

What is the difference between a square and a triangle?

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Abstract

We offer a reader-friendly introduction to the attracting edge problem (also known as the “triangle conjecture”) and its most general current solution of Limic and Tarrès (2007). Little original research is reported; rather this article “zooms in” to describe the essential characteristics of two different techniques/approaches verifying the almost sure existence of the attracting edge for the strongly edge reinforced random walk (SERRW) on a square. Both arguments extend straightforwardly to the SERRW on even cycles. Finally, we show that the case where the underlying graph is a triangle cannot be studied by a simple modification of either of the two techniques.

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1 Introduction

We briefly describe the general setting introduced, for example, in [4]. Let \mathcal{G} be a connected graph with set of vertices $V = V(\mathcal{G})$, and set of (unoriented) edges $E = E(\mathcal{G})$. The only assumption on the graph is that each vertex has at most $D(\mathcal{G})$ adjacent vertices (edges), for some $D(\mathcal{G}) < \infty$, so that \mathcal{G} is of bounded degree.

Call two vertices v, v' *adjacent* ($v \sim v'$ in symbols) if there exists an edge, denoted by $\{v, v'\} = \{v', v\}$, connecting them.

Let $\mathbb{N} = \{0, 1, \dots\}$, and let $W : \mathbb{N} \rightarrow (0, \infty)$ be the *reinforcement weight* function. Assume we are given initial *edge weights* $X_0^e \in \mathbb{N}$ for all $e \in E$, such that $\sup_e X_0^e < \infty$. Let I_n be a V -valued random variable, recording the position of the particle at time $n \in \mathbb{N}$. Set $I_0 := v_0$ for some $v_0 \in \mathcal{G}$. Let $(\mathcal{F}_n, n \geq 0)$ be the filtration generated by I .

The edge reinforced random walk (ERRW) on \mathcal{G} evolves according to a random dynamics with the following properties:

(i) if currently at vertex $v \in \mathcal{G}$, in the next step the particle jumps to a nearest neighbor of v ,

(ii) the probability of a jump from v to v' at time n is “ W -proportional” to the number of previous traversals of the edge connecting v and v' , that is,

$$\mathbb{P}(I_{n+1} = v' | \mathcal{F}_n) 1_{\{I_n = v\}} = \frac{W(X_n^{\{v, v'\}})}{\sum_{w \sim v} W(X_n^{\{v, w\}})} 1_{\{I_n = v \sim v'\}},$$

where X_n^e , $e \in E(\mathcal{G})$ equals

$$X_n^e = X_0^e + \sum_{k=0}^{n-1} 1_{\{I_k, I_{k+1}\} = e}.$$

We recommend a recent survey by Pemantle [7] as an excellent overview of processes with reinforcement: results, techniques, open problems and applications.

Let (H) be the following condition on W :

$$\sum_{k \in \mathbb{N}} \frac{1}{W(k)} < \infty. \tag{H}$$

We call any edge reinforced random walk corresponding to W that satisfies (H) a *strongly* edge reinforced random walk. Denote by

$$A := \{\exists n : \{I_k, I_{k+1}\} = \{I_{k+1}, I_{k+2}\}, k \geq n\}$$

the event that eventually the particle traverses a single (random) edge of the graph. On A we call that edge the *attracting edge*. It is easy to see that (H) is the necessary and sufficient condition for

$$\mathbb{P}(\{I_n, I_{n+1}\} = \{I_0, I_1\} \text{ for all } n) > 0.$$

This implies that (H) is necessary and sufficient for $\mathbb{P}(A) > 0$. The necessity can be seen by splitting A into a countable union of events, where each corresponds to getting attracted to a specific edge after a particular time with a specific configuration of weights on the neighbouring edges. Since A is a tail event, it seems natural to wonder whether

$$\mathbb{P}(A) = 1 \tag{A}$$

holds. The authors studied this problem in [4], and concluded that, under additional technical assumptions, (H) implies (A). In particular,

Theorem 1 ([4], Corollary 3) *If \mathcal{G} has bounded degree and if W is non-decreasing, then (H) implies (A).*

We denote by \mathcal{G}_l the *cycle* of length l , with vertices $\{0, 1, \dots, l-1\}$ and edges

$$e_i = \{i, i+1\}, \quad i = 0, \dots, l-1,$$

where $l \geq 3$, and where the addition is done modulo l .

Let us now concentrate on the case where the underlying graph \mathcal{G} is the square \mathcal{G}_4 . The next two sections demonstrate two different techniques of proving the following claim.

Theorem 2 *If $\mathcal{G} = \mathcal{G}_4$, then (H) implies (A).*

In fact we will concentrate on a somewhat simpler claim whose proof can be “recycled” (as we henceforth discuss) in order to arrive to the full statement of Theorem 2.

Proposition 3 *If $\mathcal{G} = \mathcal{G}_4$, then (H) implies*

$$\mathbb{P}(\text{all four edges are traversed infinitely often}) = 0.$$

In Section 4 we discuss the reasons why these techniques which are well-suited for $\mathcal{G} = \mathcal{G}_4$, or any graph of bounded degree without an odd cycle, cf. [9] or [3], do not extend to the setting where $\mathcal{G} = \mathcal{G}_3$ is a triangle. In fact, the following “triangle conjecture” is still open in its full generality (cf. Theorem 2 and Theorem 1) where W is a general (irregular) weight function satisfying (H):

Open Problem 4 *If $\mathcal{G} = \mathcal{G}_3$, then (H) implies (A).*

2 A continuous time-lines technique

This technique adapts a construction due to Rubin, and was invented by Davis [1] and Sellke [9]. It played a key role in his proof of the attracting edge property on \mathbb{Z}^d , and was also used by Limic [3] in order to simplify the attracting edge problem on graphs of bounded degree to the same problem on odd cycles. Denote for simplicity

$$e_i := \{i, i + 1\}, \quad i = 0, \dots, 3,$$

where addition is done modulo 4. For each $i = 0, \dots, 3$ and $k \geq 1$ let E_k^i be an exponential random variable with mean $1/W(k)$, such that $\{E_k^i, i = 0, \dots, 3, k \geq 1\}$ is a family of independent random variables. Denote by

$$T_n^i := \sum_{k=X_0^{e_i}+n}^{X_0^{e_i}+n+1} E_k^i, \quad n \geq 0, \quad T_\infty^i := \sum_{k=X_0^{e_i}}^{\infty} E_k^i, \quad i = 0, \dots, 3.$$

Note that the random variables $T_n^i, T_\infty^i, i = 0, \dots, 3, n \in \mathbb{N}$, are continuous, independent and finite almost surely (the last property is due to assumption (H)). In Figure 1, the T_n^i are shown as dots, and the “limits” $T_\infty^i, i = 0, \dots, 3$ are indicated.

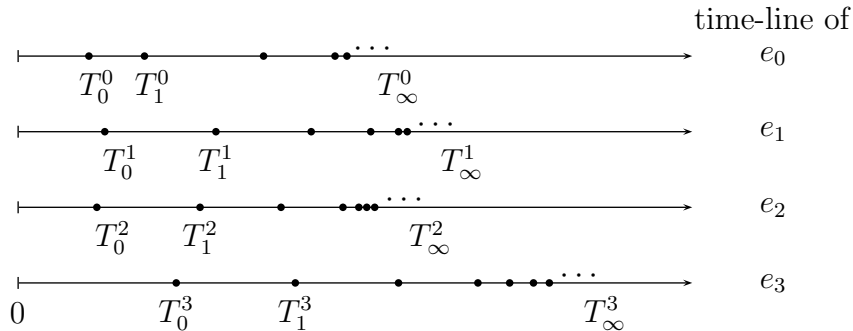


Figure 1

Here is how one can construct a realization of the edge reinforced random walk on \mathcal{G}_4 from the above data, or (informally) from the figure. Given the current position of the walk, simultaneously erase (at rate 1) the two time-lines of the incident edges in the chronological direction until encountering the next dot belonging to either of the time-lines. At this point, the walk steps into a new vertex by traversing the edge that corresponds to the time-line containing the dot. The procedure continues inductively.

We next explain why this construction indeed leads to a realization of the edge reinforced random walk, by considering carefully the first three steps. Assume for concreteness that the initial position is vertex 0 incident to the edges e_0 and e_3 . The time-lines of e_0 and e_3 are erased until the minimum of $T_0^0 = E_{X_0^{e_0}}^0$ and $T_0^3 = E_{X_0^{e_3}}^3$. In the figure this minimum happens to be T_0^0 . Thus the particle moves from 0 to 1 (traversing edge e_0) in the first step. Due to the properties of exponentials, the probability of this move is exactly $W(X_0^{e_0})/(W(X_0^{e_0})+W(X_0^{e_3}))$. The two incident edges to the current position I_1 are now e_0 and e_1 . Continue by simultaneous erasing (the previously non-erased parts of) time-lines corresponding to e_0 and e_1 until the next dot. In the figure, the dot again appears on the line of e_0 . Hence the particle traverses the edge e_0 in the second step and therefore jumps back to vertex 0. Note that again the probability of this transition matches the one of the edge reinforced random walk. Continue by the simultaneous erasure of time-lines corresponding to e_0 and e_3 . Based on the figure, the particle makes the third step across the edge e_3 , since the (residual) length of the interval on the time-line of e_3 until T_0^3 is smaller than $T_2^0 - T_1^0 = E_{X_0^{e_0}+2}^1$. The memoryless property of the exponential distribution insures that the (residual) length of the interval until T_0^3 is again distributed as exponential (rate $W(X_0^{e_3})$) random variable, independent of all other data. Hence, the transition probability again matches that of the ERRW.

Note that the above construction can be done with any number $l \geq 3$ of time-lines (corresponding to the length l of the underlying circular graph), and we make use of this generalization in Section 4.

As a by-product of the above construction, a continuous-time version of the edge reinforced random walk emerges, where the particle makes the jumps exactly at times when the dots are encountered. More precisely, if we denote by $\tilde{I}(s)$ the position of the particle at time s and if $\tau_0 = 0$ and $0 < \tau_1 < \tau_2 < \dots$ the successive jump times of \tilde{I} , then the (discrete time) ERRW constructed above, and the continuous-time version are coupled so that

$$I_k \equiv \tilde{I}(\tau_k), \quad k \geq 0.$$

It is worth noting that this continuous-time version is analogous to the Harris construction of a continuous-time Markov chain from the discrete one, yet it is different since the parameters of the exponential clocks vary. In particular, under assumption (H), the total time of evolution of the continuous-time random walk is finite.

Consider the total time of evolution for the continuous time walk,

$$T := \lim_{k \rightarrow \infty} \tau_k.$$

Note that at any time $s \geq 0$ the particle is incident to one of the edges e_0 and e_2 , and equally it is incident to one of the edges e_1 and e_3 , hence

$$\begin{aligned} T &= \sum_{i=0, i \text{ even}}^3 \text{total time spent on boundary vertices of } e_i \\ &= \sum_{i=0, i \text{ odd}}^3 \text{total time spent on boundary vertices of } e_i. \end{aligned}$$

Note that

$$\begin{aligned} &\{\text{all four edges are traversed infinitely often}\} && (1) \\ &= \{\text{the time-line of } e_i \text{ is erased up to time } T_\infty^i \text{ for each } i = 0, \dots, 3\} \\ &\subset \{T = T_\infty^0 + T_\infty^2 = T_\infty^1 + T_\infty^3\}. && (2) \end{aligned}$$

However, due to the independence and continuity of $T_\infty^0 + T_\infty^2$ and $T_\infty^1 + T_\infty^3$, the identity (2) happens with probability 0. We conclude that (1) happens with probability 0, and therefore that Proposition 3 holds.

In order to obtain the proof of Theorem 2 now note that there are essentially three possibilities remaining for the asymptotic evolution: the edge reinforced random walk visits infinitely often either one, or two adjacent, or three edges. In the latter two cases, there is at least one vertex j such that both edges e_{j-1} and e_j are traversed infinitely often. Moreover, after finitely many steps, every excursion from j starts and ends with the same edge. Now one can measure the time spent at site j from two perspectives: that of waiting to traverse edge e_{j-1} , and that of waiting to traverse edge e_j . The reader will quickly construct a variation to the above argument (alternatively, consult [9] or [3]) determining that a branching vertex exists with probability 0.

Note that this continuous time-lines technique still works on even cycles \mathcal{G}_{2k} . Indeed, given the continuous-time realization of the edge reinforced random walk constructed above, we observe that on the event that all edges are visited infinitely often,

$$T = \sum_{i=0, i \text{ even}}^{2k-1} T_\infty^i = \sum_{i=0, i \text{ odd}}^{2k-1} T_\infty^i, \quad (3)$$

where $T := \lim_{k \rightarrow \infty} \tau_k$ is the total time of evolution for the walk. As before, (3) is a consequence of the fact that T equals the total time spent on both the boundary of even and the boundary of odd edges. Now, due to independence and continuity of $\sum_{i=0, i \text{ even}}^{2k-1} T_\infty^i$ and $\sum_{i=0, i \text{ odd}}^{2k-1} T_\infty^i$, the identity (3) happens with probability 0 so that, almost surely, at least one of the edges in the cycle is visited only finitely often, and we conclude (A) as in the case of the square.

3 A martingale technique

Let, for all $n \in \mathbb{N}$,

$$W^*(n) := \sum_{k=0}^{n-1} \frac{1}{W(k)},$$

with the convention that $W^*(0) := 0$.

Assume the setting of Proposition 3. For all $n \in \mathbb{N}$, $i = 0, \dots, 3$, define the processes

$$Y_n^\pm(i) := \sum_{k=0}^{n-1} \frac{1_{\{I_k=i, I_{k+1}=i\pm 1\}}}{W(X_k^{e_i})} \quad (4)$$

$$\kappa_n^i := Y_n^+(i) - Y_n^-(i) \quad (5)$$

Clearly, κ^i is measurable with respect to the filtration $(\mathcal{F}_n, n \geq 0)$. Moreover, it is easy to check that $(\kappa_n^i, n \geq 0)$ is a martingale : on $\{I_n = i\}$, $E(\kappa_{n+1}^i - \kappa_n^i | \mathcal{F}_n)$ is equal to

$$\frac{1}{W(X_n^{e_i})} \frac{W(X_n^{e_i})}{W(X_n^{e_i}) + W(X_n^{e_{i-1}})} - \frac{1}{W(X_n^{e_{i-1}})} \frac{W(X_n^{e_{i-1}})}{W(X_n^{e_i}) + W(X_n^{e_{i-1}})} = 0.$$

Therefore the process

$$\kappa_n := \kappa_n^0 - \kappa_n^1 + \kappa_n^2 - \kappa_n^3 + \sum_{i=0}^3 (-1)^i W^*(X_0^{e_i}), \quad (6)$$

is also a martingale. Due to assumption (H), each of the four processes κ^i is a difference of bounded non-decreasing processes, and therefore has an almost sure limit as $n \rightarrow \infty$. Hence denote by κ_∞ the finite limit $\lim_{n \rightarrow \infty} \kappa_n$.

Now

$$\kappa_n = \sum_{i=0}^3 (-1)^i W^*(X_n^{e_i}). \quad (7)$$

This implies that

$$\{\text{all four edges are traversed infinitely often}\} \subset \{\kappa_\infty = 0\},$$

so that it suffices to show

$$\mathbb{P}(\mathcal{A}_\infty) = 0, \quad (8)$$

where

$$\mathcal{A}_\infty := \{\text{all four edges are traversed infinitely often}\} \cap \{\kappa_\infty = 0\}.$$

In order to prove (8), we now analyze carefully the variance of the increments of the martingale $(\kappa_n)_{n \in \mathbb{N}}$ (going down to 0, due to (H)), which will enable us to prove the nonconvergence of this martingale to 0 a.s. on the event that all edges are visited infinitely often. This technique adapts an argument proving almost sure nonconvergence towards unstable points of stochastic approximation algorithms, introduced by Pemantle [6] and generalized by Tarrès [12, 13].

Fix large n , and note that

$$\begin{aligned} \mathbb{E}((\kappa_{n+1})^2 - (\kappa_n)^2 | \mathcal{F}_n) &= \mathbb{E}((\kappa_{n+1} - \kappa_n)^2 | \mathcal{F}_n) \\ &= \mathbb{E}\left(\sum_{i=0}^3 \frac{1_{\{I_n=i, I_{n+1}=i+1\}}}{(W(X_n^{e_i}))^2} + \frac{1_{\{I_n=i, I_{n+1}=i-1\}}}{(W(X_n^{e_{i-1}}))^2} \middle| \mathcal{F}_n\right). \end{aligned} \quad (9)$$

From now on abbreviate

$$\alpha_n := \sum_{j=X_n^*}^{\infty} \frac{1}{(W(j))^2},$$

where $X_n^* = \min_{i=0, \dots, 3} X_n^{e_i}$. For $\varepsilon > 0$, define the stopping time

$$S := \inf\{k \geq n : |\kappa_k| > \varepsilon \sqrt{\alpha_n}\}. \quad (10)$$

Since

$$(\kappa_S)^2 - (\kappa_n)^2 = \sum_{k=n}^{\infty} ((\kappa_{k+1})^2 - (\kappa_k)^2) 1_{\{S > k\}},$$

by nested conditioning we obtain

$$\mathbb{E}((\kappa_S)^2 - (\kappa_n)^2 | \mathcal{F}_n) = \mathbb{E}\left(\sum_{k=n}^{\infty} \mathbb{E}[(\kappa_{k+1})^2 - (\kappa_k)^2 | \mathcal{F}_k] 1_{\{S > k\}} | \mathcal{F}_n\right),$$

so that, due to (9), we obtain

$$\begin{aligned} \mathbb{E}((\kappa_S)^2 - (\kappa_n)^2 | \mathcal{F}_n) &= \mathbb{E}\left[\sum_{k=n}^{S-1} \sum_{i=0}^3 \frac{1_{\{I_k=i, I_{k+1}=i+1\}}}{(W(X_k^{e_i}))^2} + \frac{1_{\{I_k=i, I_{k+1}=i-1\}}}{(W(X_k^{e_{i-1}}))^2} \middle| \mathcal{F}_n\right] \\ &= \sum_{i=0}^3 \mathbb{E}\left[\sum_{k=X_n^{e_i}}^{X_S^{e_i}-1} \frac{1}{W(k)^2} \middle| \mathcal{F}_n\right] \geq \alpha_n \mathbb{P}(\mathcal{A}_\infty \cap \{S = \infty\} | \mathcal{F}_n). \end{aligned} \quad (11)$$

However, $\kappa_S = 0$ on $\{S = \infty\} \cap \mathcal{A}_\infty$, also $|\kappa_S| = |\kappa_\infty| \leq \varepsilon \sqrt{\alpha_n}$ on $\{S = \infty\}$ and, on $\{S < \infty\}$, $|\kappa_S| \leq (1 + \varepsilon) \sqrt{\alpha_n}$ since the over(under)shoot of κ at time S is bounded by a term of the type $1/W(l)$ for some random $l \geq X_n^*$, so in particular it is bounded by $\sqrt{\alpha_n}$. Hence

$$\mathbb{E}((\kappa_S)^2 | \mathcal{F}_n) \leq \mathbb{E}((\kappa_S)^2 1_{\{S < \infty\} \cup \mathcal{A}_\infty^c} | \mathcal{F}_n) \leq (1 + \varepsilon)^2 \alpha_n \mathbb{P}(\{S < \infty\} \cup \mathcal{A}_\infty^c | \mathcal{F}_n). \quad (12)$$

Letting $p := \mathbb{P}(\mathcal{A}_\infty \cap \{S = \infty\} | \mathcal{F}_n)$, we conclude by combining inequalities (11) and (12) that $p \leq (1 + \varepsilon)^2(1 - p)$, or equivalently

$$\mathbb{P}(\mathcal{A}_\infty \cap \{S = \infty\} | \mathcal{F}_n) = p \leq (1 + \varepsilon)^2 / (1 + (1 + \varepsilon)^2) < 1, \quad (13)$$

almost surely.

It will be convenient to let $\varepsilon = 5$. Then note that the shifted process $(\kappa_{S+k}, k \geq 0)$ is again a martingale with respect to the filtration $\tilde{\mathcal{F}}_k := \mathcal{F}_{S+k}$. Moreover, due to (9), we have that

$$\mathbb{E}((\kappa_\infty - \kappa_S)^2 | \mathcal{F}_S) \leq 4\alpha_S \leq 4\alpha_n,$$

so that by the Markov inequality, a.s. on $\{S < \infty\}$,

$$\mathbb{P}(\mathcal{A}_\infty | \mathcal{F}_S) \leq \mathbb{P}(|\kappa_\infty - \kappa_S| > 5\sqrt{\alpha_n} | \mathcal{F}_S) \leq \frac{4\alpha_n}{25\alpha_n} = \frac{4}{25},$$

thus

$$\mathbb{P}(\mathcal{A}_\infty^c | \mathcal{F}_n) \geq \mathbb{E}[\mathbb{P}(\mathcal{A}_\infty^c | \mathcal{F}_S) 1_{\{S < \infty\}} | \mathcal{F}_n] \geq \frac{21}{25} \mathbb{P}(S < \infty | \mathcal{F}_n).$$

Note that (13) now implies

$$\mathbb{P}(\mathcal{A}_\infty^c | \mathcal{F}_n) \left(1 + \frac{25}{21}\right) \geq \mathbb{P}(\mathcal{A}_\infty^c | \mathcal{F}_n) + \mathbb{P}(S < \infty | \mathcal{F}_n) \geq 1 - (1+\varepsilon)^2 / (1+(1+\varepsilon)^2),$$

so finally

$$\mathbb{P}(\mathcal{A}_\infty^c | \mathcal{F}_n) \geq c,$$

almost surely for some constant $c > 0$. By the Lévy 0-1 law, we conclude that Proposition 3 holds.

In order to prove Theorem 2 we can proceed as in the previous section to show that no branching point is possible. In particular, we consider $i, j \in \{0, \dots, 3\}$ such that $j \neq i, i-1$, and events of the form

$$\{e_i \text{ and } e_{i-1} \text{ both traversed i.o.}\} \cap \{e_j \text{ not visited after time } n\},$$

for some finite n , and then use an appropriate modification of $(\kappa_k^i, k \geq n)$ that would have to converge to a particular limit on the above event, and show in turn that this convergence occurs with probability 0.

Note that again this martingale technique extends in the more general setting of even cycles \mathcal{G}_{2k} . Indeed, let $Y_n^\pm(i)$ and κ_n^i be defined as in (4) and (5) and, let

$$\kappa_n := \sum_{i=0}^{2k-1} (-1)^i \kappa_n^i + \sum_{i=0}^{2k-1} (-1)^i W^*(X_0^{e_i}).$$

As in equation (7),

$$\kappa_n = \sum_{i=0}^{2k-1} (-1)^i W^*(X_n^{e_i}),$$

so that

$$\{\text{all edges are traversed infinitely often}\} \subset \{\kappa_\infty = 0\}.$$

The study of the variances of the martingale increments explained in Section 3 yields similarly that $\mathbb{P}(\{\kappa_\infty = 0\}) = 0$. Hence, almost surely, at least one of the edges in the cycle is visited only finitely often and, as before, an adaptation of this argument implies (A).

4 Comparing square and triangle

In order to additionally motivate our interest in the evolution of edge reinforced random walk on cycles, we recall that the continuous time-line technique can be adapted in order to prove that, on any graph of bounded degree, almost surely, the strongly edge reinforced random walk either satisfies (A) or it eventually keeps traversing infinitely often all the edges of a (random) odd sub-cycle. The argument was given by Limic in [3], Section 2, using graph-based techniques. The martingale method could be used in a similar way to prove the above fact. In view of this, note that solving the attract-ing edge problem on odd cycles is necessary and sufficient for obtaining the solution on general bounded degree graphs.

The aim of this section is to explain why the continuous time-line and martingale techniques do not extend easily to the setting where \mathcal{G} is an odd cycle (e.g., a triangle).

4.1 Odd versus even in the time-line technique

The argument in the setting of even cycles relied on the existence of the non-trivial linear identity (3) involving independent continuous random variables. We are going to argue next that no such non-trivial linear relation (and in fact no non-linear smooth relation either) can hold with positive probability in the odd case.

Fix $l \geq 3$ and consider the set

$$\mathcal{X} := \{(x_k^i)_{k \in \mathbb{N}, i \in \{0, \dots, l-1\}} : \forall i \in \{0, \dots, l-1\}, \forall k \geq 0, x_k^i > 0, \sum_m x_m^i < \infty\}.$$

Given $\mathbf{x} = (x_k^i)_{k \in \mathbb{N}, i \in \{0, \dots, l-1\}} \in \mathcal{X}$, define

$$t_n^i \equiv t_n^i(\mathbf{x}) := \sum_{k=0}^n x_k^i, n \geq 0, t_\infty^i \equiv t_\infty^i(\mathbf{x}) := \sum_{k=0}^{\infty} x_k^i, i = 0, \dots, l-1.$$

After placing dots at points $t_0^i < t_1^i < \dots$ on the time-line of e_i , $i = 0, \dots, l-1$, and fixing the starting position ι_0 one can perform, as in Section 2, the time-line construction of the (now deterministic) walk, driven by \mathbf{x} , evolving in continuous time. If at any point the erasing procedure encounters more than one dot (on two or more different time-lines) simultaneously, choose to step over the edge corresponding to one of these time-lines in some

prescribed way, for example, to the one having the smallest index. Denote by $s^i := s^i(\mathbf{x}, \iota_0)$ the total time this deterministic walk spends visiting vertex i . Similarly, denote by

$$t^{e_i} = t^{e_i}(\mathbf{x}, \iota_0) := s^i(\mathbf{x}, \iota_0) + s^{i+1}(\mathbf{x}, \iota_0) \quad (14)$$

the total time that this deterministic walk spends waiting on the boundary vertices $i, i + 1$ of e_i . Of course, $t^{e_i}(\mathbf{x}, \iota_0) \leq t_\infty^i(\mathbf{x})$, where the equality holds if and only if e_i is traversed infinitely often. In the case of even l the identity

$$\sum_{j=0, j \text{ even}}^{l-1} t^{e_j}(\mathbf{x}, \iota_0) = \sum_{j=0, j \text{ odd}}^{l-1} t^{e_j}(\mathbf{x}, \iota_0), \quad \mathbf{x} \in \mathcal{X},$$

lead to (3) and was the key for showing that (A) occurs. Let

$$y \equiv y(\mathbf{x}, \iota_0) := \begin{pmatrix} s^0 \\ \vdots \\ s^{l-1} \end{pmatrix}, \quad z \equiv z(\mathbf{x}, \iota_0) := \begin{pmatrix} t^{e_0} \\ \vdots \\ t^{e_{l-1}} \end{pmatrix},$$

and

$$M^{(l)} := (\chi_{\{i, i+1\}}(j))_{0 \leq i, j \leq l-1},$$

where χ_B denotes a characteristic function of a set B , and the addition is done modulo l , for instance

$$M^{(5)} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then (14) states that

$$z(\mathbf{x}, \iota_0) = M^{(l)} y(\mathbf{x}, \iota_0), \quad \mathbf{x} \in \mathcal{X}.$$

Note that the determinant $\det(M^{(l)}) = 1 - (-1)^l$ can easily be computed explicitly since $M^{(l)}$ is a circular matrix. Hence $M^{(l)}$ is a regular matrix if and only if l is odd. Therefore, for odd l and fixed ι_0 , a nontrivial identity

$$\beta \cdot z(\mathbf{x}, \iota_0) = c, \quad \mathbf{x} \in \mathcal{X}, \quad (15)$$

for some $\beta \in \mathbb{R}^l \setminus \{0\}$, $c \in \mathbb{R}$, holds if and only if,

$$\beta' \cdot \mathbf{y}(\mathbf{x}, \iota_0) = c, \quad \mathbf{x} \in \mathcal{X}, \quad (16)$$

where $\beta' = (M^{(l)})^\tau \beta \in \mathbb{R}^l$ is again $\neq 0$, since the transposal of $M^{(l)}$ is also a regular matrix.

Now (16) cannot hold identically on \mathcal{X} , and we are about to show a somewhat stronger statement. Let $\mathbf{x} \in \mathcal{X}$ and fix some $r \in (0, \infty)$. Then for $j \in \{0, \dots, l-1\}$, let $\eta_r^j(\mathbf{x}) \equiv \tilde{\mathbf{x}}^{(j)} := (x_k^{i,(j)})_{k \in \mathbb{N}, i=0, \dots, l-1} \in \mathcal{X}$ be defined as follows: if $k \geq 0$, $i \in \{0, \dots, l-1\}$,

$$\tilde{x}_k^{i,(\iota_0)} := x_k^i + r \chi_{\{(\iota_0,0),(\iota_0-1,0)\}}((i,k)),$$

while for $j \neq \iota_0$, if the walk driven by \mathbf{x} visits site j for the first time by traversing e_{j-1} , let

$$\tilde{x}_k^{i,(j)} := x_k^i + r \chi_{\{(j,0),(j-1,1)\}}((i,k)),$$

otherwise let

$$\tilde{x}_k^{i,(j)} := x_k^i + r \chi_{\{(j-1,0),(j,1)\}}((i,k)). \quad (17)$$

Note that (17) comprises also the case where the walk driven by \mathbf{x} never visits site j .

Now we will modify the edge reinforced random walk by delaying the first jump out of a particular site j by some positive amount r , without changing anything else in the behaviour. Informally, the law of the modified version will be absolutely continuous with respect to the law of the original, and this will lead to a contradiction.

More precisely, for each fixed j , consider the two (deterministic time-continuous) walks: the original one that is driven by \mathbf{x} , and the new one that is driven by the transformed family $\eta_r^j(\mathbf{x})$. It is easy to check that either neither of the walks visits site j , or they both do. In the latter case, if we denote respectively by $a(\mathbf{x})$ and $a(\eta_r^j(\mathbf{x}))$ the amount of time they spend at site j before leaving, then $a(\eta_r^j(\mathbf{x})) = a(\mathbf{x}) + r$. Everything else in the

evolution of the two walks is the same. In particular, if the walk driven by \mathbf{x} ever visits j , then

$$s^j(\eta_r^j(\mathbf{x}), \iota_0) = s^j(\mathbf{x}, \iota_0) + r, \text{ and } s^i(\eta_r^j(\mathbf{x}), \iota_0) = s^i(\mathbf{x}, \iota_0), i \neq j. \quad (18)$$

Now one can simply see that if the walk driven by \mathbf{x} visits all sites at least once, then in any open neighborhood of \mathbf{x} , any identity (16) breaks due to points $\eta_r^j(\mathbf{x})$ contained in it for sufficiently small positive r .

Recall the setting of Section 2, and to simplify the notation, assume $X_0^{e_i} = 0$, $i = 0, \dots, l-1$, and specify the initial position $\tilde{I}_0 = v_0 \in \{0, \dots, l-1\}$. The point of the above discussion is that the random walk \tilde{I} is then the walk driven by a \mathcal{X} -valued random family $\mathbf{E} = (E_k^i)_{k \in \mathbb{N}, i=0, \dots, l-1}$, where the random variables E_k^i , $k \geq 0$ are independent exponentials, as specified in Section 2. If

$$A_{all} := \{\tilde{I} \text{ traverses all edges infinitely often}\},$$

then $A_{all} = \{\mathbf{E} \in \mathbf{A}_{all}\}$, where \mathbf{A}_{all} contains all $\mathbf{x} \in \mathcal{X}$ such that the deterministic walk driven by \mathbf{x} traverses all edges infinitely often. It is natural to ask whether one or more (up to countably many, note that this would still be useful) identities of the form (15) hold on \mathbf{A}_{all} , almost surely. For l even, we know that the answer is affirmative. For l odd, this is equivalent to asking whether one or more (up to countably many) identities of the form (16) hold on \mathbf{A}_{all} , almost surely. Assume

$$\mathbb{P}(A_{all} \cap \{\beta' \cdot y(\mathbf{E}, v_0) = c\}) = p(\beta', c) > 0, \quad (19)$$

for β', c as in (16). Take j such that $\beta'_j \neq 0$ (at least one such j exists). We will assume that $j \neq v_0$, the argument is somewhat similar and simpler otherwise. Denote by $\mathbf{A}_1^{j,-} \subset \mathcal{X}$ the set of all \mathbf{x} such that the walk driven by \mathbf{x} visits j for the first time by traversing e_{j-1} , and let $A_1^{j,-} = \{\mathbf{E} \in \mathbf{A}_1^{j,-}\}$. In view of (19), without loss of generality, we may assume

$$\mathbb{P}(A_{all} \cap A_1^{j,-} \cap \{\beta' \cdot y(\mathbf{E}, v_0) = c\}) \geq p(\beta', c)/2, \quad (20)$$

As a consequence of the earlier discussion in the deterministic setting we have $A_1^{j,-} = \{\mathbf{E} \in \mathbf{A}_1^{j,-}\} \subset \{\eta_r^j(\mathbf{E}) \in \mathbf{A}_1^{j,-}\}$, and

$$A_{all} \cap \{\beta' \cdot y(\mathbf{E}, v_0) = c\} \subset \{\eta_r^j(\mathbf{E}) \in \mathbf{A}_{all}\} \cap \{\beta' \cdot y(\eta_r^j(\mathbf{E}), v_0) = c + r\beta'_j\},$$

almost surely. Therefore,

$$\mathbb{P}(\{\eta_r^j(\mathbf{E}) \in \mathbf{A}_{all} \cap \mathbf{A}_1^{j,-}\} \cap \{\beta' \cdot y(\eta_r^j(\mathbf{E}), v_0) = c + r\beta'_j\}) \geq p(\beta', c)/2. \quad (21)$$

However, E_k^i are continuous and independent random variables, each taking values in any interval $(a, b) \subset (0, \infty)$, $a < b \leq \infty$ with positive probability. Moreover, since E_0^j and E_1^{j-1} are exponential (rate $W(0)$ and $W(1)$, respectively), one can easily verify that for any cylinder set $\mathbf{B} \subset \mathcal{X}$,

$$\begin{aligned} \mathbb{P}(\eta_r^j(\mathbf{E}) \in \mathbf{B} \cap \mathbf{A}_1^{j,-}) &= \\ \mathbb{P}(\{E_i^k + r\chi_{\{(j,1),(j-1,0)\}}((i,k)), k \geq 0, j = 0, \dots, l-1\} \in \mathbf{B} \cap \mathbf{A}_1^{j,-}) & \\ \leq e^{r(W(0)+W(1))} \mathbb{P}(\mathbf{E} \in \mathbf{B} \cap \mathbf{A}_1^{j,-}). & \end{aligned} \quad (22)$$

Now (22) and (21) imply

$$\mathbb{P}(\{\mathbf{E} \in \mathbf{A}_{all}\} \cap \{\beta' \cdot \mathbf{y}(\mathbf{E}, v_0) = c + r\beta'_j\}) \geq e^{-r(W(0)+W(1))} p(\beta', c)/2, \quad (23)$$

and this, together with (19), leads to a contradiction, since adding (23) over all rational $r \in (0, 1)$ would imply $\mathbb{P}(A_{all}) = \mathbb{P}(\mathbf{E} \in \mathbf{A}_{all}) = \infty$.

In the absence of a convenient linear identity (15), the reader might be tempted to look for non-linear ones. Yet, the last argument can be extended to a more generalized setting where (16) is replaced by

$$\mathbf{y}(\mathbf{x}, \iota_0) \in M, \quad \mathbf{x} \in \mathcal{X}, \quad (24)$$

for some $l-1$ -dimensional differentiable manifold $M \subset \mathbb{R}^l$. In particular, this includes the case where $F(\mathbf{y}(\mathbf{x}, \iota_0)) = 0$, $\mathbf{x} \in \mathcal{X}$, for some smooth function F with non-trivial gradient (see, for example, [10] Theorem 5-1). Indeed, assume that, in analogy to (19),

$$\mathbb{P}(A_{all} \cap \{\mathbf{y}(\mathbf{E}, v_0) \in M\}) > 0. \quad (25)$$

Then, since $\mathbf{y}(\mathbf{E}, v_0)$ is a finite random vector, due to the definition of differential manifolds (cf. [10] p. 109), there exists a point $x \in M$, two bounded open sets $U \ni x, V \subset \mathbb{R}^l$, and a diffeomorphism $h : U \rightarrow V$ such that

$$\mathbb{P}(A_{all} \cap \{\mathbf{y}(\mathbf{E}, v_0) \in M \cap U\}) =: p(M, U) > 0, \quad (26)$$

and

$$h(U \cap M) = V \cap (\mathbb{R}^{l-1} \times \{0\}) = \{\mathbf{v} \in V : v_l = 0\}.$$

Denote by \mathbf{e}_j the j th coordinate vector in \mathbb{R}^l . Then (26) can be written as

$$\mathbb{P}(A_{all}, \mathbf{y}(\mathbf{E}, v_0) \in U, h(\mathbf{y}(\mathbf{E}, v_0)) \cdot \mathbf{e}_l = 0) = p(M, U).$$

As a consequence of the Taylor decomposition, for all $j \in \{0, \dots, l-1\}$, for any $\mathbf{u} \in U$ and for all small r ,

$$h(\mathbf{u} + r\mathbf{e}_j) \cdot \mathbf{e}_l = h(\mathbf{u}) \cdot \mathbf{e}_l + r Dh(\mathbf{u}) \mathbf{e}_j \cdot \mathbf{e}_l + \text{err}(\mathbf{u}, j, r), \quad (27)$$

where for each $\mathbf{u} \in U$, $Dh(\mathbf{u})$ is the differential operator of h at \mathbf{u} , and where the error term $\text{err}(\mathbf{u}, j, r) = o(r)$ as $r \rightarrow 0$. Since h is a diffeomorphism, given any $\mathbf{u} \in U$, there exists a $j \equiv j(\mathbf{u}) \in \{0, \dots, l-1\}$ such that $Dh(\mathbf{u}) \mathbf{e}_{j+1} \cdot \mathbf{e}_l \neq 0$. Therefore (26) implies that, for some $j \in \{0, \dots, l-1\}$,

$$\mathbb{P}(A_{all}, \mathbf{y}(\mathbf{E}, v_0) \in M \cap U, Dh(\mathbf{y}(\mathbf{E}, v_0)) \mathbf{e}_{j+1} \cdot \mathbf{e}_l > 0) \geq \frac{p(M, U)}{2l}, \quad (28)$$

or

$$\mathbb{P}(A_{all}, \mathbf{y}(\mathbf{E}, v_0) \in M \cap U, Dh(\mathbf{E}(\mathbf{x}, \iota_0)) \mathbf{e}_{j+1} \cdot \mathbf{e}_l < 0) \geq \frac{p(M, U)}{2l}.$$

Without loss of generality, suppose (28) and choose $c, d \in (0, \infty)$, $c < d$, and $\delta = \delta(c) > 0$ such that

$$\begin{aligned} \mathbb{P}(A_{all}, \mathbf{y}(\mathbf{E}, v_0) \in M \cap U, Dh(\mathbf{y}(\mathbf{E}, v_0)) \mathbf{e}_{j+1} \cdot \mathbf{e}_l \in (c, d), \\ \sup_{r \in (0, \delta)} |\text{err}(\mathbf{y}(\mathbf{E}, v_0), j+1, r)|/r \leq c/2) \geq \frac{p(M, U)}{4l}. \end{aligned} \quad (29)$$

Consider the modified processes $\eta_r^j(\mathbf{E})$, $r > 0$, corresponding to this j , and note that $\mathbf{y}(\eta_r^j(\mathbf{E}), v_0) = \mathbf{y}(\mathbf{E}, v_0) + r\mathbf{e}_{j+1}$. Now, due to (27) and (29), one can choose a decreasing sequence $(r_m)_{m=1}^\infty$ of positive numbers converging to 0, so that the intervals defined by $J(r_m) := (cr_m/2, dr_m + cr_m/2)$, for each $m \geq 1$, are mutually disjoint (i.e., $J(r_m) \cap J(r_{m'}) = \emptyset$ for $m < m'$) and such that

$$\mathbb{P}(\eta_{r_m}^j(\mathbf{E}) \in \mathbf{A}_{all}, h(\mathbf{y}(\eta_{r_m}^j(\mathbf{E}), v_0)) \cdot \mathbf{e}_{j+1} \in J(r_m)) \geq \frac{p(M, U)}{4l},$$

hence

$$\mathbb{P}(\mathbf{E} \in \mathbf{A}_{all}, h(\mathbf{y}(\mathbf{E}, v_0)) \cdot \mathbf{e}_{j+1} \in J(r_m)) \geq e^{-r_m(W(0)+W(1))} \frac{p(M, U)}{4l}.$$

As in the linear case, one arrives to a contradiction.

4.2 Odd versus even in the martingale technique

The reason why the martingale technique fails on odd cycles is similar: there is no non-trivial martingale that can be expressed as a linear combination of the different $W^*(X_n^{e_i})$, $i = 0, \dots, l-1$, as in identity (7). Indeed, let us fix a time $n \in \mathbb{N}$ and let, for all $i \in \mathbb{Z}/l\mathbb{Z}$,

$$\begin{aligned} y_i &:= \mathbb{E}(Y_{n+1}^+(i) - Y_n^+(i) | \mathcal{F}_n) = \mathbb{E}(Y_{n+1}^-(i) - Y_n^-(i) | \mathcal{F}_n), \\ z_i &:= \mathbb{E}(W^*(X_{n+1}^{e_i}) - W^*(X_n^{e_i}) | \mathcal{F}_n), \end{aligned}$$

$$Y_n := \begin{pmatrix} y_0 \\ \vdots \\ y_{l-1} \end{pmatrix}, \quad Z_n := \begin{pmatrix} z_0 \\ \vdots \\ z_{l-1} \end{pmatrix}.$$

Then, for all $0 \leq i \leq l-1$,

$$z_i = y_i + y_{i+1},$$

since $W^*(X_{n+1}^{e_i}) = Y_n^+(i) + Y_n^-(i+1)$. This implies again that, almost surely,

$$Z_n = M^{(l)} Y_n, \quad n \geq 1.$$

Suppose there is a fixed vector $\beta \in \mathbb{R}^l$ such that the dot product βY_n equals 0, almost surely, for all n . Since, at each time step $n \in \mathbb{N}$, Y_n has (almost surely) only one non-zero coordinate, namely, $y_i > 0$ for $i = I_n$ and $y_j = 0$ for $j \neq I_n$, and since the walk visits each and every vertex at least once with positive probability, we see that β is necessarily the null-vector. As before, if l is odd, $M^{(l)}$ is a regular matrix, and therefore no martingale can be expressed as a non-trivial deterministic linear combination of the different $W^*(X_n^{e_i})$, $i = 0, \dots, l-1$.

However, we show in [4] that, for all $i = 0, \dots, l-1$, if t_n^i is the n -th return time to the vertex i , the process $W^*(X_{t_n^i}^{e_i}) - W^*(X_{t_n^i}^{e_{i-1}})$ approximates a martingale. The accuracy of this approximation depends on the regularity of the weight function W , hence our argument requires technical assumptions on W . In particular, the main theorem in [4] implies (A) for strongly edge reinforced random walks, where W is nondecreasing.

Even though the time-lines technique is simpler in general, one cannot adapt it similarly, since it uses the independence of random variables and is therefore unstable with respect to small perturbation.

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