $\underline{Y}$ and has discontinuities only at times $1, 2, \dots, [T_n] - 1$, where $[T_n]$ is the integer part of T_n.

Proof
We set $Z_n(k + s) = \Phi_s(Y_n(k))$ for each $k = 0, 1, \dots, [T_n] - 2$, and we set $s \in [0, 1[$ and $Z_n([T_n] - 1 + s) = \Phi_s(Y_n([T_n] - 1))$ for each $s \in [0, 1 + T_n - [T_n][$. It is not hard to see that $A(Z_n) - A(Y_n) \longrightarrow 0$, and hence, $A(\underline{Y}) = A(\underline{Z})$. □

3. The case of convex Hamiltonian systems

We assume the hypotheses in Section 1.1, and we prove that the symplectic definitions of Section 2 agree with the standard definitions of Section 1. This proves that the theory of Section 2 is not trivial, at least in this case. This also ends the proof of Theorem 1.10.

3.1

In Section 3, we consider a Hamiltonian function $H : T^*M \times \mathbb{T} \longrightarrow \mathbb{R}$ satisfying the hypotheses in Section 1.1. We set $N = M \times \mathbb{T}$. We denote by (P, t, E) the points of T^*N, and we set $G(P, t, E) = E + H(P, t) : T^*N \longrightarrow \mathbb{R}$. We denote by $h(q, t; q', t')$ the Peierls's barrier associated to H in Section 1, and we denote by $\tilde{h}(P, t, E; P', t', E')$ the barrier associated to G in Section 2.

3.2

Before we state the main result of Section 3, some terminology is necessary. If $u : M \longrightarrow \mathbb{R}$ is a continuous function, then we say that $P \in T_q^*M$ is a proximal superdifferential of u at point q (or simply a superdifferential) if there exists a smooth function $f : M \longrightarrow \mathbb{R}$ such that $f - u$ has a minimum at q and $df_q = P$. Clearly, if u is differentiable at q and if P is a proximal superdifferential of u at q, then $P = du_q$.

3.3

PROPOSITION

We have the relation

$$h(q, t; q', t') = \min_{P \in T_q^*M, P' \in T_{q'}^*M} \tilde{h}\big(P, t, -H(P, t); P', t', -H(P', t')\big).$$

In addition, if the minimum is reached at (P, P'), then P is a superdifferential of the function $h(\cdot, t; q', t')$ at point q and $-P'$ is a superdifferential of the function $h(q, t; \cdot, t')$ at point q'.

Proof

Let us fix two points (q, t) and (q', t') in $N = M \times \mathbb{T}$. We claim that the inequality

$$\tilde{h}(P, t, E; P', t', E') \geqslant h(q, t; q', t')$$

holds for each $(P, t, E) \in T^*_{(q,t)}N$ and each $(P', t', E') \in T^*_{(q',t')}N$. If $\tilde{h}(P, t, E; P', t', E') = +\infty$, then there is nothing to prove. Otherwise, let us fix $\epsilon > 0$. There exists a preorbit $\underline{Y} = Y_n(s) : [0, T_n] \longrightarrow T^*N$ between (P, t, E) and (P', t', E') such that $A(\underline{Y}) \leqslant \tilde{h}(P, t, E; P', t', E') + \epsilon$ (resp., $A(\underline{Y}) \leqslant -1/\epsilon$ in the case where $\tilde{h}(P, t, E; P', t', E') = -\infty$). In view of Lemma 2.14, it is possible to assume that the discontinuity points T_n^i of Y_n satisfy $T_n^{i+1} \geqslant T_n^i + 1$. Let us write

$$Y_n(s) = \big(P_n(s), \tau_n(s), E_n(s)\big)$$

and $q_n(s) = \pi(P_n(s))$. Let δ_n^i be the real number closest to $T_n^{i+1} - T_n^i$ among those that satisfy $\tau_n(T_n^i) + \delta_n^i = \tau_n(T_n^{i+1})$.

We have

$$\begin{aligned} A(Y_n) &= \sum_{i=0}^{N_n} \int_{T_n^i}^{T_n^{i+1}} L\big(q_n(s), \dot q_n(s), s + \tau_n(T_n^i) - T_n^i\big)\, ds \\ &\geqslant \sum_{i=0}^{N_n} F\big(q(T_n^i), \tau_n(T_n^i); q(T_n^{i+1}-), T_n^{i+1} - T_n^i\big). \end{aligned}$$

It is known that the functions $F(q, t; q', s)$ are Lipschitz on $\{s \geqslant 1\}$ (see e.g., [1, Section 3.2]. We have

$$\begin{aligned} &\sum_{i=0}^{N_n} \big| F\big(q_n(T_n^i), \tau_n(T_n^i); q_n(T_n^{i+1}-), T_n^{i+1} - T_n^i\big) - F\big(q_n(T_n^i), \tau_n(T_n^i); q_n(T_n^{i+1}), \delta_n^i\big)\big| \\ &\quad\leqslant C \sum_{i=0}^{N_n-1} D\big(q_n(T_n^{i+1}-), \tau_n(T_n^{i+1}-); q_n(T_n^{i+1}), \tau_n(T_n^{i+1})\big) \\ &\quad\leqslant C \sum_{i=0}^{N_n-1} D\big(Y_n(T_n^{i+1}-), Y_n(T_n^{i+1})\big) \longrightarrow 0. \end{aligned}$$

As a consequence, we have

$$\begin{aligned} A(\underline{Y}) &\geqslant \liminf \sum_{i=0}^{N_n} F\big(q(T_n^i), \tau_n(T_n^i); q(T_n^{i+1}), \delta_n^i\big) \\ &\geqslant \liminf F\Big(q_n(0), \tau_n(0); q_n(T_n), \sum_{i=0}^{N_n} \delta_n^i\Big) \geqslant h(q, t; q', t'), \end{aligned}$$

and hence, $\epsilon + \tilde h(P, t, E; P', t', E') \geqslant h(q, t; q', t')$ (resp., $-1/\epsilon \geqslant h(q, t; q', t')$). Since this holds for all $\epsilon > 0$, we have $\tilde h(P, t, E; P', t', E') \geqslant h(q, t; q', t')$, as desired.

Conversely, let us consider a sequence T_n such that $T_n \longrightarrow \infty$, $t + T_n \bmod 1 = t'$, and

$$h(q, t; q', t') = \lim_{n \longrightarrow \infty} F(q, t; q', T_n).$$

Let $q_n(s) : [0, T_n] \longrightarrow M$ be a curve such that

$$\int_0^{T_n} L\big(q_n(s), \dot q_n(s), s+t\big)\, ds = F(q, t; q', T_n).$$

Since the curve q_n is minimizing the action, there exists a Hamiltonian trajectory

$$Y_n(s) = \big(P_n(s), t+s, E_n(s)\big) : [0, T_n] \longrightarrow T^*N$$

whose projection on M is the curve q_n. In addition, by well-known results on minimizing orbits (see [10]), there exists a compact subset of T^*M that contains the images of all the curves $P_n(s)$. As a consequence, we can assume, taking a subsequence if necessary, that the sequences $P_n(0)$ and $P_n(T_n)$ have limits $P \in T_q^*M$ and $P' \in T_{q'}^*M$. The sequence $\underline{Y} = Y_n$ is then a preorbit between $(P, t, -H(P, t))$ and $(P', t', -H(P', t'))$, and its action is

$$A(\underline{Y}) = \lim A(Y_n) = \lim \int_0^{T_n} L\big(q_n(s), \dot q_n(s), t+s\big)\, ds = h(q, t; q', t').$$

As a consequence, we have

$$\tilde h\big(P, t, -H(P, t); P', t', -H(P', t')\big) \leqslant h(q, t; q', t').$$

This ends the proof of the first part of the proposition.

Let now $Y = (P, t, E) \in T_q^*M \times T^*\mathbb{T}$ and $Y' = (P', t', E') \in T_{q'}^*M \times T^*\mathbb{T}$ be points such that $h(q, t; q', t') = \tilde h(Y; Y')$. Let $q(s)$ be the projection on M of the orbit $\Phi_s(Y)$. Using Property 2.5 and Section 1.5, we get

$$\begin{aligned}\tilde h(Y, Y') &= \tilde h\big(\Phi_s(Y), Y'\big) + \int_0^s \lambda_{\Phi_\sigma(Y)}\big(V_G\big(\Phi_\sigma(Y)\big) - G\big(\Phi_\sigma(Y)\big)\, d\sigma \\ &\geqslant h\big(q(s), t+s; q', t'\big) + \int_0^s L\big(q(\sigma), \dot q(\sigma), t+\sigma\big)\, dt \\ &\geqslant h(q, t; q', t') = \tilde h(Y, Y').\end{aligned}$$

As a consequence, all the inequalities are equalities. We obtain that the curve $q(s)$ is minimizing in the expression

$$h(q, t; q', t') = \min\Big(h\big(q(s), t+s; q', t'\big) + \int_0^s L\big(q(\sigma), \dot q(\sigma), t+\sigma\big)\, dt\Big).$$

Fathi has proved that $-P$ is then a superdifferential of the function $h(\cdot, t; q', t')$ at q. The properties at (q', t') are treated in a similar way. □

3.4

COROLLARY

If H satisfies the hypotheses of Section 1.1, then $m(H) \leqslant \tilde{m}(G)$.

3.5

COROLLARY

If H satisfies the hypotheses of Section 1.1 and if $m(H) = 0$, then $\tilde{m}(G) = 0$, and we have $\tilde{\mathscr{A}}_s(G) = \tilde{\mathscr{A}}(H)$. In addition, we have

$$\tilde{h}(X_0, t_0, E_0; X_1, t_1, E_1) = h\big(\pi(P_0), t_0; \pi(P_1), t_1\big)$$

for each (P_0, t_0, E_0) and (P_1, t_1, E_1) in $\tilde{\mathscr{A}}(H)$.

Proof

Let (P, t, E) be a point of T^*N and let $q = \pi(P)$. If $(P, t, E) \in \tilde{\mathscr{A}}_s(G)$, then

$$\tilde{h}(P, t, E; P, t, E) = 0,$$

so that $h(q, t; q, t) \leqslant 0$. On the other hand, since we have $h(q, t; q, t) \geqslant m(H) = 0$, we conclude that $h(q, t; q, t) = 0$, and hence, $(q, t) \in \mathscr{A}(H)$. As a consequence, the function $h(q, t; \cdot, t)$ is differentiable at q (see Section 1.6), and $\big(\partial_3 h(q, t; q, t), t - H(\partial_3 h(q, t; q, t), t)\big) \in \tilde{\mathscr{A}}(H)$. Since $\tilde{h}(P, t, E; P, t, E) = h(q, t; q, t)$, the point P is a superdifferential of $h(q, t; \cdot, t)$ at q, and we must have $P = \partial_3 h(q, t; q, t)$. Moreover, we have $G(P, t, E) = H(P, t) + E = 0$, and hence, $(P, t, E) \in \tilde{\mathscr{A}}(H)$.

Conversely, assume that $(P, t, E) \in \tilde{\mathscr{A}}(H)$. We then have $E = -H(P, t)$. In addition, $h(q, t; q, t) = 0$, the functions $h(q, t; \cdot, t)$ and $h(\cdot, t; q, t)$ are differentiable at q, and we have $P = \partial_3 h(q, t; q, t) = -\partial_1 h(q, t; q, t)$. Now let $X \in T_q^*M$ and $X' \in T_q^*M$ be such that

$$\tilde{h}\big(X, t, -H(X, t); X', t', -H(X', t')\big) = h(q, t; q, t).$$

Then $-X$ is a superdifferential at q of $h(\cdot, t; q, t)$, and X' is a superdifferential at q of $h(q, t; \cdot, t)$. It follows that $X = P = X'$. Hence, we have $\tilde{h}(P, t, E; P, t, E) = h(q, t; q, t) = 0$. This proves that $\tilde{m}(G) = 0$ and that $(P, t, E) \in \tilde{\mathscr{A}}_s(G)$.

Finally, let $(P_0, t_0, E_0) \in T_{q_0}^*M \times T^*\mathbb{T}$ and $(P_1, t_1, E_1) \in T_{q_1}^*M \times T^*\mathbb{T}$ be two points of $\tilde{\mathscr{A}}(H)$. We have $E_0 = -H(P_0, t_0)$ and $E_1 = -H(P_1, t_1)$. Furthermore, the function $h(q_0, t_0; \cdot, t_1)$ is differentiable at q_1, with $\partial_3 h(q_0, t_0; q_1, t_1) = P_1$, and the function $h(\cdot, t_0; q_1, t_1)$ is differentiable at q_0, with $\partial_1 h(q_0, t_0; q_1, t_1) = -P_0$. Since $-P_0$ and P_1 are then the only superdifferentials of $h(\cdot, t_0; q_1, t_1)$ and $h(q_0, t_0; \cdot, t_1)$, we conclude that $\tilde{h}(P_0, t_0, E_0; P_1, t_1, E_1) = h(q_0, t_0; q_1, t_1)$. $\square$

3.6

COROLLARY

If H satisfies the hypotheses of Section 1.1 and if $m(H) = 0$, then $\tilde{\mathscr{M}}_s(G) = \tilde{\mathscr{M}}(H)$.

3.7

COROLLARY

If H satisfies the hypotheses of Section 1.1 and if $m(H) = 0$, then $\tilde{\mathscr{N}}_s(G) = \tilde{\mathscr{N}}(H)$.

Proof

It is enough to prove that if (P_0, t_0, E_0) and (P_1, t_1, E_1) belong to $\tilde{\mathscr{A}}_s(G)$ and $q_0 = \pi(P_0)$, $q_1 = \pi(P_1)$, then

$$\tilde{\mathscr{I}}_s(P_0, t_0, E_0; P_1, t_1, E_1) = \tilde{\mathscr{I}}(q_0, t_0, q_1, t_1).$$

Let (P, t, E) be a point of $\tilde{\mathscr{I}}_s(P_0, t_0, E_0; P_1, t_1, E_1)$. We then have $G(P_0, t_0, E_0) = G(P, t, E) = 0$, and hence, $E = -H(P, t)$. Furthermore, the inequalities

$$\begin{aligned} h(q_0, t_0; q_1, t_1) &= \tilde{h}\big(P_0, t_0, -H(P_0, t_0); P_1, t_1, -H(P_1, t_1)\big) \\ &= \tilde{h}\big(P_0, t_0, -H(P_0, t_1); P, t, E\big) \\ &\quad + \tilde{h}\big(P, t, E; P_1, t_1, -H(P_1, t_1)\big) \\ &\geqslant h(q_0, t_0; q, t) + h(q, t; q_1, t_1) \geqslant h(q_0, t_0; q_1, t_1) \end{aligned}$$

are all equalities. As a consequence, the point (q, t) belongs to the set $\mathscr{I}(q_0, t_0; q_1, t_1)$, and the differentials $\partial_3 h(q_0, t_0; q, t)$ and $\partial_1 h(q, t; q_1, t_1)$ exist. We have $\partial_3 h(q_0, t_0; q, t) = -\partial_1 h(q, t; q_1, t_1)$, and the point

$$(X, t, e) = \big(\partial_3 h(q_0, t_0; q, t), t, -H(\partial_3 h(q_0, t_0; q, t), t)\big)$$

belongs to $\tilde{\mathscr{I}}(q_0, t_0; q_1, t_1)$, as follows from our definition of the Mañé set. Since

$$\tilde{h}\big(P_0, t_0, -H(P_0, t_0); P, t, -H(P, t)\big) = h(q_0, t_0; q, t),$$

the point P must be a superdifferential of $h(q_0, t_0; \cdot, t)$ at q, and hence, $P = X$. We have proved that $(P, t, E) \in \tilde{\mathscr{I}}(q_0, t_0; q_1, t_1)$.

Conversely, assume that $(P, t, E) \in \tilde{\mathscr{I}}(q_0, t_0; q_1, t_1)$, so that $E = -H(P, t)$. Then

$$h(q_0, t_0; q, t) + h(q, t; q_1, t_1) = h(q_0, t_0; q_1, t_1)$$

and

$$P = \partial_3 h(q_0, t_0; q, t) = -\partial_1 h(q, t; q_{1,1}).$$

In addition, since (q_0, t_0) and (q_1, t_1) belong to $\mathscr{A}(H)$, the differential $P_0 = \partial_1 h(q_0, t_0; q, t)$ exists for all q and satisfies $(P_0, t_0, -H(P_0, t_0) \in \tilde{\mathscr{A}}(H))$. Similarly, setting $P_1 = \partial_3 h(q, t; q_1, t_1)$, we have $(P_1, t_1, -H(P_1, t_1) \in \tilde{\mathscr{A}}(H))$. We conclude that

$$\tilde{h}\big(P_0, t_0, -H(P_0, t_0); P, t, E\big) = h(q_0, t_0; q, t)$$

and

$$\tilde{h}\big(P, t, E; P_1, t_1, -H(P_1, t_1)\big) = h(q, t; q_1, t_1).$$

As a consequence, setting $E_0 = -H(P_0, t_0)$ and $E_1 = -H(P_1, t_1)$, we have

$$\tilde{h}(P_0, t_0, E_0; P, t, E) + \tilde{h}(P, t, E; P_1, t_1, E_1) = \tilde{h}(P_0, t_0, E_0; P_1, t_1, E_1). \qquad \square$$

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