Interacting Particle Systems

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Chapter 1

INTRODUCING IPS

Interacting particle systems (IPS for short) is a large and active field of probability theory devoted to the rigorous study of certain models composed of a large or infinite number of particles living on a lattice and evolving as a whole as a Markov process.
The field started at the end of the 1960's with seminal works by F.Spitzer and R.L.Dobrushin, and by 1975 four classic models had already been introduced: the stochastic Ising model, the contact process, the exclusion process, and the voter model. The first book on IPS, which is still one of the reference books was written by T.Liggett in 1985 [Lig85]. The original motivation for the field came from statistical mechanics, a branch of mathematical physics that studies the the collective behavior of systems composed of a large number of particles (atoms, molecules, droplets, grains...) by using a probabilistic model encoding the microscopic interactions with the goal to understand the macroscopic laws (see [FV17] for an excellent introduction to the topics). A key idea in statistical mechanics is indeed, though real systems evolve according to deterministic laws, a stochastic description is well suited in the presence of a large number of microscopic components, due to the fact that following the deterministic laws is impossible and the knowledge of the initial configuration inevitably contains some alea. A central object in statistical mechanics is the *Gibbs distribution* that provides the probability of observing a certain microscopic state of the system when it is in equilibrium at a certain temperature. Studying the properties of this distribution when the system size is very large (or infinite) as well as numerically sampling configurations according to this distribution is often a particularly hard task especially in presence of a *phase transition*, namely when for the physical system an abrupt change of the macroscopic behavior occurs for a small change of a control parameter (e.g. the temperature). This phenomenon, from the statistical mechanics view point, corresponds to an emergence of long range correlations in the probability distribution when approaching the

transition and to the non-uniqueness of the Gibbs distribution.

After these pioneering works the field of IPS rapidly expanded with the introduction of many other models. It was rapidly understood that these models, besides allowing to sample the equilibrium Gibbs measure, can also be used to *model the evolution of physics systems out of equilibrium* either in the pre-asymptotic regime of approach to equilibrium or for systems constantly driven out of equilibrium (e.g. by their boundary conditions). Furthermore, IPS rapidly turned out to have interesting applications as *models of collective complex behavior* in many other fields besides physics, including *biology* (models for spread of infections), *social sciences* (e.g. opinion dynamics models) and *economics*.

To summarize, IPS were born as auxiliary models in the framework of statistical mechanics and rapidly evolved as an independent field at the border among probability theory and various fields of applied mathematics. Though the theory of Markov processes was already well established when IPS were introduced in the '70s, studying IPS turned out to be particularly challenging. Indeed, issues that arise for IPS due to the presence of a large or infinite number of particles needed developing new tools and the field is still evolving today with many beautiful issues for several paradigmatic models being still open.

In these lectures my aim is to give an introduction to IPS. We will start by constructing the processes, then we will focus on two classic models: the stochastic Ising model and the contact process. Studying these two models we will have the occasion to meet some of the tools that have been developed for IPS in particular coupling and duality, and to review the basic questions: determining the large time behavior, the invariant laws and their domain of attraction, the speed of convergence to equilibrium. Many important facets of the IPS field will not be covered by this mini-course. A crucial missing part are *scaling limits*, which link the evolution of the microscopic discrete stochastic IPS with some macroscopic continuous equations. These are either PDE or stochastic PDE depending on whether one is looking at the law of large numbers or the central limit theorem scaling. This part has been in particularly very much developed for the so called exclusion type IPS, those for which particles are locally conserved, namely elementary moves correspond to jump of particles. The interested reader may have a look at the classic books [Spo91] or [KL99] to have an idea of the vastness of this subject.

1.1 An informal definition of the most popular IPS

1.1 Notation

To define an IPS we have to choose a (finite or infinite) lattice, namely a countable vertex set Λ and edge set E, and a finite local (on site) state space S and to specify the rates at which transition occurs from two different configuration in the state space $X := S^\Lambda$. The dynamics follows a Markov process whose elementary moves will always correspond to the modification of the configuration on a finite number of sites and the rate at which they occur depend on the configuration on a certain finite neighbourhood of the to-be-updated sites [1].

We will use the greek letters σ and η to denote configurations, i.e. elements of X and denote by $(\eta_t)_{t \geqslant 0}$ the Markov process on the space S^Λ. Given a lattice site $x \in \Lambda$ and configuration η, we denote by $\eta(x)$ the configuration at site x. We also adopt the notation $x \sim y$ to say that $|x-y| = 1$, or there is an edge on the graph from x to y. In these notes we will deal always with the case in which S contains only two possible states, more precisely $S = \{0,1\}$ (or $S = \{\pm 1\}$ for the stochastic Ising model) and $\Lambda \subset \mathbb{Z}^d$. For $S = \{0,1\}$ we denote by δ_0, δ_1 the measures concentrated on the configuration in which all sites have occupation variable equal to 0 and to 1, respectively. Analogous definition for the measures δ_+ and δ_- when $S = \{+,-\}$. If $|\Lambda|$ is finite, saying that the transition $\eta \to \eta'$ with $\eta, \eta' \in \Omega$ occurs at rate $r(\eta, \eta')$ means that, when $t \downarrow 0$, it holds

$$P(\eta_t = \eta' | \eta_0 = \eta) = r(\eta, \eta') t + o(t).$$

If $|\Lambda|$ is infinite, the probability for a specific configuration to occur is typically zero, and one gets informally the intuitive meaning of rate by replacing on the left hand side with $P(\eta_t|_V = \eta'_V | \eta_0 = \eta)$ with $V \subset \Lambda$ and $|V|$ large but finite.

We say that a measure μ is an *invariant law* for an IPS if, when the system at time zero is distributed according to μ, then at all later times it is also distributed according to μ. A more precise definition will be given in chapter 3. In order to define the models we need a few more definitions

Definition 1.1.1 (Exponential variable). *We say that X is an exponential variable of parameter c with $c \in \mathbb{R}^+$ is X is a real positive random variable with cumulative distribution $F(x) = 1 - e^{-cx}$ which yields a mean value $\mathbb{E}(X) = c^{-1}$. For $T \in (0, \infty)$ we will use the short notation an exponential time of mean T to denote an exponential variable of parameter $1/T$.*

[1] Note that in general, though the whole process is Markovian, the single particle evolution is not (due to interactions).

Definition 1.1.2 (Flipped configuration and exchanged configuration)**.** *Let the onsite configuration space be $S = \{0, 1\}$ and fix a configuration $\eta \in X$ and a site $x \in \Lambda$. Then we call configuration η flipped at x the configuration $\eta^x \in X$ defined as follows*

$$\eta^x(y) = \begin{cases} 1 - \eta(y) & \text{if } y = x \\ \eta(y) & \text{if } y \neq x \end{cases} \tag{1.1.1}$$

If instead $S = \{-1, +1\}$ we let

$$\eta^x(y) = \begin{cases} -\eta(y) & \text{if } y = x \\ \eta(y) & \text{if } y \neq x \end{cases} \tag{1.1.2}$$

For $S = \{0, 1\}$ pr $S = \{\pm 1\}$, $\eta \in X$ and $x, y \in \Lambda$ we also call configuration η exchanged at x, y the configuration $\eta^{x,y}$ defined as follows

$$\eta^{xy}(z) = \begin{cases} \eta(y) & \text{if } z = x \\ \eta(x) & \text{if } z = y \\ \eta(z) & \text{otherwise} \end{cases} \tag{1.1.3}$$

1.2 Contact process (CP)

CP is a model of spread of infection. The on-site configuration space is $S = \{0, 1\}$, with 0 (resp. 1) representing healthy (resp. infected) individuals. Here

- Infected individuals become healthy after an exponential time of mean 1, independently of the others (namely the recovery times of different infected individuals are independent).
- an healthy individual at site x in configuration η becomes infected after an exponential time of mean $1/(\lambda N_x(\eta))$ with $\lambda \geqslant 0$ a parameter that is called the *infection rate* and N_x the number of infected nearest neighbours of x.

More precisely, when the system is in configuration η, it flips to η^x after an exponential time of mean $1/r(x, \eta)$ with

$$r(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1 \\ \lambda \sum_{y, y \sim x} \eta(y) & \text{if } \eta(x) = 0 \end{cases} \tag{1.1.4}$$

It is easily seen that if the initial configuration contains only healthy individuals we will always have only healthy individuals, namely δ_0 is an invariant law.

What happens if we start with some infections? do infections typically survive at later times?

The answer depends on the value of the infection rate λ and on the lattice.

If Λ is finite, then any initial configuration is eventually attracted to δ_0, which is the unique invariant law. This is an easy consequence of the finiteness of the state space and of the irreducibility of the dynamics (revise finite state Markov chain theory).

If instead $S = \mathbb{Z}^d$ CP on $\Lambda = \mathbb{Z}^d$ undergoes a phase transition, namely there exists $\lambda_c(d)$ depending on d and with $0 < \lambda_c(d) < \infty$ such that

- for $\lambda < \lambda_c(d)$: δ_0 is the unique invariant measure and all initial measures are attracted to δ_0;
- for $\lambda > \lambda_c(d)$: there are other invariant measures besides δ_0. We shall see that an important role is played by the measure towards which the process is attracted starting from a configuration in which all individuals are infected which in this regime does not coincide with δ_0;

In the following lectures we will prove the above results and furthermore we will prove that λ_c is also the value at which the infection survival probability starting from a single infected sites starts to be positive, namely $\theta(\lambda) = 0$ for $\lambda < \lambda_c$ and $\theta(\lambda) > 0$ for $\lambda > \lambda_c$ where

$$\theta(\lambda) := \mathbb{P}_{\bar{\eta}}[\eta_t \neq \vec{0} \ \forall t \geqslant 0]$$

with $\bar{\eta}$ the configuration which is 1 in the origin and zero elsewhere and $\vec{0}$ the configuration which is 0 on all sites. Furthermore the function θ is continuous, strictly increasing and concave. Proving these statements is not easy, for example proving continuity at λ_c, namely $\theta(\lambda_c) = 0$ was proved only in 1990 almost twenty years after the model was introduced. Concerning the value of the (dimension dependent) λ_c the only avaialble results are upper and lower bounds.

1.3 Voter model (VM)

VM is a model of opinion spread. Again, $S = \{0, 1\}$, and here 0 and 1 represent voters for two different parties, say 0 is a republican voter, 1 a democrat voter. The dynamics evolves as follows: after an exponential time of mean 1, the voter at site x chooses uniformly at random one of its neighbours and adopts its opinion.

More precisely, when the system is in configuration η, it flips to η^x after an exponential time of mean $1/r(x, \eta)$ with

$$r(x, \eta) = \frac{1}{2d} \sum_{y, y \sim x} 1_{\eta(y) \neq \eta(x)} \tag{1.1.5}$$

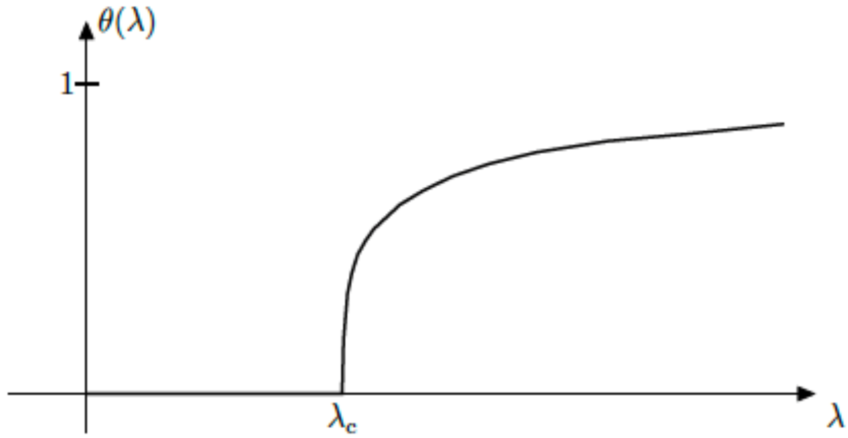

Figure 1.1: Survival probability of CP in dimension 1 as a function of the infection probability.

From the above definition it follows immediately that for VM both δ_0 and δ_1 are invariant measures.

What happens if $\Lambda = \mathbb{Z}^d$ and we start from a mixture of opinions? can we preserve a mixture of opinions or are we deemed to a totalitarian situation?

Here the answer strongly depends on the spatial dimension

- for $d = 1, 2$: δ_0 and δ_1 are the only two extremal invariant measures : the process is attracted to a single opinion state;
- for $d \geqslant 3$ there is a whole family of extremal invariant measures (that are ergodic under translations): a mixture of opinions can survive.

1.4 The Stochastic Ising model (SIM)

SIM is a dynamical version of the Ising model (the celebrated model for magnetism that has been introduced in 1925 by Ising to model ferromagnetic material) which was introduced in 1963 by Glauber and very much studied since the seminal works of Dobrushin in the 1970s. The usual convention is to let the onsite space state be $S = \{+1, -1\}$ [2]. Here sites represents atoms in a ferromagnetic material, e.g. iron, and ± 1 are the two possible orientations (up and down) of the spin on each atom. The elementary moves of the dynamics are spin flips and the rates are chosen to take into account the fact that a spin "prefers" to be aligned with its nearest

[2] One can of course easily rephrase the model on the state space $S = \{0, 1\}$

neighbours. More precisely the spin at site x in configuration η it flips to η^x after an exponential time of mean $1/r(x,\eta)$ with

$$r(x,\eta) = e^{-\beta \sum_{y\sim x} \eta(x)\eta(y)} \equiv e^{-2d\beta+2\beta \tilde{N}_x(\eta)}$$

where $\tilde{N}_x(\eta)$ is the number of spins n.n. to x whose spin is not aligned with x and β is a positive constant is called inverse temperature since in the physical interpretation it corresponds to J/kT with k the Boltzmann's constant, J the energy difference between aligned and not aligned spins and T the temperature. Notice that

- the larger β, the strongest the bias to align spins
- the higher the number of non aligned neighbours, the highest the flip rate
- if $\beta = 0$ (= infinite temperature) SIM is an independent spin dynamics with a unique invariant measures, the product measure with $\mu_x(+1) = \mu_x(-1) = 1/2$
- δ_{+1} and δ_{-1} are no more invariant laws.

We will see that the invariant measures of SIM coincide with the Gibbs measure of the Ising model and

- in $d = 1$ SIM has a unique invariant measure ;
- for $d \geqslant 2$ there exists $\beta_c(d)$ with $0 < \beta_c(d) < \infty$ separating the regime ($\beta < \beta_c$) in which we have a unique invariant measure and the regime ($\beta > \beta_c$) in which uniqueness is broken. This corresponds to the ferromagnetic/paramagnetic phase transition in real materials;
- if we define the spontaneous magnetization, $m(\beta)$ as $m(\beta) := \lim_{t\to\infty} \mathbb{E}_+(\eta_t(0))$ where $\mathbb{E}_+$ is the expectation under the process started from the up configuration (i.e. from η s.t. $\eta(x) = 1$ for all $x \in \Lambda$), it holds $m(\beta) = 0$ for all β in $d = 1$ and for $\beta \leqslant \beta_c$ for $d \geqslant 2$, while $m(\beta) > 0$ for $\beta > \beta_c$. Furthermore when $m(\beta) = 0$ there is no long range order in the large time limit, namely the correlation bvetween the value of the spin at site $0i$ and j thends to zero as the distance from i and j goes to infinity.

An alternative interpretation of the Ising model is as a model for collective decision making. Each site is a person that has to decide his (binary) state. It does so according to a utility function: if we set $\beta > 0$ it is more advantageous to make the same choice as the neighbour we take, instead for $\beta < 0$ it is more advantageous to make the opposite decision. In the physics interpretation the choice $\beta < 0$ is also meaningful: it models antiferromagnetic materials.

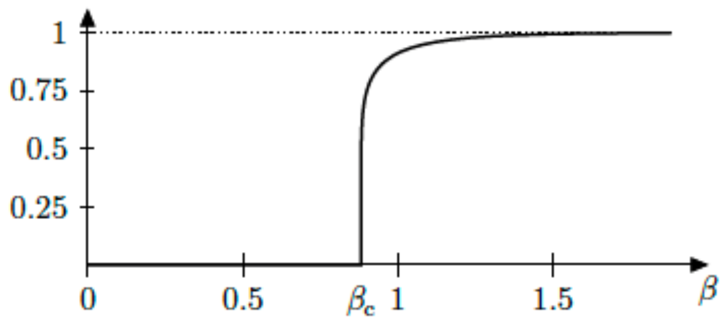

Figure 1.2: Spontaneous magnetization of SIM in dimension 2 as a function of the inverse temperature.

1.5 Friedrickson-Andersen 1 spin facilitated model (FA-1f) and other KCM

FA-1f is an IPS used to model the liquid glass transition that occurs for rapidly cooled liquids when we approach the dynamical arrest to the amorphous solid glass state. Here $S = \{0, 1\}$: 0 represents facilitating sites, i.e. regions that are not dense and thus facilitate motion, 1 represent highly packed regions. The dynamics evolves as follows: each site waits the ring of an exponential clock of mean time one and then "tries" to update its value. I say "tries", because when the clock on site x rings, before updating the configuration at x we have to check whether a certain *local constraint* is satisfied: at least 1 of the nearest neighbours of x should be empty. Then

- if the constraint is satisfies the configuration at x is updated to 0 at rate q and to 1 at rate $1 - q$ and we go to the next clock ring
- otherwise no update occurs and we go to the next clock ring

More precisely the spin at site x in configuration η it flips to η^x after an exponential time of mean $1/r(x, \eta)$ with

$$r(x, \eta) = c_x(\eta)(q\eta(x) + (1 - q)(1 - \eta(x))) \tag{1.1.6}$$

with

$$c_x(\eta) = (1 - \prod_{y, y \sim x} \eta(y)) \tag{1.1.7}$$

Notice that

- the rate to update the configuration on a given site does not depend in the configuration on that site, but only on the state of its neighbours (at variance with SIM);
- the completely filled configuration is blocked, so δ_1 is an invariant measure;
- the completely empty configuration is not an invariant measure (unless $q = 1$);

FA-1f model belongs to a class of IPS called the *kinetically constrained models or KCM*. These can be obtained by varying the choice of the constraint that allows the update (changing the neighbourhood, changing the threshold value..). The only requirement is that the constraint has finite range and does not depend on the configuration on the to-be-updated site. For example, two other very much studied KCM are

- the East model on $\mathbb{Z}$ for which the constraint to update x requires $x + 1$ to be empty
- the FA-2f model on $\mathbb{Z}^d$ with $d \geqslant 2$ for which the constraint to update x requires at least 2 empty nearest neighbours.

In formulas, the rate at which η flips to η^x is given by (1.1.6) albeit with

$$c_x(\eta) = (1 - \eta(x+1)) \quad \text{for East} \tag{1.1.8}$$

and

$$c_x(\eta) = \begin{cases} 1 & \text{if } \sum_{y:y\sim x}(1-\eta_x) \geqslant 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{for FA-2f} \tag{1.1.9}$$

We will see in Chapter 3 that FA-1f and all other KCM have, beside the trivial measure concentrated on the completely filled configuration, another invariant measure which depends on q but not on the constraints.

1.6 Simple Symmetric Exclusion process (SSEP)

SSEP is a models in which particles can move and never disappear (it is conservative). $S = \{0, 1\}$, 1 are particles and 0 are empty sites. After an exponential time of mean 1, a particle chooses uniformly at random a nearest neighbours and "tries" to jump there. I say "tries" because it has to check whether the arrival site is empty (i.e. to satisfy the exclusion constraint). If it is the case the jump occurs, otherwise the particle does not change position. Namely, η is updated to η^{xy} after an exponential time of mean $1/r(x, y, \eta)$ with

$$r(x,y,\eta) = \begin{cases} \frac{1}{2d}\mathbb{1}_{\eta(x)\neq\eta(y)} & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases} \tag{1.1.10}$$

Here

- δ_0 and δ_1 are invariant measures;
- for any density parameter $\rho \in [0,1]$ the Bernoulli(ρ) product measure

$$\pi_\rho := \prod_i \rho^{\eta(i)}(1-\rho)^{1-\eta(i)}$$

  is an invariant measure.

The name of this model comes from the following features:

- simple = jumps to nearest neighbours;
- symmetric= equal rate to jump to any of the empty nearest neighbours;
- exclusion : occupancy by a multiple number of particle is not allowed

Several variations of SSEP have been considered: long jumps, non symmetric rates (ASEP), totally asymmetric rates (TASEP), multiple occupancy ...

1.7 Other notable examples

- Potts model. The onsite state space is $S = \{1, 2 \dots q\}$ with $q \geqslant 2$. Here each site $x \in \Lambda$ at rate one updates its occupation variable. The new value, s' is chosen in S with probability

$$\frac{e^{\beta N_{x,s'}(\eta)}}{\sum_{s \in S} e^{\beta N_{x,s'}(\eta)}}$$

  where $\beta \in \mathbb{R}$ is a fixed parameter called (as for SIM) inverse temperature, and for any triple $s \in S, x \in \Lambda, \eta \in X$ we let $N_{x,s}(\eta)$ be the number of sites nearest neighbours of x which, in the configuration η, have occupation variable equal to s. Here in dimension $d \geqslant 2$ there exists $\beta_c(d)$ with $0 < \beta_c(d) < \infty$ such that for $\beta \leqslant \beta_c$ there is a unique invariant measure while for $\beta \geqslant \beta_c(d)$ there are q invariant measures.

- Biased voter model. Here $S = \{0, 1\}$ and the move $1 \to 0$ occurs with rate equal to the fraction of 0 neighbours but the move $0 \to 1$ occurs with rate $(1 + s)$ times the fraction of 1 neighbour with $s > 0$. This model is relevant as a model of evolution of two genetic types one of which (type 1) is more fit than the other and hence reproduces at a lerger rate. At rate one an organism dies and it is replaced by a clone of one of its nearest neighbour chosen randomly albeit not uniformly (there is a bias favouring type 1). Here (at variance with the standard VM) even starting with a single 1 if s is sufficiently high 1's might survive.

- another class of widely studied models are reaction diffusion models. Here $S = \{0, 1\}$ and occupied sites are called particles. Models in this class include
  - coalescing random walks. Here each particle jumps at rate one to a uniformly chosen neighbour and when two particle meets at the same site they coalesce
  - annihilating random walks. Here each particle jumps at rate one to a uniformly chosen neighbour and when two particle meets at the same site both particles die
  - branching and coalescing random walks. Here each particle jumps at rate one to a uniformly chosen neighbour and gives birth at rate $\lambda > 0$ to a new particle on a uniformly chosen nearest neighbours. When two particle meets at the same site they coalesce

Exercise 1. *Prove that for $q = 2$ the above definition of the Potts model corresponds to the definition of SIM properly time rescaled.*

IPS are easy to simulate numerically. If you want to play with some already built programs for CP, VM, SIM or POTTS you might have a look at this webpage *https://mate.dm.uba.ar/ leo-rolla/simulations/*

Remark 1.1.3. *All models considered above have the property that in the large time limit they converge to an invariant measure. This is not always the case, there are cases of IPS with periodic behavior.*

1.8 Phase transitions

Figures (1.2) and (1.4) are examples of phase transitions: an abrupt change of behavior occurs for CP and for SIM by varying smoothly the control parameter (λ and β respectively). The value at which the change of behavior occurs is called critical point. As we have explained in the sections below and we will see in detail in the following, the critical point separates a regime (for $\lambda < \lambda_c$ and $\beta < \beta_c$) in which there is a unique invariant law (which is attained at large times) from a regime in which there are more invariant laws. For both models at the critical point the order parameter ($\theta(\lambda)$ and $m(\beta)$) are continuous and there is a single invariant law. These two property correspond to a phase transition which is dubbed by physicists *second order (or* *continuous) phase transition*. The same is true for the Potts models when $q < \bar{q}(d)$ where the threshold $\bar{q}(d)$ equals 4 in dimension $d = 2$. Instead, for $q \geqslant \bar{q}(d)$ the order parameter has a jump at criticality and the model displays multiple invariant laws at this point. This phenomenon correspond to a phase transition which is dubbed by physicists *forstorder (or discontinuous) phase transition*. Proving whether a phase transition is first or second order is often a hard task. For CP it was proven 20 years after the introduction of the model (though the result had been conjectured much earlier based on numerical simulations). For the Ising model the result was proved in 1944 by Onsager in $d = 2$ [Ons44] by proving an explicit solution of the model. In higher dimensions the model is not explicitly solvable and the result was proven only 70 (!) years later [AS15].

Second order phase transition are associated to the occurrence of a power law behavior near criticality and to the occurrence of universal critical exponents.

For the Ising model it holds

$$m(\beta) \sim (\beta - \beta_c)^{\nu} \text{ as } \beta \downarrow \beta_c$$

where ν is a critical exponent, which is given by $\nu = 1/8$ in dimension 2 (this follows from the exact solution of Onsager), $\nu \sim 0.326$ in dimension 3 and $\nu = 1/2$ in $d \geqslant 4$.

For the contact process, it holds

$$\theta(\lambda) \sim (\lambda - \lambda_c)^{\nu} \text{ as } \lambda \downarrow \lambda_c$$

with a critical exponent $\nu \sim 0.276$ for $d = 1$; $\nu \sim 0.583$ for $d = 2$; $\nu \sim 0.813$ for $d = 3$ and $\nu = 1$ for $d \geqslant 4$.

In theoretical physics, renormalization group theory is used to explain these critical exponents and calculate them. According to this theory (which is not mathematically rigorous), critical exponents have a certain degree of *universality*. For example, if we define a modified model CP in which infected sites can infect also non-nearest neighbour up to a finite range, the new model will have a different critical point, but the critical exponent ν will have the same value independent of the range. Also, in two dimensions changing from the square lattice to, e.g., the triangular lattice has no effect on ν. So far, there is no mathematical theory that can explain critical behavior, except in high dimensions (where one uses a technique called the lace expansion) and in a few two-dimensional models.

Chapter 2

CONSTRUCTING IPS

We will now proceed to construct a continuous time Markov process $(\eta_t)_{t\geqslant 0}$ with $\eta_t \in X$ that evolves according the IPS dynamics informally stated in the previous chapter. We will do this for simplicity of notation only for *spin IPS*, namely those IPS for which the elementary moves are of the form $\eta \to \eta^x$ (for exclusion type processes like SSEP the procedure is similar). Informally, we wish to construct a Markov process that satisfies

$$P(\eta_{t+\delta} = \eta^x | \eta_t = \eta) = \delta r(x, \eta) + o(\delta) \tag{2.0.1}$$

If the lattice Λ is finite, it is not difficult to check that such a construction is feasible for any choice of the rates, provided the rates are finite. However, if $|\Lambda|$ is infinite, the process might not be well defined due to the fact that many spin flip might occur at the same time. Indeed we will see that in order for the process to be well defined we should impose proper conditions not only on the boundedness of the rates but also on their range, i.e. on their spatial support.

2.1 The finite volume case: Poisson (or graphical) construction

Let's proceed step by step and start by formally constructing a Markov process which satisfies (2.0.1) when Λ is finite.

1.1 CP on finite volume

Let's consider for simplicity the case of the contact process, whose rates are defined in (1.1.4).

Recall that $X = \{0,1\}^\Lambda$ and define a set $\mathcal{M} = \{\mathcal{H}_x\}_{x\in\Lambda} \cup \{\mathcal{I}_{x,y}\}_{x,y\in\Lambda, y\sim x}$, as follows

- $\mathcal{H}_x : X \to X$ is the transformation that heals site x namely sets its value to 0 and leaves the other sites unchanged. Namely $\mathcal{H}_x\eta(k) = \eta(k)$ if $k \neq x$ and $\mathcal{H}_x\eta(y) = 0$.

- $\mathcal{I}_{x,y} : X \to X$ is the transformation that infects y if x is infected. Namely $\mathcal{I}_{x,y}\eta(k) = \eta(k)$ if $k \neq y$ and $\mathcal{I}_{x,y}\eta(y) = \max(\eta(y), \eta(x))$.

We associate to each map $m \in \mathcal{M}$ a sequence of i.i.d random variables $(\sigma_m^{(k)})_{k \geqslant 1}$ that are exponentially distributed and of mean $1/r_m$, where

$$r_{\mathcal{H}_x} = 1, \ \ \forall x \in \Lambda$$

$$r_{\mathcal{I}_{x,y}} = \lambda \ \ \forall x, y \in \Lambda, y \sim x$$

Then we define for $m \in \mathcal{M}$ the random times $(t_m^{(i)})_{i \geqslant 1}$

$$t_m^{(i)} := \sum_{k=1}^{i} \sigma_k$$

that we call *arrival times* of the map m.

Lemma 2.1.1. *Fix $m \in \mathcal{M}$. The random set of its arrival times $(t_m^{(i)})_{i \geqslant 1}$ satisfies the following*

(i) Fix $s, t \in \mathbb{R}^+$ with $0 \leqslant s \leqslant t < \infty$. Then the number of arrival times that fall in the time interval $[s, t]$ is Poisson distributed with mean $(t - s)$, namely

$$P(N_{s,t} = k) = e^{-(t-s)} \frac{\lambda^k}{k!}$$

where the integer random variable $N_{s,t}$ is defined as $N_{s,t} := |[s,t] \cap \{\cup_{i \geqslant 1} t_m^{(i)}\}|$

(ii) Fix $n \in \mathbb{N}$ and $(s_i, t_i)_{i=1}^n$ with $0 \leqslant s_i \leqslant t_i < \infty$ and such that $[s_i, t_i] \cap [s_j, t_j] = \emptyset$ for all $i \neq j$. Then the n random variables $(N_{s_i,t_i})_{i=1}^n$ are independent.

The proof of this lemma follows immediately from the definition of arrival times [1].

We let

$$\begin{aligned} \Delta &:= \cup_{m \in \mathcal{M}} \cup_{\ell \in \mathbb{N}} (m, t_\ell(m)) \\ \Delta_{s,t} &:= \Delta \cap (\mathcal{M} \times (s, t]). \end{aligned} \tag{2.1.1}$$

[1] The result can be restated by saying that $(t_m^{(i)})_{i \geqslant 1}$ form a Poisson Point Set in $[0, \infty)$ of intensity $r_m dt$ with dt the Lebesgue measure. See [Swaa] Section 1.6 for a rigorous definition of Poisson Point Set.

Namely Δ is the set of all couples (m,u) where m is a map and u is one of its arrival times, and $\Delta_{s,t}$ is the set of all couples (m,u) where m is a map, u belongs to the interval $[s,t]$ and it is one of the arrival times of the map m. Note that, thanks to the fact that $|\mathcal{M}|<\infty$ and $r_m<\infty$ for each m

- for any $t<\infty$ $|\Delta_{s,t}|$ is finite with probability one
- for any couple $(m,\tau),(m',\tau')\in\Delta_{s,t}$ it holds $\tau\neq\tau'$.

Thus, with probability one, we can re-order $\Delta_{s,t}$ in increasing order of the arrival times

$$\Delta_{s,t}:=\{(m_1,\tau_1),\dots(m_n,\tau_n)\},\quad \text{with}\quad \tau_1<\dots<\tau_n$$

Let $\psi_{\Delta_{s,t}}:X\to X$ be the composition of the maps in reverse order

$$\psi_{\Delta_{s,t}}(\eta):=m_n\cdot\dots\cdot m_1(\eta) \tag{2.1.2}$$

with the convention $\psi_{\Delta_{s,t}}=\mathbb{1}$ if $\Delta_{s,t}=\emptyset$.
Note that for $s\leqslant u\leqslant t$ it holds

$$\psi_{\Delta_{u,t}}\cdot\psi_{\Delta_{s,u}}=\psi_{\Delta_{s,t}}.$$

The easiest way to understand this definition is by making a drawing as in Fig. 2.1: on the column over site x I mark with a *cross* each arrival time of $\mathcal{H}_x$, and with an *arrow* from $x\to x+1$ each arrival times of $\mathcal{I}_{x,x+1}$ and with an arrow from $x\to x-1$ each arrival times of $\mathcal{I}_{x,x-1}$.
We are now ready to construct the IPS.

Theorem 2.1.2. *Let $\eta\in X$ and set*

$$\eta_t^\eta:=\psi_{\Delta_{0,t}}(\eta),\ \ t\geqslant 0$$

with Δ and $\Delta_{0,t}(\Delta)$ defined as in (2.1.1) *and $\psi_{\Delta_{0,t}}$ defined in* (2.1.2). *Then*

- *$(\eta_t^\eta)_{t\,\geqslant\,0}$ is a Markov process on the space $\mathcal{D}_X[0,\infty]$ of cadlag functions from $[0,\infty)$ to X with initial condition $\eta_0^\eta=\eta$*
- *for any $f\in\mathcal{C}(X)$ with $\mathcal{C}(X)$ the space of continuous real functions on X equipped with the supremum norm $||f||:=\sup_{x\in X}|f(x)|$, it holds*

$$\lim_{t\to 0}\frac{\mathbb{E}^\eta(f(\eta_t))-f(\eta)}{t}=\sum_{m\in\mathcal{M}}r_m(f(m(\eta))-f(\eta))=\sum_x r(x,\eta)(f(\eta^x)-f(\eta)). \tag{2.1.3}$$

where we denote by $\mathbb{E}^\eta$ the mean over the Markov process $(\eta_t^\eta)_{t\,\geqslant\,0}$.

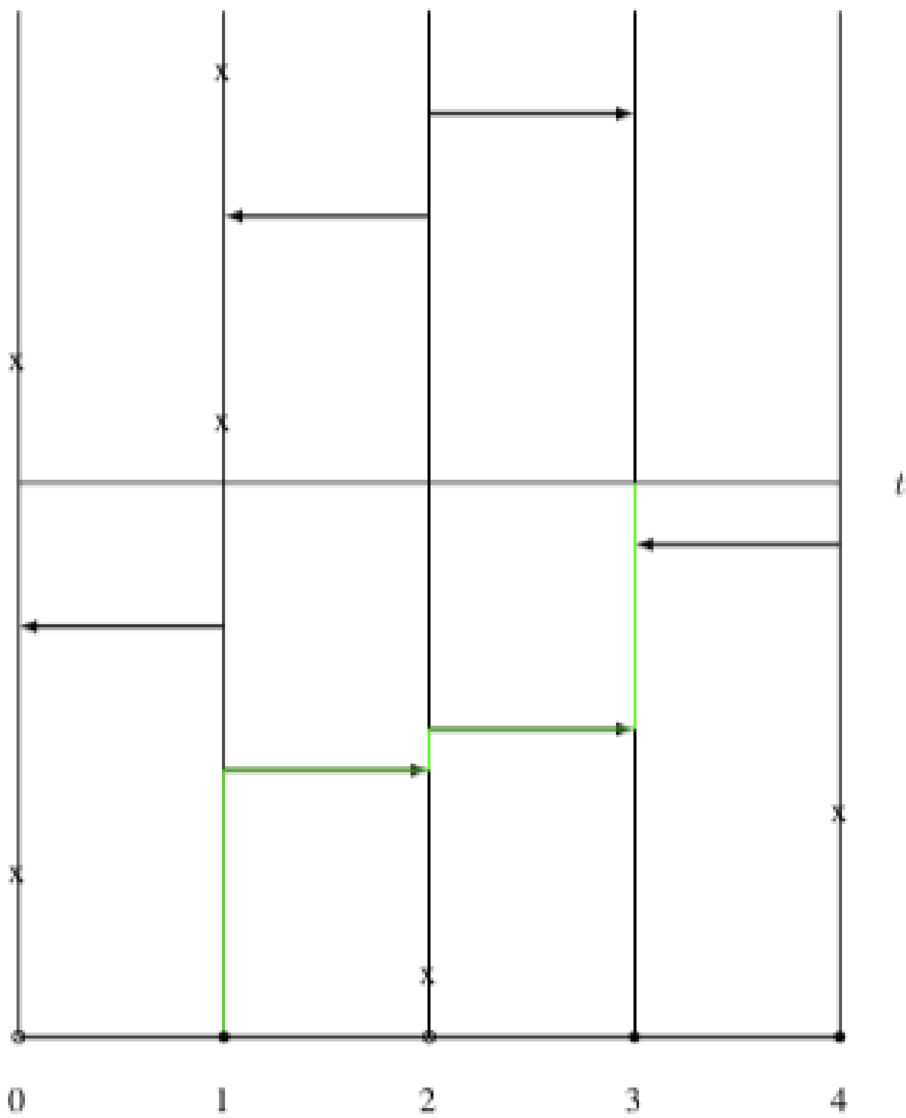

Figure 2.1: Graphical construction for CP in $d = 1$ on the finite volume $\Lambda := [0, 4]$. Here $\psi_{\Delta_{0,t}} = \mathcal{I}_{4,3}\,\mathcal{I}_{1,0}\,\mathcal{I}_{2,3}\,\mathcal{I}_{1,2}\,\mathcal{H}_4\,\mathcal{H}_0\,\mathcal{H}_2$. We highlight in green the path of influence from $(1, 0)$ to $(3, t)$. If we let $\eta(0) = \eta(2) = 0$ and $\eta(1) = \eta(3) = \eta(4) = 1$ we have $\psi_{\Delta_{0,t}}(\eta) = \eta'$ with $\eta'(x) = 1$ for $x \in [0, 3]$ and $\eta'(4) = 0$. If instead $\eta(1) = \eta(3) = \eta(4) = 1$ and $\eta(1) = 0$ we get $\eta'(3) = 1$ and $\eta'(1) = \eta'(2) = \eta'(4) = 0$.

The above theorem says in particular that the process satisfies the informal condition (2.0.1). Indeed, if we let $f : X \to \{0,1\}$ $f(\sigma) := 1_{\sigma=\eta^x}$ and apply (2.1.3), using $\mathbb{E}^\eta(1_{\eta_t=\eta^x}) = P(\eta_t = \eta^x)$ and $f(\eta) = 0$ and $\sum_y r(y,\eta) f(\eta^y) = r(x,\eta)$ we get

$$\lim_{t\to 0} \frac{P(\eta_t = \eta^x)}{t} = r(x,\eta).$$

Here and in the following, when confusion does not arise, we let $\eta_t^\eta = \eta_t$ for simplicity of notation. Note that the initial config η can be any configuration, also a random one, but if it is random it should be independent on Δ.

Proof. By definition η_t has paths that are cadlag (right continuous and left limited). So to prove it is a Markov process we have to prove that the Markov property holds, namely that

$$\mathbb{E}^\eta(f(\eta_t)|\mathcal{F}_s) = \mathbb{E}^{\eta_s} f(\eta_{t-s}) \tag{2.1.4}$$

where for all $s \geqslant 0$, $\mathcal{F}_s$ is the σ-algebra

$$\mathcal{F}_s := \sigma(\eta_{s'} : s' \in [0,s]).$$

Note that, for $s \leqslant t$, it holds by definition $\mathcal{F}_s \subset \mathcal{F}_t$ so that $(\mathcal{F}_t)_{t \geqslant 0}$ is a filtration. Thanks to the independence of the sets of arrival times on distinct time intervals (see Lemma 2.1.1 (i)) and the independence from the initial configuration of the arrival times, (2.1.4) can be easily proven. We are left with proving (2.1.3). From the condition on the finiteness of the sum of rates, $\sum_{m\in\mathcal{M}} r_m < \infty$, and using Lemma 2.1.1 it follows that the probability that two or more arrival times fall in the interval $[0,t]$ is $O(t^2)$ and the probability that a single arrival time falls of a chosen map m falls in this interval is $r_m t + O(t^2)$. This implies that

$$\mathbb{E}^\eta(f(\eta_t)) = f(\eta) + t \sum_{m\in\mathcal{M}} r_m(f(m(\eta)) - f(\eta)) + O(t^2) \tag{2.1.5}$$

which yields (2.1.3).

□

1.2 IPS on finite volume: the general case

Exercise 2. *Show that all other IPS mentioned in Section 1.1 can be constructed on a finite volume Λ along analogous lines as done above for CP. The difference will be the choice of the maps and of the associated rates.*

Remark 2.1.3. *The representation of an IPS in terms of maps is not unique, namely choosing $\mathcal{M}$ and $(r_m)_{m\in\mathcal{M}}$ determine uniquely the process but the converse is not true.*

2.2 The infinite volume case: Poisson (or graphical) construction

We shall now extend the construction of the previous section to the infinite volume setting and see that it actually makes sense for all the models defined in section 1.1. Let $\Lambda \subset \mathbb{Z}^d$ be an infinite volume, S a finite on-site configuration space, and $X = S^{\Lambda}$. Consider a countable set $\mathcal{M}$ of maps $m : X \to X$, and a set of bounded positive rates $\{r_m\}_{m \in \mathcal{M}}$. In analogy to the finite volume case we would like to construct a Markov process $(\eta_t)_{t \geqslant 0}$ such that (2.1.3) holds. If we try to proceed as for the finite volume case, the first problem we encounter is that $\sum_m r_m = \infty$ so $\{t : (t, m) \in \Delta\}$ is dense in $\mathbb{R}^+$ and it is now not possible to order the elements of $\Delta_{s,t}$ according to their arrival times.

The key observation is to notice that the maps and the rates of the processes that interest us are defined in such a way that *with high probability only finitely many points of $\Delta_{0,t}$ are necessary to determine the value of the process at a given space time point* (x, t). Thus it will actually be possible to order these finely many "relevant" space time points according to their arrival times and proceed essentially as for the finite volume case.

In order to formalise the above observation we should introduce the notion of path of influence. For concreteness, we start by treating the case of CP on $\mathbb{Z}$, then we will extend the procedure to general models.

2.1 CP on $\mathbb{Z}$

In this section we let $\Lambda = \mathbb{Z}$ and define $\mathcal{M} = \cup_{x \in \mathbb{Z}} \mathcal{H}_x \cup_{x,y \in \mathbb{Z}, x \sim y} \mathcal{I}_{x,y}$ with $\mathcal{H}_x$ and $\mathcal{I}_{x,y}$ defined as for the finite volume CP. Let us introduce for each map $m \in M$ the sets $D(m) \subset \Lambda$ and $\{R_i(m)\}_{i \in \Lambda} \subset \Lambda$ as follows

$$D(m) := \{i \in \Lambda : \exists \eta \in X : \eta(i) \neq m(\eta)(i)\} \tag{2.2.1}$$

and, for any $i \in \Lambda$, we let

$$R_i(m) := \{j \in \Lambda : \exists \eta \in X \text{ s.t. } m(\eta)(i) \neq m(\eta^j)(i)\}. \tag{2.2.2}$$

In words $D(m)$ is the set of sites whose value can be possibly changed by m and $R_i(m)$ is the sets of sites that are m-relevant for i.

It is not difficult to verify that it holds

$$D(\mathcal{H}_x) = \{x\}, \;\; D(\mathcal{I}_{x,y}) = \{y\} \tag{2.2.3}$$

and

$$R_z(\mathcal{H}_x) = \begin{cases} \emptyset & \text{if } z = x \\ z & \text{otherwise} \end{cases} \tag{2.2.4}$$

$$R_z(\mathcal{I}_{x,y}) = \begin{cases} \{x, y\} & \text{if } z = y \\ z & \text{otherwise} \end{cases} \tag{2.2.5}$$

Definition 2.2.1 (Paths)**.** *A path in Λ is a pair of functions (γ_{t-}, γ_t) defined in an interval $[s, u]$ with $s \leqslant u$ and taking values in Λ that verify*

$$\lim_{t\downarrow\to t_0} \gamma_{t-} = \gamma_{t_0} \quad \textit{for } t_0 \in [s, u) \tag{2.2.6}$$

$$\lim_{t\uparrow\to t_0} \gamma_t = \gamma_{t_0-} \quad \textit{for } t_0 \in (s, u] \tag{2.2.7}$$

Note that γ_{t_0-} can be different from γ_{t_0}.

Definition 2.2.2 (Paths of influence)**.** *Fix $\Delta \in \mathcal{M} \times [0, \infty)$ a realisation of the Poisson point processes associated to the maps in $\mathcal{M}$ as in the finite volume case. For any $i, j \in \mathbb{Z}$ and $0 \leqslant s \leqslant u$ we say that there is a path of influence from (i, s) to (j, u) iff $\gamma_{s-} = i$, $\gamma_u = j$ and*

- *whenever $\gamma_{t-} \neq \gamma_t$ for $t \in [s, u]$ necessarily there exists $m \in \mathcal{M}$ s.t. $(m, t) \in \Delta$, $\gamma_t = D(m)$ and $\gamma_{t-} \in R_{\gamma_t}(m)$*
- *for each $(m, t) \in \Delta$ with $t \in [s, u]$ and $\gamma_t \in D(m)$ it holds $\gamma_{t-} \in R_{\gamma_t(m)}$.*

We write $(i, s) \to (j, u)$ for the event there is a path of influence from (i, s) to (j, u). and $(i, s) \not\to (j, u)$ for the complementary event. We also set, for any finite $A \subset \mathbb{Z}$, $\{(i, s) \to A \times \{u\}\}$ if there exists $j \in A$ s.t. $(i, s) \to (j, u)$ and let

$$\xi_s^{A,u} := \{i \in \Lambda : (i, s) \to A \times \{u\}\} \tag{2.2.8}$$

with the convention $\xi_u^{A,u} = A$.

For example, in Fig. 2.1, it holds $(2, 0) \not\to (3, t)$ and $(1, 0) \to (3, t)$. Furthermore, if we let $s = 0$, $u = t$ and $A = \{3, 4\}$ it holds $\xi_s^{A,u} = \{1, 3\}$.

Remark 2.2.3. *The above definition is equivalent to saying that, if we make a graphical representation of Δ as in the finite volume case, the existence of a path of influence from (i, s) to (j, s) is equivalent to the existence of a path in the graphical representation that, when following the positive time direction,*

- *either grows vertically (namely $\gamma_t = \gamma_{t-}$) or it moves horizontally of one step to the right or to the left (so that in this case $\gamma_t = \gamma_{t-} \pm 1$*
- *moves horizontally only if it meets an arrow (but can also go upward when meeting an arrow)*
- *never meets a cross.*

Remark 2.2.4. *Note that, for any $A \subset \mathbb{Z}$ and for any $0 \leqslant s \leqslant t$ the value at time t of the process on all sites belonging to A can be constructed by the knowledge of the value of the configuration at time s only on the sites belonging to $\xi_s^{A,t}$.*

The following result will play a key role

Lemma 2.2.5.

(i) For any finite $A \subset \mathbb{Z}$ it holds [2]

$$\mathbb{E}[|\xi_s^{A,u}|] \leqslant |A| e^{(2\lambda-1)(u-s)} \qquad 0 \leqslant s \leqslant u$$

(ii) For each $i \in \Lambda$ and $s \leqslant u$, the set

$$\{(m,t) \in \Delta_{s,u} : \mathcal{D}(m) \times \{t\} \to (i,u)\}$$

is finite almost surely.

Proof. Proof of (i). Fix A and set for simplicity of notation

$$\xi_t := \xi_{u-t}^{A,u}.$$

The idea is to use the fact that $(\xi_t)_{t \geqslant 0}$ is a Markov process and to prove using its generator

$$\frac{\partial \mathbb{E}(|\xi_t|)}{\partial t} \leqslant (2\lambda - 1) E[|\xi_0|] \tag{2.2.9}$$

In order to prove this result we start by a cut-off procedure. Let $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of finite sets such that $\Lambda_n \uparrow \Lambda$. Pick n large enough so that $A \subset \Lambda_n$ and set

$$\xi_t^n := \{i \in \Lambda : (i, u-t) \to_n A \times \{u\}\} \tag{2.2.10}$$

[2] here the mean $\mathbb{E}$ is over the randomness in Δ

where $\to_n$ denotes the presence of a path of influence that stays inside Λ_n. Observe that $\xi_t^n \subset \xi_t^m$ for $n < m$ and

$$\xi_t^n \uparrow \xi_t \quad \text{for all } t \geqslant 0.$$

It is not difficult to verify that $(\xi_{u-t}^n)_{t\in[0,u]}$ is a Markov process taking values in the finite space of subsets of Λ_n and with generator acting on function $f : \Lambda_n \to \mathbb{R}$ as

$$\mathcal{L}_n f(B) = \sum_{m\in M_n} r_m(f(B^m) - f(B))$$

where $\mathcal{M}_n := \{m \in M : D(m) \cap \Lambda_n \neq \emptyset\}$ and

$$B^m = B \setminus i \quad \text{if} \quad m = \mathcal{H}_i$$

$$B^m = B \quad \text{if} \quad m = \mathcal{I}_{ij} \quad \text{and} \quad j \notin B$$

$$B^m = B \cup \{i\} \quad \text{if} \quad m = \mathcal{I}_{ij} \quad \text{and} \quad j \in B$$

Let $(P_t^n)_{t\,\geqslant\,0}$ be the associated semigroup and define the function $g : \Lambda_n \to \mathbb{R}$ as

$$g(B) := |B|.$$

Then

$$\mathcal{L}_n g(B) = \sum_{m\in M_n} r_m(g(B^m) - g(B)) \leqslant \sum_{i\in B}(|B| + 1 - |B|) + 2\lambda(|B| + 1 - |B|) = |B|(2\lambda - 1) \tag{2.2.11}$$

Let $K = 2\lambda - 1$ Notice now that

$$\frac{\partial}{\partial t}(e^{-Kt}P_t^n\, g) = -Ke^{-Kt}P_t^n g + e^{-Kt}P_t^n \mathcal{L}_n g = e^{-Kt}P_t^n(\mathcal{L}_n g - Kg) \leqslant 0$$

and therefore

$$e^{-Kt}P_t^n\, g \leqslant e^{-Kt}P_t^n\, g|_{t=0} = g(\xi_0^n) = |A|$$

where we used the fact that by definition

$$\xi_0^n = A.$$

This yields

$$\mathbb{E}(|\xi_t^n|) \leqslant |A|e^{Kt} \quad \forall t \in [0, u].$$

Now letting $n \to \infty$ so that $\xi_t^n \to \xi_t = \xi_{u-t}^{u,A}$ the result of point (i) is proven.

The proof of point (ii) follows along similar lines. $\square$

Fix $i \in \mathbb{Z}$ and $0 \leqslant s \leqslant u < \infty$, in view of Lemma 2.2.5, we can order the relevant arrival times as

$$\{(m,t) \in \Delta_{s,u} : D(m) \times \{t\} \to (i,u)\} = \{(m_1,t_1), \dots (m_n,t_n)\}$$

with $t_1 < \cdots < t_n$. Then we define

$$\psi_{\Delta_{s,u}}(\eta)(i) = m_n \circ \cdots \circ m_1(\eta)(i) \tag{2.2.12}$$

and we can define the probability kernels[3]

$$\{p_t(\eta,\cdot)\}_{t\,\geqslant\,0} := \{\mathbb{P}(\psi_{\Delta,0,t}(\eta) \in \cdot)\}_{t\,\geqslant\,0}. \tag{2.2.13}$$

Then, along the same lines as Theorem 2.1.2 for the finite volume case, we can prove the following

Theorem 2.2.6.

(i) The probability kernels $\{p_t(\eta,\cdot)\}_{t\,\geqslant\,0}$ define a continuous transition probability, namely they satisfy the following properties

- *$(x,t) \to P_t(,\cdot)$ is a continuous map from $X \times [0,\infty) \to \mathcal{P}(X)$;*
- *$\int_X p_s(x,dy)p_t(y,\cdot) = p_{s+t}(x,\cdot)$ for all $x \in X$ and $s,t \geqslant 0$*
- *$p_0(x,\cdot) = \delta_x$ for all $x \in X$*

As a consequence, they can be used define a Feller semigroup $(P_t)_{t\,\geqslant\,0}$ by letting

$$P_t f(\eta) := \int_X p_t(\eta, d\eta')f(\eta') \quad \textit{for any } f \in B(X).$$

(ii) Let $\mathcal{L}$ be the generator of the semigroup $(P_t)_{t\,\geqslant\,0}$, namely the operator acting as

$$\mathcal{L}f(\eta) := \lim_{t\to 0} \frac{P_t f(\eta) - f(\eta)}{t}, \tag{2.2.14}$$

where the limit is intended in the topology of the supremum norm on $\mathcal{C}(X)$ and the operator is defined on functions $f \in \mathcal{D}(\mathcal{L})$ with $D(\mathcal{L}) \subset \mathcal{C}(X)$ the set of functions for which the above limit exists. Then the action of $\mathcal{L}$ on functions that depend on finitely many coordinates, i.e. on functions that satisfy

$$\sup_{\eta\in X} \sum_x |f(\eta^x) - f(\eta)| < \infty$$

[3] A probability kernel on X is a function K from $X \times \mathcal{B}(X)$ (with $\mathcal{B}(X)$ the Borel sigma field generated by the open sets of X) to $[0,1]$ s.t. (i) for all $x \in X$, $K(x,\cdot)$ is a probability measure on X and (ii) for all $A \in \mathcal{B}(X)$, $K(\cdot,A)$ is a real measurable function on X.

can be written as follows

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}} r(x,\eta)(f(\eta^x) - f(\eta)) \tag{2.2.15}$$

with $r(x,\eta)$ the rates of CP (1.1.4)*.*

The Markov process $(\eta_t)_{t\,\geqslant\,0}$ with values on X and cadlag sample paths from $[0,\infty)$ to X associated to the Feller semigroup P_t defined above is the contact process on $\mathbb{Z}$.

2.2 Some reminders from Markov process theory

Let $B(X)$ be the set of real, bounded $\mathcal{B}(X)$-measurable functions on X and $C(X)$ the set of continuous functions. Given a *continuous transition probability* (see Theorem **??** (i) for a definition) the associated *Feller semigroup* $(P_t)_{t\,\geqslant\,0}$ is defined by letting

$$P_t f(\eta) := \int_X p_t(\eta, d\eta') f(\eta') \quad \text{for any } f \in B(X).$$

From the continuity of the transition probability it follows that the collection of operators $(P_t)_{t\,\geqslant\,0}$ is a collection of linear operators that verify the following properties:

(i) if $f \in C$ then for any $t \geqslant 0$ it holds $P_t f \in C$

(ii) $\lim_{t\to 0} ||P_t f - f|| = 0$

(iii) $P_s P_t = P_{s+t} f$

(iv) $P_0 f = f$

(v) if $f \geqslant 0$ then $P_t f \geqslant 0$

(vi) $P_t 1 = 1$

Conversely any collection $(P_t)_{t\,\geqslant\,0}$ of linear operators $P_t : C(X) \to C(X)$ satisfying the six properties (i)-(vi) is called a *Feller semigroup* and it corresponds to a unique continuous transition probability on X. Given a Feller semigroup we can construct a stochastic process $(\eta_t)_{t\,\geqslant\,0}$ with values on X and cadlag sample paths from $[0,\infty)$ to X by letting $\eta_0 = \eta$ and setting for any $f \in \mathcal{C}(X)$

$$\mathbb{E}^\eta(f(\eta_t)|\mathcal{F}_s) = P_{t-s} f(\eta_s) \quad \text{a.s. } \forall s \leqslant t. \tag{2.2.16}$$

where for $s \geqslant 0$, we let $\mathcal{F}_s$ be the σ-algebra generated by $(\eta_t)_{t \geqslant 0}$, namely

$$\mathcal{F}_s := \sigma(\eta_{s'} : s' \in [0, s])$$

and $\mathbb{E}^\eta$ denote the mean over the process. Note that, for $s \leqslant t$, it holds by definition $\mathcal{F}_s \subset \mathcal{F}_t$ so that $(\mathcal{F}_t)_{t \geqslant 0}$ is indeed a filtration. This stochastic process is a Markov process. Indeed it satisfies the *strong Markov property*, namely if we denote by $\mathbb{P}^\eta$ the law of the process started from η it holds

$$\mathbb{P}^\eta(\eta_{t+\delta} \in A | \mathcal{F}_t) = P^{\eta_t}(\eta_\delta \in A) \tag{2.2.17}$$

a.s. for every $\eta \in X$, $A \in \mathcal{F}$, $t \geqslant 0, \delta \geqslant 0$. This can be proven easily setting $f = 1_A$ and using (2.2.16) which yields

$$\mathbb{P}^\eta(\eta_{t+\delta} \in A|\mathcal{F}_t) = \mathbb{E}^\eta(f(\eta_{t+\delta}|\mathcal{F}_t)) = P_\delta(f(\eta_t)) = \mathbb{E}^{\eta_t}(f(\eta_\delta)) = \mathbb{P}^{\eta_t}(\eta_\delta \in A).$$

2.3 IPS on infinite volume: the general case

Let m be a map from $X \to X$ where $X = S^\Lambda$ and Λ is an infinite volume (typically, $\Lambda = \mathbb{Z}^d$). We define $D(m) \subset \Lambda$ as in(2.2.1) and, for $i \in \Lambda$, $\{R_i(m)\} \subset \Lambda$ as in (2.2.2). We say that a map m is *local* if $D(m)$ is finite and $R_i(m)$ is finite for all $i \in D(m)$.

Exercise 3. *Verify that all the possible elementary moves for each of the processes defined in Chapter 1.1 can be recast in term of local maps.*

For the Voter Model (VM), whose rates as defined by equation (1.1.5), a natural choice of maps to make the Poisson construction is $\mathcal{M} = \{\mathcal{V}_{x,y}\}_{x \in \mathbb{Z}^d, y \sim x}$ with $r_m = 1/(2d)$ for each m and $\mathcal{V}_{x,y}$ defined as the transformation that sets the variable on y equal to $\eta(x)$ and leaves the other sites unchanged, namely

$$\mathcal{V}_{x,y}(\eta)(z) = \begin{cases} \eta(x) & \text{if } z = y \\ \eta(z) & \text{otherwise} \end{cases}. \tag{2.2.18}$$

Verify that for this maps it holds $D(\mathcal{V}_{x,y}) = \{y\}$,

$$R_z(\mathcal{V}_{x,y}) = \begin{cases} x & \text{if } z = y \\ z & \text{otherwise} \end{cases} \tag{2.2.19}$$

Let $\mathcal{M}$ be a countable set of local maps and $(r_m)_{m \in M}$ be non negative constants and Δ the Poisson point processes constructed as for the CP by associating to each map $m \in \mathcal{M}$ a Poisson point process on $\mathbb{R}^+$ with intensity $r_m dt$ with dt the Lebesgue measure and taking the union over all maps.

We should now state two assumptions on the rates that will be crucial in the construction of the process

(A1) $\sup_{i \in \Lambda} \sum_{m \in \mathcal{M}: i \in D(m)} r_m < \infty$

(A2) $K < \infty$, where

$$K := \sup_{i \in \Lambda} \sum_{m: i \in \mathcal{D}(m)} r_m(|R_i(m)| - 1) < \infty \tag{2.2.20}$$

Define paths of influence as in Definition 2.2.2 and, for each times $0 \leqslant s \leqslant u < \infty$ and each set $A \subset \Lambda$ defined the set $\xi_s^{A,u} \subset \Lambda$ as in (2.2.8) .

The following result is a key ingredient for the construction of the IPS on infinite volume

Lemma 2.2.7 (Exponential bound on paths of influence and finitely many relevant clock rings)**.**

Suppose that (A1) and (A2) hold. Then

- *for any finite $A \subset \Lambda$, it holds*

$$\mathbb{E}[|\xi_s^{A,u}|] \leqslant |A| e^{K(u-s)} \qquad 0 \leqslant s \leqslant u$$

- *for each $i \in \Lambda$ and $s \leqslant u$, the set*

$$\{(m,t) \in \Delta_{s,u} : \mathcal{D}(m) \times \{t\} \to (i,u)\}$$

is finite almost surely

Assumption (A1) guarantees that in each finite time interval there are only finitely many events that may change the state of any fixed lattice site. Assumption (A2) is necessary to guarantee that the influence coming (in the same time interval) from all other lattice points is also under control. The proof follows analogous line as Lemma 2.2.5 , a full proof can be found in [Swaa] Section 1.6.

In view of this result, for a fixed $i \in \Lambda$ and $0 \leqslant s \leqslant u < \infty$, we can a.s. order the relevant arrival times as

$$\{(m,t) \in \Delta_{s,u} : D(m) \times \{t\} \to (i,u)\} = \{(m_1,t_1), \dots (m_n,t_n)\}$$

with $t_1 < \cdots < t_n$ and define

$$\psi_{\Delta,s,u}(\eta)(i) = m_n \circ \cdots \circ m_1(\eta)(i).$$

Then, along the same lines as the proof of Theorem 2.1.2 the following can be proven

Theorem 2.2.8. *Fix a countable collection $\mathcal{M}$ of local maps and $(r_m)_{m\in\mathcal{M}}$ non negative constants that satisfy assumptions (A1) and (A2) above. Let*

$$p_t(\eta,\cdot) := \mathbb{P}(\psi_{\Delta,0,t}(\eta) \in \cdot)$$

with Δ a Poisson point process on $\mathcal{M} \times [0,\infty)$ with intensity $r_m dt$. Then $\{p_t(\eta,\cdot)\}_{t\geqslant 0}$ is a continuous transition probability and the corresponding Markov process has a generator that acts on functions that depend on finitely many coordinate as

$$\mathcal{L}f(\eta) = \sum_{m\in\mathcal{M}} r_m(f(m(\eta)) - f(\eta)).$$

Exercise 4. *Go back to exercise 3 and verify that the transition rates of the local maps that you constructed for the different models of section 1.1 satisfy assumptions (A1)–A(3).*

Remark 2.2.9. *The Poisson construction provides not only a rigorous construction of IPS but also a very powerful tool to couple processes started in different initial configurations and/or evolving with different parameters. In case of processes started from different initial conditions, the idea is to couple them by the using the same realisation of the Poisson processes for the arrival times of the maps. Two questions that may be easily solved using the powerful coupling tool provided by the graphical construction are stated below in Exercise 5 and 6. An alternative rigorous construction of IPS via the generator (instead of the Poisson processes) is also possible. It may be found on Liggett's book [Lig85] or Swart's lecture notes ([Swaa] or [Swab]).*

Given $\eta, \sigma \in X$ we say that σ dominates η, and denote this as $\eta < \sigma$, if for all $x \in \Lambda$ it holds $\eta(x) \leqslant \sigma(x)$.

Exercise 5. *Consider two CP starting from two different initial configurations, η_1, η_2 such that $\eta_1 < \eta_2$. Fix $i \in \Lambda$ and $t \geqslant 0$ prove that*

$$\text{if it holds } \mathbb{E}^{\eta_1}(\eta_t(i)) > 0 \text{ then necessarily } \mathbb{E}^{\eta_2}(\eta_t(i)) > 0.$$

Exercise 6. *Fix $\eta \in \{0,1\}^{\mathbb{Z}}$ and $0 < \lambda_1 < \lambda_2 < \infty$. Consider two CP process started from η, the first one with infection rate λ_1, the second one with infection rate λ_2. Let $\mathbb{E}_1$ (resp. $\mathbb{E}_2$) be the mean over the first (resp, second) CP process. Fix $t \geqslant 0$ and $x \in \Lambda$, prove that a.s. $\mathbb{E}_1(\eta_t(x)) \leqslant \mathbb{E}_2(\eta_t(x))$.*

If you have a hard time figuring out how to do properly Exercise 5 and 6, I suggest re-trying after studying next chapter (in particular after understanding the notion of coupling and how to use this tool).

Chapter 3

SOME USEFUL TOOLS AND GENERAL RESULTS

3.1 Some additional notation

Recall that S is a finite on-site configuration space, $\Lambda \subset \mathbb{Z}^d$ is a finite or infinite volume, and X is the configuration space $X = S^{\Lambda}$. We endow X with the product topology [1], so that it is a compact metrizable space [2], namely there exists a countable collection of compact sets whose union forms the space. This follows by Tychonoff' theorem using the fact that S is compact. For $(\eta_n)_{n\in\mathbb{B}}$ a sequence of elements of X we set $\lim_{n\to\infty}\eta_n \to \eta$ iff we have point-wise convergence, namely $\eta_n(x) \to \eta(x)$ for all $x \in \Lambda$.

We will also call $\mathcal{B}(X)$ the Borel sigma-field generated by the open subsets of X, $B(X)$ the set of real, bounded $\mathcal{B}(X)$-measurable functions on X, and $\mathcal{C}(X)$ are the continuous functions [3]on X. Finally, we denote by $\mathcal{P}(X)$ the space of probability measures on X endowed with the weak topology [4]. Since X is compact, $\mathcal{P}(X)$ is also compact.

For $\eta \in X$ and $\mu \in \mathcal{P}(X)$, we denote by

- $\mathbb{P}^{\eta}$ and $\mathbb{E}^{\eta}$ respectively the law of the IPS started at η and the expectation over this process;

- $(P_t)_{t\,\geqslant\,0}$ the semigroup of the Markov process, which is defined by its action on $f \in \mathcal{C}$

$$P_t f(\eta) := \mathbb{E}^{\eta}(f(\eta_t)) \quad \forall t \geqslant 0$$

[1]Recall that open sets in the product topology are (finite or infinite) unions of sets of the form $\pi_{i\in I}U_i$ with $U_i \neq$ the whole space X_i only for a finite number of indexes.

[2]We can for example choose as metric $d(\eta,\xi) = \sum_{x\in\Lambda} 2^{-|x|}\mathbb{1}_{\eta(x)\neq\xi(x)}$

[3]Note that $\mathcal{C}(X)$ can be viewed as a Banach space with norm $||f|| = \sup_{\eta\in X}|f(\eta)|$.

[4]Namely, we let $\mu_n \to \mu$ for $n \to \infty$ iff for all $f \in \mathcal{C}$ it holds $\int f d\mu_n \to \int f d\mu$, where $\mathcal{C} = \mathcal{C}(X)$ is the set of real continuous functions on X viewed as a Banach space with norm $||f|| = \sup_{\eta\in X}|f(\eta)|$.

- $\mathbb{P}^{\mu}$ the law of the IPS with initial distribution μ, i.e.

$$\mathbb{P}^{\mu} = \int_X \mathbb{P}^{\eta} \mu(d\eta)$$

  and

$$\mathbb{E}^{\mu}(f(\eta_t)) = \int_X E^{\eta}(f(\eta_t))\mu(d\eta) = \int_X P_t f d\mu$$

- μP_t the the distribution at time t of the process started from μ, i.e. the measure satisfying [5] for all $f \in \mathcal{C}$

$$\int_X f d(\mu P_t) := \int_X P_t f d\mu$$

From now on we will drop the index X from the integral over the whole configuration space, namely we set for simplicity of notation $\int f d\mu := \int_X f d\mu$.

3.2 Invariant (or stationary) measures

Definition 3.2.1 (Invariant (or stationary) measures)**.** *We say that $\mu \in \mathcal{P}$ is invariant if*

$$\int P_t f d\mu = \int f d\mu, \quad \forall\, t \geqslant 0, \quad \forall\, f \in \mathcal{C}(X)$$

namely if

$$\mu P_t = \mu, \ \ \forall t \geqslant 0.$$

We denote by $\mathcal{I}$ the set of invariant measures. As a consequence, for any $\mu \in \mathcal{I}$ and for any measurable set A, it holds

$$\mathbb{P}^{\mu}(\eta_s \in A) = \mathbb{P}^{\mu}(\eta_{s+t} \in A), \quad \forall s, t \geq 0.$$

which follows immediately using $\mathbb{P}^{\mu}(\eta_s \in A) = \int P_s 1_A d\mu$.

The invariant measures satisfy the following properties:

Theorem 3.2.2 (Properties of $\mathcal{I}$)**.**

(i) $\mathcal{I}$ is a compact and convex subset of $\mathcal{P}(X)$;

[5] The fact that this relation determines μP_t uniquely is a consequyence of the Riesz representation theorem.

(ii) Given an initial measure π, if the weak limit $\lim_{t\to\infty} \pi P_t$ exists, i.e. if exists μ t.q.

$$\lim_{t\to\infty} \int P_t f d\pi = \mu(f), \ \ \forall f \in \mathcal{C}$$

then $\mu \in \mathcal{I}$;

(iii) $\mathcal{I}$ is non empty;

(iv) $\mu \in \mathcal{I}$ iff $\mu(\mathcal{L}f) = 0$ for any $f \in \mathcal{D}(\mathcal{L})$ with $\mathcal{D}(\mathcal{L})$ the domain [6] of the generator $\mathcal{L}$, namely the sets of continuous functions for which the limit (2.2.14) *exists.*

Proof. (i) Since $\mathcal{I}$ is a subset of the compact set $\mathcal{P}(X)$ we only have to show that it is closed to prove that it is compact. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be such that

1. $\mu_n \in \mathcal{I}$ for all n;
2. there exists $\mu \in \mathcal{P}$ such that $\lim_{n\to\infty} \mu_n = \mu$.

By definition it holds

(a) $\mu_n = \mu_n P_t \ \ \forall n$;

(b) $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$ for any $f \in \mathcal{C}$;

(c) if $f \in \mathcal{C}$, it holds $P_t f \in \mathcal{C}$ for any $t \geqslant 0$

Therefore

$$\int P_t f d\mu = \lim_{n\to\infty} \int P_t f d\mu_n = \lim_{n\to\infty} \int f d\mu_n = \int f d\mu,$$

which implies that $\mu \in \mathcal{I}$. We used: (b) and (c) to obtain the first equality; (a) to obtain the second equality; and (b) to obtain the third equality. Convexity of $\mathcal{I}$ follows by definition;

$$\int P_s f d\mu = \lim_{t\to\infty} \int P_t(P_s f) d\pi = \lim_{t\to\infty} \int P_{t+s} f d\pi = \int f d\mu$$

where we used point (c) above and the fact that a Feller semigroup verifies (see Section 2.1) $P_t P_s f = P_{t+s} f$ for any $s, t \geqslant 0$ and $f \in \mathcal{C}$.

[6] See [Lig85] for a formal definition of the domain. For practical purposes you can think of the domain as being the sets of local functions, i.e. continuous functions that depend on finitely many coordinates.

(iii) Fix a measure $\mu \in \mathcal{P}$ and a sequence of real positive numbers $\{T_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} T_n = \infty$. Define a sequence of measures $\{\mu_n\}_{n\in\mathbb{N}}$ by letting for all $f \in \mathcal{C}(X)$

$$\int f d\mu_n := \frac{1}{T_n}\int_0^{T_n} dt \int f d(\mu P_t) = \frac{1}{T_n}\int_0^{T_n} dt \int P_t f d\mu \tag{3.2.1}$$

The compactness of $\mathcal{P}$ yields the existence of a converging subsequence, namely the existence of a measure $\tilde{\mu} \in \mathcal{P}$ and an increasing sequence $\{a_i\}_{i\in\mathbb{N}}$ with $a_i \in \mathbb{N}$, $\forall i$ s.t.

$$\lim_{n\to\infty} \tilde{\mu}_{a_n} = \tilde{\mu} \tag{3.2.2}$$

We will now prove that $\tilde{\mu}$ is an invariant measure, which implies in particular that $\mathcal{I}$ is non empty. Note that

$$\begin{aligned}\int P_s f d\tilde{\mu} = \lim_{n\to\infty}\int P_s f d\mu_{a_n} = \lim_{n\to\infty} T_{a_n}^{-1}\int_0^{T_{a_n}} dt \int P_t P_s f d\mu = \\ \lim_{n\to\infty} T_{a_n}^{-1}\int_0^{T_{a_n}} dt \int P_{t+s} f d\mu = \lim_{n\to\infty} T_{a_n}^{-1}\int_s^{T_{a_n}+s} g(\tau)d\tau = \\ \lim_{n\to\infty} T_{a_n}^{-1}\left[\int_0^{T_{a_n}} g(\tau)d\tau + \int_{T_{a_n}}^{T_{a_n}+s} g(\tau)d\tau - \int_0^s g(\tau)d\tau\right]\end{aligned} \tag{3.2.3}$$

where

$$g(\tau) := \int P_\tau f d\mu$$

By noticing that $|g(\tau)| \leqslant \sup_{\eta\in X}|f(\eta)|$ we get

$$\lim_{n\to\infty} T_{a_n}^{-1}\left|\int_{T_{a_n}}^{T_{a_n}+s} g(\tau)d\tau - \int_0^s g(\tau)d\tau\right| = 0$$

which, inserted in (3.2.3), yields

$$\int P_s f d\tilde{\mu} = \lim_{n\to\infty} T_{a_n}^{-1}\int_0^{T_{a_n}}\int P_\tau f d\mu = \lim_{n\to\infty}\int f d\mu_{a_n} = \int f d\tilde{\mu} \tag{3.2.4}$$

where in the second equality we used (3.2.1) and in the third equality we used (3.2.2).

(iv) Let $f \in \mathcal{D}(\mathcal{L})$ and $\mu \in \mathcal{I}$. Then

$$\int \mathcal{L} f d\mu = \lim_{t\to 0}\frac{\int P_t f d\mu - \int f d\mu}{t} = 0$$

and the *if* condition is proven. In order to prove the *only if* condition suppose that for any function $f \in \mathcal{D}(\mathcal{L})$ it holds $\mu(\mathcal{L}f) = 0$. In order to prove that $\mu \in \mathcal{I}$ we proceed as follows. Fix $g \in \mathcal{C}$ and $\lambda \geqslant 0$, and let $f_1 = f_1(g,\lambda) \in \mathcal{D}$ be defined via

$$(\mathbb{1} - \lambda\mathcal{L})f_1 = g$$

(the existence of f_1 is guaranteed by Theorem 3.2.3). By integrating and using the hypothesis $\mu(\mathcal{L}f_1) = 0$ we get

$$\int f_1 d\mu = \int g d\mu.$$

We extend the above definition letting for $n \geqslant 1$

$$f_n = (\mathbb{I} - \lambda\mathcal{L})^{-n} g.$$

By iterating the above argument we get

$$\int f_n d\mu = \int g d\mu. \tag{3.2.5}$$

Therefore, letting $\lambda := \frac{t}{n}$ we get

$$\lim_{n\to\infty} \int (\mathbb{I} - \frac{t}{n}\mathcal{L})^{-n} g d\mu = \int g d\mu. \tag{3.2.6}$$

On the other hand, using again Theorem 3.2.3 we have

$$\lim_{n\to\infty} (\mathbb{I} - \frac{t}{n}\mathcal{L})^{-n} f = P_t f$$

which, together with (3.2.6) implies $\mu \in \mathcal{I}$.

□

Theorem 3.2.3 (Hille Yoshida)**.** *There is a one to one correspondence between Markov semigroups and generators given as follows*

$$\mathcal{L}f = \lim_{t\to 0} \frac{P_t f - f}{t} \quad \forall f \in \mathcal{D}(\mathcal{L})$$

$$P_t f = \lim_{n\to\infty} (\mathbb{I} - \frac{t}{n}\mathcal{L})^{-n} f \quad \forall\ f \in \mathcal{C}(X),\ t \geqslant 0$$

where $\mathcal{D}(\mathcal{L}) \subset C(X)$ *is the set of functions for which* $\lim_{t\to 0} \frac{P_t f - f}{t}$ *exists. Furthermore*

- *for* $f \in D(\mathcal{L})$ *it holds* $P_t f \in \mathcal{D}(\mathcal{L})$
- *the following backward forward equation holds*

$$\frac{d}{dt} P_t f = P_t(\mathcal{L}f) = \mathcal{L}(P_t f) \quad \forall f \in \mathcal{D}(X)$$

- *for any* $\lambda \geqslant 0$ *and* $g \in \mathcal{C}(X)$*, there exists a function* $f \in D(\mathcal{L})$ *s.t.* $(\mathbb{I} - \lambda\mathcal{L})f = g$*. Furthermore this function satisfies* $f := \int_0^\infty e^{-t} P_{\lambda t}\, g dt$*.*

We will not provide a prove of the Hille Yoshida theorem, the interested reader can find it Chapter 1 of [**?**].

3.3 Ergodicity

Definition 3.3.1 (Ergodicity and phase transitions)**.** *We say that an IPS is ergodic*[7] *if the following two conditions hold:*

(i) there is only one invariant measure, $\mathcal{I} = \{\mu\}$

(ii) $\lim_{t\to\infty} \pi P_t = \mu$ *for any* π

We say that a phase transition occurs for an IPS if:

(a) the definition of the IPS contains a parameter (temperature, density, ...) that can vary in a (finite or infinite) real interval $I = [a, b]$

(b) if we denote by λ *the parameter, there exists a value* $\lambda_c \in I$ *such that the IPS is ergodic for* $\lambda < \lambda_c$ *and it is not ergodic for* $\lambda > \lambda_c$ *or vice versa. In this case, we call* λ_c *the critical value.*

Remark 3.3.2 (Irreducibility and Ergodicity on finite volume)**.** *Given a Markov processes on a finite state* X*,* $|X| < \infty$ *we say that it is irreducible if for any couple* $(\eta, \eta') \in X \times X$ *it holds* $\mathbb{P}^\eta(\eta_t = \eta') > 0$ *for some* $t \geqslant 0$*. A finite state irreducible Markov process has a unique stationary measure. If the following stronger requirement holds* $\exists t > 0$ *s.t. for any couple* $(\eta, \eta') \in X \times X$ *it holds* $\mathbb{P}^\eta(\eta_t = \eta') > 0$ *(note the inversion of the two conditions in the assumption!) then the process is ergodic, namely convergence to the stationary measure holds starting from any initial condition.*

Instead, for processes with infinite state space X, it is not enough to exhibit for each couple $(\eta, \eta') \in X \times X$ a (possibly infinite) chain of moves with positive transition rate connecting η to η' to deduce the uniqueness of the stationary measure. For example, for SIM in the low temperature regime there is more than one invariant measure, and yet the existence of a chain of moves that connect any two configurations is guaranteed by the fact that the rate at which we can change the value of the spin at a given site is strictly positive in any configuration.

Theorem 3.3.3 (A sufficient condition for ergodicity)**.** *Consider an IPS with local maps satisfying assumptions (A1) and (A2) stated in Section 2.3 and let* K *be defined as in* (2.2.20)*.*
If $K < 0$ *the IPS is ergodic.*

[7]The term *ergodic* (which is the current jargon in IPS) can be misleading. Indeed the term ergodic usually denotes a process for which all events that are invariant under time shifts have probability either zero or one. Actually the stationary process of an ergodic (in the sense of definition 3.3.1) IPS is ergodic in this sense. However, the converse is not true.

The proof of the above theorem is left as an exercice. It follows easily from the construction of the Markov process in Section 2.3 and using the result from Lemma 2.2.7. Hint: if there does not exist an influence path starting at time zero and reaching time t, the configuration at time t does not depend on the initial configuration ...

3.4 Reversibility

Definition 3.4.1 (Reversible measure)**.** *We say that $\mu \in \mathcal{P}$ is a reversible measure for the process if*

$$\int f P_t g \, d\mu = \int g \, P_t f \, d\mu, \quad \forall t \geqslant 0, \forall f, g \in \mathcal{C}(X)$$

Remark 3.4.2 (Reversibility vs stationarity)**.** *A reversible measure is necessarily invariant. Indeed, letting $g := 1$ in the above definition, we get the condition for stationarity .*

Theorem 3.4.3. *A measure μ is reversible iff for all $f, g \in \mathcal{D}$ it holds*

$$\int f \mathcal{L} g \, d\mu = \int g \mathcal{L} f \, d\mu$$

namely iff $\mathcal{L}$ is self-adjoint w.r.t. μ.

Exercise 7. *Prove Theorem 3.4.3 along analogous lines as Theorem 3.2.2 (iv).*

We can extend the definition of the IPS to negative times $(\eta_t)_{t \in \mathbb{R}}$ by setting

$$\mathbb{E}[f(\eta_t) | \mathcal{F}_s] = P_{t-s} f(\eta_s), \quad \forall f \in \mathcal{C} \ \forall s \ \text{ s.t. } -\infty < s \leqslant t$$

With this definition, if μ is stationary it holds

$$\mathbb{P}^\mu(\eta_t \in \cdot) = \mu(\cdot), \quad \forall t \in \mathbb{R}, \tag{3.4.1}$$

namely the process with distribution μ at time 0 preserves this distribution at any time.

Lemma 3.4.4. *If μ is reversible, the processes $\{\eta_t\}_{t \geqslant 0}$ and $\{\eta_{-t}\}_{t \geqslant 0}$ started at time zero from μ have the same law, i.e.*

$$\mathbb{P}^\mu((\eta_{-t})_{t \in \mathbb{R}} \in \cdot) = \mathbb{P}^\mu((\eta_t)_{t \in \mathbb{R}} \in \cdot), \tag{3.4.2}$$

namely the process is time reversible.

Proof. Let P_t^* be the adjoint operator of P_t on $L^2(X,\mu)$ (the space of functions with $\mu(f^2)<\infty$) which is defined by requiring that for all f,g it holds

$$\mu(gP_t^*f)=\mu(fP_tg) \tag{3.4.3}$$

Note that P_t^* is the semigroup of the time-reversed process, indeed for all g it holds

$$\mu(gP_t^*f)=\int fP_tg d\mu=\mathbb{E}^\mu(f(\eta_0)g(\eta_t))=\mathbb{E}^\mu(\mathbb{E}(f(\eta_0)g(\eta_t)|\eta_t))=\int\mathbb{E}(f(\eta_0|\eta_t=\xi)g(\xi)\mu(d\xi)$$

thus

$$P_t^*f(\eta)=\mathbb{E}(f(\eta_0)|\eta_t=\eta)).$$

Since Definition 3.4.1 together with the definition of P_t^* in (3.4.3) imply that μ is reversible iff $P_t=P_*t$, this means that μ is a reversible measure iff the law of the stationary process equals the law of the time reversed process $\square$

From the above remark it is easily seen that CP cannot have reversible measure.

Lemma 3.4.5 (Invariance and reversibility in finite volume)**.** *Given a Markov process on X with $|X|<\infty$, there exists $c(\eta,\eta')\geqslant 0$ s.t. $c(\eta,\eta)=0$ and the action of the generator on any continuous function can be written as*

$$\mathcal{L}f(\eta)=\sum_{\eta'\in X}c(\eta,\eta')(f(\eta')-f(\eta))$$

The following holds

- *$\mu\in\mathcal{P}(X)$ is invariant iff*

$$\sum_\eta[\mu(\eta)c(\eta,\xi)-\mu(\xi)c(\xi,\eta)]=0\ \ \forall\xi$$

- *$\mu\in\mathcal{P}(X)$ is reversible iff*

$$\mu(\xi')c(\xi',\xi)=\mu(\xi)c(\xi,\xi')\ \ \forall\xi,\xi'.$$

  Note that this condition, which is called detailed balance, corresponds to requiring that each term is zero in the sum appearing in the stationarity condition.

Proof. Theorem 3.2.2(iv) and the finiteness of X imply that μ is invariant iff $\mu(\mathcal{L}\mathbb{1}_\xi)=0$ for any $\xi\in X$. Then note that $\mathcal{L}\mathbb{1}_\xi(\eta)=1_{\eta\neq\xi}c(\eta,\xi)-1_{\eta=\xi}\sum_{\eta'}c(\xi,\eta')$ and therefore $\mu(\mathcal{L}\mathbb{1}_\xi)=\sum_{\eta\neq\xi}\mu(\eta)c(\eta,\xi)-\mu(\xi)\sum_{\eta'}c(\xi,\eta')$. Theorem 3.4.3 in finite volume implies that reversibility holds iff $\mu(\mathbb{1}_\xi\mathcal{L}\mathbb{1}_{\xi'})=\mu(\mathbb{1}_{\xi'}\mathcal{L}\mathbb{1}_\xi)$ for any two configuraitions ξ,ξ'. $\square$

Exercise 8. *Prove that FA-1f model (see Section 1.5) on $\mathbb{Z}^d$, for any $q \in (0,1]$ and for any $d \geqslant 1$ has, besides the trivial invariant measure δ_1, another invariant measure (which depends on q) . Note that this in particular implies that this model is never ergodic (see definition 3.3.1).*

[**Hint.** *Consider FA-1f on a finite interval $[a,b]$ with empty boundary condition on $b+1$ and $a-1$. This model is irreducible and therefore has a unique invariant measure, which is easily seen not to be δ_1. Search for the explicit form of this invariant measure. Notice that there is a measure that satisfies the detailed balance condition stated in Remark 3.4.5, which is therefore the unique invariant measure of this finite volume process. Which is this measure? From the knowledge of the finite volume reversible measure try to guess which is a reversible invariant measure for the model on $\mathbb{Z}^d$ and verify that it satisfy the sufficient and necessary condition for reversibility of Theorem 3.4.3]*

3.5 Monotonicity or attractivness

For $X = \{0,1\}^{\mathbb{Z}^d}$ we define the following *partial order*

$$\eta \leqslant \xi \quad \text{iff} \quad \eta(x) \leqslant \xi(x) \quad \forall x \in \mathbb{Z}^d$$

We say that a function $f : X \to \mathbb{R}$ is *increasing* if

$$\eta \leqslant \xi \qquad \text{implies} \quad f(\eta) \leqslant f(\xi)$$

and we let $\mathcal{N} \subset \mathcal{C}$ be the set of continuous increasing functions. For example, for any $A \subset \mathbb{Z}^d$ with $|A| < \infty$, the function $f_A(\eta) := \prod_{x \in A} \eta(x)$ is increasing.

Remark 3.5.1. *Given μ_1, μ_2 two probability measures on S, if it holds*

$$\mu_1(f) = \mu_2(f) \ \ \forall f \in \mathcal{N}$$

then $\mu_1 = \mu_2$. This follow from the observation that we can rewrite any function $g : X \to \mathbb{R}$ as $f_1 - f_2$ with f_1 and f_2 increasing.

Given μ_1, μ_2 two probability measures on S we say that μ_1 is *stochastically dominated* by (or stochastically smaller than) μ_2 and we write $\mu_1 \leqslant \mu_2$ if the following holds:

$$\int f d\mu_1 \leqslant \int f d\mu_2 \quad \forall f \in \mathcal{N}$$

Recall that $\delta_0 \in \mathcal{P}(X)$ (respectively $\delta_1 \in \mathcal{P}(X)$) is the measure concentrated on the configuration in which all sites have occupation variables equal to zero (respectively to one). Therefore from the above definition it follows immediately that for any $\mu \in \mathcal{P}(X)$ it holds

$$\delta_0 \leqslant \mu \leqslant \delta_1 \tag{3.5.1}$$

Definition 3.5.2. *We say that an IPS is monotone (or attractive) if $\mu_1 \leqslant \mu_2$ implies $\mu_1 P_t \leqslant \mu_2 P_t$ for all $t \geqslant 0$.*

Exercise 9. *Prove that Definition 3.5.2 is equivalent to the following: "We say that an IPS is monotone if for any $f \in \mathcal{N}$ and any $t \geqslant 0$, the function $P_t f$ also belongs to $\mathcal{N}$."*
[Hint: use that fact that for any $\mu \in \mathcal{P}(X)$, any function f and for any time $t \geqslant 0$ it holds $\int f d(\mu P_t) = \int P_t f d\mu$.

Recall that we call *spin IPS* an interacting particle system for which the configuration space is of the form S^Λ with onsite space $S = \{0, 1\}$ (or any other two state space) and all elementary moves are all of the form $\eta \to \eta^x$ with η^x the configuration in which only the value at site x has been changed w.r.t. configuration η. Therefore the corresponding generator takes the form

$$\mathcal{L} f(\eta) = \sum_{x \in \Lambda} r(x, \eta)(f(\eta^x) - f(\eta) \tag{3.5.2}$$

Theorem 3.5.3. *A spin IPS is monotone iff the following holds: for any couple of configurations η, ξ that satisfy $\eta \leqslant \xi$ it holds*

(i) $r(x, \eta) \leqslant r(x, \xi)$ *if* $\eta(x) = \xi(x) = 0$

(ii) $r(x, \eta) \geqslant r(x, \xi)$ *if* $\eta(x) = \xi(x) = 1$

Note that these conditions on the rates roughly say that an occupation variable tries to align with its neighbours, hence the name "attractive". Indeed (i) requires that the rate to flip from 0 to 1 is higher in a configuration that has more ones, while (ii) requires that the rate to flip from 1 to 0 is higher in a configuration that has more zeros.

Exercise 10. *Use Theorem 3.5.3 to prove that*

- *CP, VM are monotone*
- *FA-1f model is* not *monotone.*

Exercise 11. *Consider SIM under the change of variables* $\eta \in \{\pm 1\}^\Lambda \to \tilde{\eta} \in \{0,1\}^\Lambda$ *with* $\tilde{\eta}(x) = \frac{1-\eta(x)}{2}$ *for all* x *and prove using Theorem 3.5.3 that it is monotone.*

Before giving the proof of Theorem 3.5.3 we should understand better the notion of stochastic domination among measure by introducing the notion of *coupling*.

Definition 3.5.4 (Coupling). *A coupling of two random variables is a joint construction of the variables on a common probability space. More precisely, given* $\mu_1 \in \mathcal{P}(X)$ *and* $\mu_2 \in \mathcal{P}(X)$*, a coupling is a measure* μ *on* $X \times X$ *whose marginals are* μ_1 *and* μ_2 *, i.e. such that for* $i \in [1,2]$ *and any* $A \subset X$ *it holds* $\mu(\{\eta : \eta^{(i)} \in A\}) = \mu_i(A)$*, where for* $\eta \in X \times X$ *we denote by* $\eta^{(1)}$*) (respectively* $\eta^{(2)}$ *the first (respectively the second) coordinate of* η*.*

Theorem 3.5.5 (Strassen). *Given* μ_1*,* μ_2 *on* X*, it holds*

$$\mu_1 \leqslant \mu_2$$

iff $\exists$ *a coupling* μ *on* $X \times X$ *s.t.*

$$\mu\{\eta = (\eta^1, \eta^2) : \eta^1 \leqslant \eta^2\} = 1$$

Proof. A direction of the proof is easy. Fix f an increasing function. If a coupling μ with the property $\mu\{\eta : \eta^{(1)} \leqslant \eta^{(2)}\} = 1$ exists, with probability 1 w.r.t. μ it holds $f(\eta^{(1)}) \leqslant f(\eta^{(2)})$. Therefore

$$\mu_1(f) = \int f(\eta^{(1)}) d\mu(\eta) \leqslant \int f(\eta^{(2)}) d\mu(\eta) \leqslant \mu_2(f)$$

The other direction is more tricky, full proof on [Lig85] (Theorem 2.4, pag 72). $\square$

Exercise 12. *Fix* $p_1, p_2 \in [0,1]$ *with* $p_1 < p_2$*. Let* $X = \{0,1\}^\Lambda$ *with* $|\Lambda| < \infty$ *and*

$$\mu_i = \prod_{x \in \Lambda} p_i^{\eta(x)} (1-p_i)^{1-\eta(x)} \quad \text{for } i \in \{1,2\}.$$

Construct a coupling μ *of* μ_1 *and* μ_2 *such that* $\mu\{\eta = (\eta^1, \eta^2) : \eta^1 \leqslant \eta^2\} = 1$*.* [**Hint.** *Case* $|\Lambda| = 1$*. Let* z *be a uniform random variable on the interval* $[0,1]$*. If you set* $\eta^1 = \mathbb{1}_{z<p_1}$ *and* $\eta^2 = \mathbb{1}_{z<p_2}$ *it follows that* $\eta^1 \leqslant \eta^2$ *and it is easily checked that* η^1 *is distributed with* μ_1 *and* η^2 *is distributed with* μ_2*. Therefore we have provided the coupling. It is now very easy to extending the coupling to the case* $|\Lambda| > 1$*.*]

Neither Definition 3.5.4 nor Theorem 3.5.5 give an efficient way to check whether, given $\mu_1, \mu_2 \in \mathcal{P}$, one of the two measures is stochastically dominated by the other. A precious result is the following sufficient condition. For $\eta, \xi \in X$ let $\eta \vee \xi \in X$ and $\eta \wedge \xi \in X$ be the configurations defined by

$$\eta \vee \xi(x) = \max(\eta(x), \xi(x)), \quad \eta \wedge \xi(x) = \min(\eta(x), \xi(x))$$

Theorem 3.5.6 (Holley theorem)**.** *Given μ_1, μ_2 that assign a strictly positive probability to any point in X, if it holds*

$$\mu_1(\eta \wedge \xi)\mu_2(\eta \vee \xi) \geqslant \mu_1(\eta)\mu_2(\xi) \quad \forall \eta, \xi \in X$$

then it holds

$$\mu_1 \leqslant \mu_2.$$

We refer the reader to [Lig85] pag. 75 for a detailed proof. The strategy of the proof is to construct a Markov chain (η_t, ξ_t) on $X \times X$ with starting point η, ξ s.t. $\eta \leqslant \xi$ and preserving this property during the evolution and such that the first (resp. second) marginal is a Markov chain with stationary measure μ_1 (resp. μ_2).

We are now ready to prove the necessary and sufficient condition for an IPS to be monotone.

Proof of Theorem 3.5.3. We should prove that

(a) any IPS satisfying conditions (i) and (ii) is necessarily monotone

(b) any monotone IPS satisfies conditions (i) and (ii)

Proof of (a). Fix $\mu_1, \mu_2 \in \mathcal{P}$ s.t. $\mu_1 \leqslant \mu_2$. Our goal is to show that (i) and (ii) imply that for any $t > 0$ it holds $\mu_1 P_t \leqslant \mu_2 P_t$. To this aim we construct a coupling of $\mathbb{P}^{\mu_1}, \mathbb{P}^{\mu_2}$ that preserves the partial order at any fixed time, namely a probability $P(\{\eta_t^1\}_{t \geqslant 0}, \{\eta_t^2\}_{t \geqslant 0})$ with marginals $\mathbb{P}^{\mu_1}$ and $\mathbb{P}^{\mu_2}$ and such that $P(\{\eta_t^1 \leqslant \eta_t^2 \quad \forall t \geqslant 0\}) = 1$. If we exhibit such a coupling, then the result follows by Theorem 3.5.5.
In order to construct the coupling with the desired properties notice that, since $\mu_1 \leqslant \mu_2$ and again thanks to Theorem 3.5.5, there exists a distribution μ on $X \times X$ that is a coupling for μ_1, μ_2 and that satisfies $\mu(\eta^1 \leqslant \eta^2) = 1$. Consider the Markov process $(\eta_t \times \xi_t)_{t \geqslant 0}$ on the configuration space $X \times X$ with initial configuration $\eta_0 \times \xi_0$ distributed according to μ and with elementary moves corresponding to (i) either the simultaneous change of η and ξ on the same site; or (ii) only η changes on a single site; or (iii) only ξ changes on a single site x. These moves occur at the following rates

1) $(\eta, \xi) \to (\eta^x, \xi^x)$ at rate $r(x,\eta)\mathbb{1}_{\eta(x)=\xi(x)=0} + r(x,\xi)\mathbb{1}_{\eta(x)=\xi(x)=1}$;

2) $(\eta, \xi) \to (\eta^x, \xi)$ at rate $(r(x,\eta) - r(x,\xi))\,\mathbb{1}_{\eta(x)=\xi(x)=1} + r(x,\eta)\mathbb{1}_{\eta(x)=0}\mathbb{1}_{\xi(x)=1}$;

3) $(\eta, \xi) \to (\eta, \xi^x)$ at rate $(r(x,\xi) - r(x,\eta))\,\mathbb{1}_{\eta(x)=\xi(x)=0} + r(x,\xi)\mathbb{1}_{\eta(x)=0}\mathbb{1}_{\xi(x)=1}$

It is immediate to verify that

- if $\eta \leqslant \xi$, each elementary move preserves this partial order. Thus since the event $\eta \leqslant \xi$ has probability one under the initial distribution, at any time it holds $\eta_t \leqslant \xi_t$ with probability one.
- (i) and (ii) and the partial order among η and ξ guarantee that all the transition rates of the constructed process are non negative, thus the process is well defined
- each marginal process evolves according to the correct transition rates

Proof of (b). Consider a monotone spin IPS. Fix a site $x \in \Lambda$ and define a function $f : X \to \mathbb{R}$ by letting $f(\eta) = \eta(x)$. Since f is increasing, by monotonicity of the process also $P_t f$ is increasing. Choose two configurations η, ξ s.t.

$$\eta(x) = \xi(x) \quad \text{and} \quad \eta \leqslant \xi.$$

Then it holds

$$\mathcal{L}f(\eta) - \mathcal{L}f(\xi) = \lim_{t \to 0} \frac{P_t f(\eta) - P_t f(\xi) - f(\eta) + f(\xi)}{t} \leqslant 0 \tag{3.5.3}$$

where we used the fact that: $f(\eta) = \eta(x) = \xi(x) = f(\xi)$, and $P_t f(\eta) \leqslant P_f f(\xi)$ (recall that $\eta \leqslant \xi$ and $P_t f$ is increasing). Furthermore it holds

$$\mathcal{L}f(\eta) = \sum_y r(y,\eta)(f(\eta^y) - f(\eta)) = r(x,\eta)(1 - 2\eta(x)) \tag{3.5.4}$$

Putting (3.5.4) and (3.6.7) together yields

$$r(x,\eta)(1 - 2\eta(x)) \leqslant r(x,\xi)(1 - 2\xi(x))$$

which immediately imply the validity of conditions (i) and (ii).

□

Theorem 3.5.7 (Invariant measures for monotone spin IPS)**.** *For a monotone spin IPS it holds*

(a) $\delta_0 P_s \leqslant \delta_0 P_t$ *for all* $s \in [0, t]$

(b) $\delta_1 P_s \geqslant \delta_1 P_t$ *for all* $s \in [0, t]$

(c) $\delta_0 P_t \leqslant \mu P_t \leqslant \delta_1 P_t$ *for all* $t \geqslant 0$ *and any* μ

(d) $\lim_{t\to\infty} \delta_0 P_t$ *and* $\lim_{t\to\infty} \delta_1 P_t$ *exist. We let* $\underline{\nu} := \lim_{t\to\infty} \delta_0 P_t$ *and* $\bar{\nu} := \lim_{t\to\infty} \delta_1 P_t$

(e) let $\mu \in \mathcal{P}$. *If* $\nu := \lim_{t\to\infty} \mu P_t$ *exists, it holds* $\underline{\nu} \leqslant \nu \leqslant \bar{\nu}$

(f) $\underline{\nu}$ *et* $\bar{\nu}$ *are extremal on* $\mathcal{I}$

Proof.

(a) Fix s, t with $s \leqslant t$. From (3.5.1) it follows that $\delta_0 \leqslant \delta_0 P_{t-s}$. Due to monotonicity this order is preserved at any later time, thus $\delta_0 P_s \leqslant \delta_0 P_{t-s} P_s = \delta_0 P_t$;

(b) analogous to (a)

(c) Follows immediately using (3.5.1) and the fact that the IPS is monotone;

(d) Fix any increasing sequence of times $\{t_n\}_{n\in\mathbb{N}}$ and let $\mu_n := \delta_0 P_{t_n}$. By point (a) we have that $\mu_n \leqslant \mu_m$ for any $n < m$, thus $\lim_{n\to\infty} \mu_n$ exists and belongs to $\mathcal{P}$ due to compactness. Suppose that we fix two increasing sequences of times $\{t_n^1\}_{n\in\mathbb{N}}$ and $\{t_n^2\}_{n\in\mathbb{N}}$ and call μ_1 and μ_2 the corresponding limit measures. Then it follows that for any $f \in \mathcal{N}$ it holds $\mu_1(f) = \mu_2(f)$, which implies $\mu_1 = \mu_2$ by Remark 3.5.1. We proceed analogously to prove the existence of $\lim_{t\to\infty} \delta_1 P_t$

(e) it follows from (c), (d) and monotonicity

(f) $\underline{\nu}$ et $\bar{\nu}$ are invariant thanks to Theorem 3.2.2. To prove extremality we proceed by contradiction. Suppose that $\bar{\nu}$ is not extremal, namely suppose that $\exists \mu_1, \mu_2 \in \mathcal{I}$ with μ_1 and μ_2 different from $\bar{\nu}$ and $\alpha \in (0,1)$ s.t. $\bar{\nu} = \alpha\mu_1 + (1-\alpha)\mu_2$. Since μ_1, μ_2 are invariant measures they can be obtained as infinite limit of a process started with themselves, thus (e) implies $\mu_1, \mu_2 \leqslant \bar{\nu}$. Therefore for any $f \in \mathcal{N}$ it holds $\mu_i(f) \leqslant \bar{\nu}(f)$ and $\bar{\nu}(f) = \alpha\mu_1(f) + (1-\alpha)\mu_2(f)$ which implies $\mu_1(f) = \mu_2(f) = \bar{\nu}(f)$. Thus we deduce that for any $f \in \mathcal{N}$ it holds $\mu_1(f) = \mu_2(f) = \bar{\nu}(f)$ which implies by Remark 3.5.1 that $\mu_1 = \mu_2 = \bar{\nu}$ thus contradicting the hypothesis.

□

The interested reader might have a look at Theorem 3.13 p.152 in [Lig85] which proves that

$$r(x,\eta) + r(x,\eta^x) > 0 \;\; \forall x \in \mathbb{Z}, \forall \eta \in X : \eta(x-1) \neq \eta(x+1)$$

is a sufficient condition for one-dimensional ($\Lambda = \mathbb{Z}$) attractive IPS that guarantees that the only extremal invariant measures are $\underline{\nu}$ and $\bar{\nu}$. This condition is satisfied by CP, SIM, and VM.

Corollary 3.5.8. *For a monotone spin IPS the following three conditions are equivalent*

1. *the process is ergodic*
2. *$\mathcal{I}$ is a singleton*
3. *$\bar{\nu} = \underline{\nu}$*

3.6 Duality

Duality is a very useful tool that allows sometimes to connect two different IPS expressing the law of one process in term of the other and vice versa.
A first example: consider VM in d=1 and focus on the evolution of the position of the boundaries separating islands of 0's and 1's. It is not difficult to realise that these boundaries evolve as simple symmetric annihilating random walks on $\mathbb{Z}$: when two boundaries meet they annihilate and otherwise each boundary moves as a random walks jumping at rate 1/2 to each of its 2 nearest neighbours. So one can translate the probability law of one-dimensional VM in terms of the law for one-dimensional simple symmetric annihilating random walks. These two systems are dual one to the other [8].

Let us start by giving an abstract definition of duality. We will later provide specific examples.

Definition 3.6.1 (Duality and Self-duality)**.** *Given two Markov processes $(\eta_t)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$ on space states X and Y, and given $H(\eta,\xi)$ a bounded measurable function on $X\times Y$, we say that $(\eta_t)_{t\geqslant 0}$ and $(\xi_t)_{t\geqslant 0}$ are dual to each other w.r.t. H if*

$$\mathbb{E}^{\eta} H(\eta_t,\xi) = \mathbb{E}^{\xi} H(\eta,\xi_t), \quad \forall \eta\in X, \xi\in Y, t\geqslant 0 \tag{3.6.1}$$

We say that $(\eta_t)_{t\geqslant 0}$ is self dual if it is dual w.r.t. H to a process $(\xi_t)_{t\geqslant 0}$ that has the same law as $(\eta_t)_{t\geqslant 0}$.

[8] Actually duality here holds configuration wise, namely in a stronger sense than the one given by Definition 3.6.1.

Theorem 3.6.2. *[A class of duality relations for spin IPS]*
Let

$$X = \{0,1\}^{\mathbb{Z}^d}, \quad Y := \{A : A \subset \mathbb{Z}^d, |A| < \infty\}$$

and

$$H(\eta, A) := \prod_{x\in A}(1-\eta(x)) \ \textit{for} \ A \neq \emptyset, \quad H(\eta, \emptyset) := 1 \tag{3.6.2}$$

Fix $c : \mathbb{Z}^d \to \mathbb{R}$ and $p : \mathbb{Z}^d \times Y \to [0,1]$ satisfying

(A1) $\sup_x c(x) < \infty$

(A2) $c(x) \geqslant 0 \ \forall x \in \mathbb{Z}^d$,

(A3) $p(x, A) \geqslant 0 \ \ \forall x \in \mathbb{Z}^d, \forall A \in Y$

(A4) $\sum_{B\in Y} p(x, B) = 1 \ \ \forall x \in \mathbb{Z}^d$

(A5) $\sup_x c(x) \sum_A |A| p(x, A) < \infty$.

Fix $\eta \in X$, $A \in Y$ and define two Markov processes $(\eta_t)_{t\,\geqslant\,0}$ on X and $(A_t)_{t\,\geqslant\,0}$ on Y as follows

- *$(\eta_t)_{t\,\geqslant\,0}$ is the spin IPS with $\eta_0 = \eta$ and rates*

$$r(x, \eta) := c(x)\left[\eta(x)\sum_{A\subset Y} p(x, A)H(\eta, A) + (1-\eta(x))\sum_{A\subset Y} p(x, A)(1 - H(\eta, A))\right] \tag{3.6.3}$$

- *$(A_t)_{t\,\geqslant\,0}$ is the Markov process with $A_0 = A$ and rate $q(A, B)$ to go from state A to B defined by*

$$q(A, B) := \sum_{x\in A} c(x) \sum_{F:(A\setminus\{x\})\cup F = B} p(x, F) \quad \textit{for} \ \ A \neq \emptyset \tag{3.6.4}$$

$$q(\emptyset, B) = 1 \ \textit{if} \ B = \emptyset,$$
$$q(\emptyset, B) = 0 \ \textit{if} \ B \neq \emptyset$$

Notice that assumption (A5) guarantees that for any $t \geqslant 0$ it holds $|A_t| < \infty$, thus $(A_t)_{t\,\geqslant\,0}$ is a Markov process on Y. Then $(\eta_t)_{t\,\geqslant\,0}$ and $(A_t)_{t\,\geqslant\,0}$ are dual to each other w.r.t. the function H defined in (3.6.2).

Proof of Theorem 3.6.2. Let $\mathcal{L}$ and $\bar{\mathcal{L}}$ be the generator of $(\eta_t)_{t\,\geqslant\,0}$ and $(A_t)_{t\,\geqslant\,0}$, so that

$$\mathcal{L}H(\eta, A) = \sum_{x\in\mathbb{Z}^d} r(x, \eta)[H(\eta^x, A) - H(\eta, A)] \tag{3.6.5}$$

$$\bar{\mathcal{L}}H(\eta, A) = \sum_{B \in Y} q(A, B)[H(\eta, B) - H(\eta, A)] \tag{3.6.6}$$

Using the form of the rates (see (3.6.3) and (3.6.4)), assumption (A4) and the form of the function H (see (3.6.2)) we get

$$\mathcal{L}H(\eta, A) = \sum_{x \in A} c(x) \left[(2\eta(x) - 1) \sum_{F \subset Y} p(x, F) H(\eta, F) + (1 - \eta(x)) \right] [H(\eta^x, A) - H(\eta, A)] =$$

$$\sum_{x \in A} c(x) \left[\eta(x) \left(\sum_{F \subset Y} p(x, F) H(\eta, F) \right) H(\eta, A \setminus x) + (1 - \eta(x)) \left(\sum_{F \subset Y} p(x, F) H(\eta, F) - 1 \right) H(\eta, A) \right]$$

Noticing that for $x \in A$ it holds

$$H(\eta, A)(1 - \eta(x)) = H(\eta, A \setminus x)(1 - \eta(x)) = H(\eta, A)$$

we get

$$\mathcal{L}H(\eta, A) = \sum_{x \in A} c(x) \left[\left(\sum_{F \subset Y} p(x, F) H(\eta, F) \right) H(\eta, A \setminus x) - H(\eta, A) \right] \tag{3.6.7}$$

Using (3.6.2) and (3.6.4) we get

$$H(\eta, F) H(\eta, A \setminus x) = H(\eta, A \setminus x \cup F)$$

and

$$\sum_{B} q(A, B) = \sum_{x \in A} c_x$$

which, inserted in (3.6.7) yield

$$\mathcal{L}H(\eta, A) = \bar{\mathcal{L}}H(\eta, A) \ \ \forall \eta \in X, \, \forall A \in Y \tag{3.6.8}$$

Therefore, letting P_t (respectively $\bar{P}_t$) be the semigroup associated to $\mathcal{L}$ (respectively to $\bar{\mathcal{L}}$) and using Theorem 3.2.3 which gives $\mathcal{L}P_t f = P_t(\mathcal{L}f)$ we get

$$\begin{aligned} \frac{d}{dt} \mathbb{E}^{\eta} H(\eta_t, A) = \frac{d}{dt} P_t H(\eta, A) = \mathcal{L}(P_t H(\eta, A)) = P_t(\mathcal{L}H(\eta, A)) = \\ P_t(\bar{\mathcal{L}}H(\eta, A)) = \bar{\mathcal{L}} P_t H(\eta, A) = \bar{\mathcal{L}} \left(\mathbb{E}^{\eta} H(\eta_t, A) \right) \end{aligned} \tag{3.6.9}$$

and

$$\frac{d}{dt} \mathbb{E}^{A} H(\eta, A_t) = \frac{d}{dt} \bar{P}_t H(\eta, A) = \bar{\mathcal{L}} \bar{P}_t H(\eta, A_t) = \bar{\mathcal{L}} \left(\mathbb{E}^{A} H(\eta, A_t) \right) \tag{3.6.10}$$

The above equations (3.6.9) and (3.6.10) say that the quantities $\mathbb{E}^A H(\eta, A_t)$ and $\mathbb{E}^{\eta} H(\eta_t, A)$, which are equal at time 0 since $\eta_0 = \eta$ and $A_0 = A$, satisfy the same differential equation

$\frac{d}{dt}t = \bar{\mathcal{L}}f$. Since the assumptions (A1)-(A5) guarantee that the process generated by $\mathcal{L}$ is unique it follows that

$$\mathbb{E}^\eta H(\eta_t, A) = \mathbb{E}^A H(\eta, A_t) \quad \text{for all} \quad t \geqslant 0$$

□

Remark 3.6.3. *Let $\emptyset$ denote the configuration $\eta \in X$ s.t. $\eta(x) = 0$ for all $x \in \mathbb{Z}^d$. The configuration $\emptyset$ is a trap for the process $(\eta_t)_{t \geqslant 0}$. More precisely, if $\eta_s = \emptyset$, then it holds $\eta_t = \emptyset$ for all $t \geqslant s$ with probability one. Indeed if $\eta = \emptyset$ it holds $r(x, \eta) = 0$ for all $x \in \mathbb{Z}^d$ (use $H(\emptyset, A) = 1$ for all A). The same is true for the process $(A_t)_{t \geqslant 0}$, for which the configuration $\emptyset$ (now denoting as an element of Y) is a trap.*

Corollary 3.6.4. *Fix $\eta \in X$ and $A \in Y$, and let $\mathbb{P}^\eta$ and $\mathbb{P}^A$ be the laws defined in Theorem 3.6.2. It holds*

$$\mathbb{P}^\eta(\eta_t(x) = 0 \; \forall x \in A) = \mathbb{P}^A(\eta(x) = 0 \, \forall x \in A_t) \quad \forall \, t \geqslant 0$$

Proof. The proof follows easily by noticing that

$$\mathbb{E}^\eta \left(\prod_{x \in A} (1 - \eta_t(x)) \right) = \mathbb{P}^\eta \left(\eta_t(x) = 0 \; \forall x \in A \right)$$

and

$$\mathbb{E}^A \left(\prod_{x \in A_t} (1 - \eta(x) \right) = \mathbb{P}^A \left(\eta(x) = 0 \; \forall x \in A_t \right)$$

□

Exercise 13. *Prove that if we let*

$$c(x) := 1 + 2d\lambda, \qquad p(x, A) := \frac{1}{1 + 2d\lambda} \mathbb{I}_{A = \emptyset} + \frac{\lambda}{1 + 2d\lambda} \sum_{y : y \sim x} \mathbb{I}_{A = \{x, y\}}$$

the two functions satisfy (A1)–(A5) of Theorem 3.6.2. Furthermore using (3.6.2), (3.6.3) *and* (3.6.4) *it follows that $(\eta_t)_{t \geqslant 0}$ is CP(d, λ), i.e. the contact process with infection parameter λ on $\mathbb{Z}^d$ and $(A_t)_{t \geqslant 0}$ is the process evolving as the subset of $\mathbb{Z}^d$ containing all the infected sites of a CP(d, λ).*

Hint. *The present choice of c and p together with* (3.6.3) *and* (3.6.4) *yield*

$$r(x, \eta) = \eta(x) + (1 - \eta(x))\lambda \sum_{y : y \sim x} \eta(y)$$

$$q(A, B) = |\{x : x \in A, A \setminus x = B\}| + \lambda |\{(x, y) : x \in A, A \cup y = B, y \sim x\}|$$

Remark 3.6.5. *As a by-product of Exercise 13 we have proven that CP is self-dual. For an alternative proof of self-duality for the contact process the interested reader might read section 2.1 of [Swaa] (see in particular Lemma 2.1 therein)* [9]*.*

Exercise 14 (Duality of the Voter model with coalescing random walks)**.** *Let*

$$c(x) := 1, \quad p(x, A) := \frac{1}{2d} \sum_{y: y \sim x} \mathbb{1}_{A=\{y\}}.$$

Prove that

- *these functions satisfy (A1)–(A5) of Theorem 3.6.2*
- *with this choice $(\eta_t)_{t \geqslant 0}$ is the voter model (VM) while $(A_t)_{t \geqslant 0}$ evolves on Y with rate*

$$q(A, B) = |\{(x, y) : x \in A, B = (A \setminus \{x\}) \cup \{y\}, y \sim x\}|$$

- *$A_{tt \geqslant 0}$ can be equivalently described as a Markov process $(\sigma_t)_{t \geqslant 0}$ on X by making the identification $\sigma(X)(x) = 1$ iff $x \in A$ evolving with generator*

$$\mathcal{L}_{Crw} f(\eta) = \sum_x \frac{1}{2d} \sum_{y, y \sim x} (f(\bar{\eta}^{xy}(x) - f(\eta))$$

  where we let

$$\bar{\eta}^{xy}(z) := \begin{cases} 0 & \text{if } z = x \\ \max(\eta(x), \eta(y)) & \text{if } z = y \\ \eta(z) & \text{otherwise} \end{cases} \tag{3.6.11}$$

  In words, this correspond to random walks that move at rate 1 to a randomly chosen neighbour, coalesce when they meet and otherwise evolve independently.

[9] The proof presented by Swart is completely graphical (and less abstract) and based on a simple observation. Draw the occurrences of the arrival times of the infection and healing maps as described in Section 2.1. For $A \subset \mathbb{Z}^d$ and $s, t \geqslant 0$, let $\eta_t^{A,s}$ be the set of points $i \in \mathbb{Z}^d$ s.t. $\exists y \in A$ with $(y, s) \to (i, A)$. Let also $\eta_t^{+,A,s}$ be the set of points $i \in \mathbb{Z}^d$ s.t. there exists $y \in A$ with $(i, s-t) \to (y, s)$. Then the law of $\eta_t^{+,A,s}$ and $\eta_t^{A,s}$ coincide. The proof of the above result can be done as follows: (1) take a piece of paper, (2) draw a realisation of the arrows and crosses corresponding to the arrival times of the infection and healing maps of CP (see Section 2.1) , (3) turn the paper upside down, (4) invert the direction of each infection arrow and put a $-$ sign in front of each time (so that e.g. an original horizontal line at time 10 is now at time -10), (5) notice that thanks to the fact that infections from $i \to i+1$ have the same rate as infections from $i \to i-1$, the crosses and arrows that you see now are still distributed as for a contact process, (6) notice that a path of influence occurs now from $j, -(s+t)$ iff in your original (non upside down) picture a path of influence was occurring from i, s to $j, s+t$.

Remark 3.6.6 (Duality of the Voter model with annihilating random walks)**.** *An IPS can have different dual processes (with respect to different functions H). For example, the VM is also dual to annihilating random walks. More let* $(\sigma_t)_{t \geqslant 0}$ *be a Markov process on* $(0,1)^{\mathbb{Z}^d}$ *with generator*

$$\mathcal{L}_{Arw} f(\sigma) = \sum_x \frac{1}{2d} \sum_{y, y\sim x} (f(\bar{\sigma}^{xy}(x) - f(\sigma))$$

where

$$\tilde{\sigma}^{xy}(z) := \begin{cases} 0 & \text{if } z = x \\ \sigma(x) + \sigma(y) \ \ mod(2) & \text{if } z = y \\ \sigma(z) & \text{otherwise }. \end{cases} \tag{3.6.12}$$

For any $\eta, \sigma \in X$ *it holds*

$$\mathbb{E}^{\eta}\left[(-1)^{\eta_t \cdot \sigma}\right] = \mathbb{E}^{\sigma}\left[(-1)^{\eta \cdot \sigma_t}\right]$$

with $(\eta_t)_{\geqslant 0}$ *the VM and* $\eta \cdot \sigma \subset X$ *the configuration which equals point-wise the product of the two configurations,*

$$\eta \cdot \sigma(x) = \eta(x)\sigma(x). \tag{3.6.13}$$

The following theorem is the key ingredient to prove that for the contact process the critical value of λ separating the regime in which CP started from a single infection dies out from the regime in which it survives coincides with the critical value separating the ergodic and non ergodic regimes. We postpone this proof (see Theorem 4.1.4) to the next chapter where the contact process is studied in detail. Below, we will show another application of this theorem to the case of the voter model (see Theorem 3.6.10).

Theorem 3.6.7. *[Duality and Ergodicity] Let* $(\eta_t)_{t \geqslant 0}$ *and* $(A_t)_{t \geqslant 0}$ *be defined as in Theorem 3.6.2. Then* $(\eta_t)_{t \geqslant 0}$ *is ergodic iff* $(A_t)_{t \geqslant 0}$ *is ergodic, namely iff it holds*

$$P^A(\tau < \infty) = 1 \ \ \forall A \in Y$$

where τ *is a random time corresponding to the extinction time of* A_t*, namely*

$$\tau := \inf\{t \geqslant 0 : A_t = \emptyset\}.$$

The importance of this theorem is seen by noticing that $(A_t)_{t \geqslant 0}$ lives on the state space Y of *finite* subsets of $\mathbb{Z}^d$, therefore extinction is in general easier to prove .

Corollary 3.6.8. *VM is not ergodic, namely* $\delta_0 = \underline{\nu} \neq \bar{\nu} = \delta_1$.

Proof. By definition coalescing random walk never die out, namely starting from any finite number of random walks there is at least one walker at any subsequent time, therefore it holds $P^A(\tau < \infty) = 0$. This, together with Theorem 3.6.7, implies that VM is not ergodic. Of course this could have been proved immediately by noticing that starting from the configuration completely one and the configuration completely zero are traps for VM and therefore $\underline{\nu} \neq \bar{\nu}$ $\square$

Proof of Theorem 3.6.7.

- If it holds

$$P^A(\tau < \infty) = 1 \ \ \forall A \in Y,$$

then using Theorem 3.6.2, we write for any μ, any A and any $t \geqslant 0$

$$\mathbb{P}^\mu(\eta_t(x) = 0\, \forall x \in A) = \int \mathbb{E}^\eta H(\eta_t, A)\mu(d\eta) = \int \mathbb{E}^A H(\eta, A_t)\mu(d\eta) =$$

$$= \mathbb{P}^A(\tau \leqslant t) + \int \mathbb{P}^A(\eta(x) = 0\, \forall x \in A_t, \tau > t)\mu(d\eta)$$

Now letting $t \to \infty$ and using the hypothesis we get

$$\lim_{t\to\infty} \mathbb{P}^\mu(\eta_t(x) = 0\, \forall x \in A) = 1 \ \ \forall \mu \in \mathcal{P}(X), A \in Y$$

Thus $\lim_{t\to\infty} \mu P_t = \delta_0$ and so $(\eta_t)_{t\,\geqslant\,0}$ is ergodic.

- Suppose that we know that $(\eta_t)_{t\,\geqslant\,0}$ is ergodic, thus $\lim_{t\to\infty} \delta_1 P_t = \delta_0$ indeed ergodicity the upper invariant measure equals the lower invariant measure, and the latter is δ_0 since the completely empty configuration is a trap for the dynamics (see Remark 3.6.3) By using the same formulas as before with the choice $\mu = \delta_1$

$$\mathbb{P}^{\delta_1}(\eta_t(x) = 0 \ \ \forall x \in A) = \mathbb{P}^A(\tau \leqslant t) + \int \mathbb{P}^A(\eta(x) = 0\, \forall x \in A_t, \tau > t)\delta_1(d\eta)$$

The second term in the r.h.s. is zero since $A_t \neq \emptyset$ on the event $\tau > t$ and δ_1 is concentrated on all one configuration. If we now let $t \to \infty$ and use the hypothesis $\lim_{t\to\infty} \delta_1 P_t = \delta_0$, we get

$$1 = \mathbb{P}^A(\tau \leqslant \infty) = \mathbb{P}^A(\tau \leqslant \infty)$$

$\square$

Theorem 3.6.9. *Let $(A_t)_{t\geqslant 0}$ be defined as in Theorem 3.6.7. If*

$$\sup_x \left(c(x) \sum_{F\in Y} p(x,F)(|F|-1) \right) = \omega < \infty$$

then $(A_t)_{t\geqslant 0}$ is ergodic.

Proof. Let $f : Y \to \mathbb{R}$ be defined setting $f(A) = |A|$ Then it holds

$$\mathcal{L}f(A) = \sum_{B\subset Y} q(A,B)(|B|-|A|) = \sum_{x\in A} c(x) \sum_{F\in Y} p(x,F)[|(A\setminus\{x\}\cup F)-|A|) \leqslant \tag{3.6.14}$$

$$\leqslant \sum_{x\in A} c(x) \sum_{F\in Y} p(x,F)(|F|-1) \leqslant \omega|A| = \omega f(A) \tag{3.6.15}$$

□

Exercise 15. *Use Theorem 5.0.9 to prove that for $\lambda < 1/2d$ the contact process is ergodic.*

As anticipated, using Theorem 3.6.7 we can prove easily ergodicity results for some processes. An example is the following result for the voter model.

Theorem 3.6.10. *Let $(\eta_t)_{t\geqslant 0}$ be the VM on $\mathbb{Z}^d$ started at time zero from a Bernoulli(p) product measure, μ_p, with $p\in[0,1]$. For $d=1,2$ it holds*

$$\lim_{t\to\infty} \mu P_t = (1-p)\delta_0 + p\delta_1$$

For $d\geqslant 3$

$$\lim_{t\to\infty} \mu P_t = \nu_p \quad \text{with } \nu_p(\vec{0}) = \nu_p(\vec{1}) = 0$$

with $\vec{0}$ (resp. $\vec{1}$) the completely empty (resp. completely filled) configuration.

Proof. Fix $\sigma \in \{0,1\}^{\mathbb{Z}^d}$, we call (see the notation in the hint for this exercise) $(\sigma_t)_{t\geqslant 0}$ the coalescing r.w. process evolved starting from σ. Notice that the limit $\lim_{t\to\infty}|\sigma_t|$ exists, since for coalescing r.w. $|\sigma_t|$ is non increasing in time. Let

$$N(\sigma) := \lim_{t\to\infty} |\sigma_t|,$$

by general results for random walks (recurrence in $d=1,2$ and transience for $d\geqslant 3$) it holds that

$$\begin{cases} N(\sigma) = 1 & \text{a.s. in } d=1,2 \\ \mathbb{P}\left[N(\sigma)\geqslant 2\right] > 0 & \text{for all } \sigma \text{ s.t. } |\sigma|\geqslant 2. \end{cases} \tag{3.6.16}$$

Then using the duality relation (3.6.1) connecting the VM process $(\eta_t)_{t \geqslant 0}$ and the coalescing random walks guaranteed by Theorem 3.6.2 and Exercise 14, we deduce

$$\mathbb{P}^\eta(|\eta_t \cdot \sigma| = 0) = \mathbb{P}^\sigma(|\eta \cdot \sigma_t| = 0)$$

where the notation $\cdot$ has been defined in (3.6.13). Therefore $\mu_p P_t$ converges weakly to a probability law ν_p characterised by [10]

$$\int P^\eta(|\eta_t \cdot \sigma| = 0) d\mu_p(\eta) = \mathbb{E}\left[(1-p)^{|\sigma_t|}\right] \tag{3.6.17}$$

Letting $t \to \infty$ and using (3.6.16) we get $\nu_p = (1-p)\delta_0 + p\delta_1$ for $d = 1, 2$ Furthermore, again using general results on random walks it can be proven that in $d \geqslant 3$ for any $n \geqslant 1$ and any $\epsilon > 0$ we can find a configuration σ with $|\sigma| = n$ and all walkers sufficiently far from each other so that $P(N(\sigma) = n \geqslant 1 - \epsilon)$. As a consequence using (**??**) and letting $t \to \infty$ we get

$$\nu_p(\vec{0}) \leqslant (1-\epsilon)(1-p)^n + \epsilon$$

where $\vec{0}$ is the completely empty configuration. This, thanks to the arbitrariness of n and ϵ, yields $\nu_p(\vec{0}) = 0$ for any $p > 0$. By symmetry between 1's and 0's. we also have $\nu_p(\vec{1}) = 0$ for any $p < 1$. $\square$

[10] Here we use Stone - Weierstrass theorem that guarantees that, for any μ, ν probability laws on the sigma algebra of subsets of $\mathbb{Z}^d$ if for any finite $B \subset \mathbb{Z}^d$ it holds $\int \mu(dA) \mathbb{1}_{A \cap B = \neq \emptyset} = \int \nu(dA) \mathbb{1}_{A \cap B = \neq \emptyset}$ then $\mu = \nu$.

Chapter 4

CONTACT PROCESS

4.1 Main results

For $\lambda \geqslant 0$ and $d \in \mathbb{N}$ we call CP(d, λ) the contact process on $\mathbb{Z}^d$ with infection rate λ which has been constructed in Chapter 1. From Chapter 3 we already know that CP(d, λ) is a monotone spin IPS and the lower invariant measure is independent on λ and is concentrated in the completely empty configuration, $\underline{\nu} = \delta_0$. The upper invariant measure depends instead on λ and we denote it by $\bar{\nu}_\lambda$.

Exercise 16. *Fix $\lambda_1, \lambda_2 > 0$ with $\lambda_1 \leqslant \lambda_2$. Prove that $\bar{\nu}_{\lambda_1} \leqslant \bar{\nu}_{\lambda_1}$. As a consequence, $\rho(\lambda) := \bar{\nu}_\lambda(\eta(x))$ is non decreasing in λ.*

[**Hint.** *Use the graphical construction to couple the process with infection rate λ_1 and the process with infection rate λ_2. Both CP(d, λ_1) and CP(d,λ_2) have healing rate 1, so we can take the same realisation of the arrival times for the healing maps. Infection maps instead have different rates for the two process. Recall that for CP(d, λ), for any oriented couple of neighbouring sites, the arrival times of the infection maps form a Poisson Point Set (PPS) of intensity λ on $\mathbb{R}_+$. Find a way to construct a coupling of a PPS of intensity λ_1 and a PPS of intensity λ_2 such that whenever an arrival time occurs for the first PPS it also occur for the latter.*]

We denote by $\mathbb{P}^\eta_{d,\lambda}$ the law of CP(d, λ) started by η and (with slight abuse of notation) for any $A \subset \mathbb{Z}^d$ we denote by $\mathbb{P}^A_{d,\lambda}$ the law of the CP(d, λ) started from a configuration η_A with $\eta_A(x) = 1$ iff $x \in A$. Note that a configuration $\eta \in X$ can be either be identified by specifying the occupation variable for each site, namely by specifying $\eta(x)_{x\in\mathbb{Z}^d}$ or equivalently by specifying the set of its occupied sites. Therefore in the following when for $A \subset \mathbb{Z}^d$ we write $\eta = A$ it means that $\eta(x) = 1$ if $x \in A$ and $\eta(x) = 0$ if $x \in A^c$. We are now ready to introduce the notion of *survival probability*, that is the probability that starting from a single infected site infection survives at any time, namely

$$\theta_d(\lambda) := \mathbb{P}^{\{x\}}_{d,\lambda}(\eta_t \neq \emptyset \, \forall t \geqslant 0). \tag{4.1.1}$$

By recalling the graphical construction of the contact process performed in Section 2.1 (and in particular Definition 2.2.2, Remark 2.2.4 and Theorem 2.2.6) it is easy to verify that

$$\theta_d(\lambda) = \mathcal{P}\{(x,0) \to \infty\} \tag{4.1.2}$$

with $\mathcal{P}$ the probability over $\Delta \subset \mathcal{M} \times [0,\infty)$, i.e. on the Poisson point processes associated to the healing and infection maps and we recall that $\{(x,0) \to \infty\}$is the event that there a path of influence from $(x,0)$ to at least one point with time coordinate t for all $t \geqslant 0$. Notice that by translation invariance the r.h.s. does not depend on x.

Definition 4.1.1. *If $\theta_d(\lambda) > 0$ we say that CP(d,λ) survives, otherwise we say that it dies out. We also define the critical infection rate for survival, $\bar\lambda_c(d)$, as*

$$\bar\lambda_c(d) := \sup\{\lambda \geqslant 0 : \text{ CP}(d,\lambda) \text{ dies out}\}.$$

Exercise 17. *Use the graphical construction to prove that for any λ_1, λ_2 with $\lambda_1 < \lambda_2$*

- *if CP(d,λ_2) dies out, then CP(d,λ_1) also dies out;*
- *if CP(d,λ_1) survives, then CP(d,λ_2) also survives*

In other words, it holds

$$\bar\lambda_c(d) = \inf\{\lambda \geqslant 0 : \text{ CP}(d,\lambda) \text{ survives}\}.$$

Remark 4.1.2. *We could have defined survival starting from any finite sets of infections, it would have been an equivalent definition. Namely for any finite non-empty set $A \subset \mathbb{Z}^d$ the following holds*

- $\mathbb{P}^{\{A\}}_{d,\lambda}(\eta_t \neq \emptyset \ \forall t \geqslant 0) = 0$ *if* $\lambda < \bar\lambda_c(d)$
- $\mathbb{P}^{\{A\}}_{d,\lambda}(\eta_t \neq \emptyset \ \forall t \geqslant 0) > 0$ *if* $\lambda > \bar\lambda_c(d)$

Indeed from the graphical construction it holds

$$\mathbb{P}^{\{A\}}_{d,\lambda}(\eta_t \neq \emptyset \ \forall t \geqslant 0) = \mathcal{P}((A,0) \to \infty)$$

with

$$\{(A,0) \to \infty\} := \cup_{j \in A}\{(j,0) \to \infty\}$$

And therefore

$$\theta_d(\lambda) = \mathcal{P}((x,0) \to \infty) \leqslant \mathcal{P}((A,0) \to \infty) \leq \sum_{j \in A} \mathcal{P}((j,0) \to \infty) = |A|\theta_d(\lambda)$$

Exercise 18. *Use the graphical construction to prove that the contact process is additive, namely that for any η, ξ the CP(d, λ) started at $\eta \vee \xi$ has the same law of $(\eta_t \vee \xi_t)_{t \geqslant 0}$, the max at each time among the CP(d, λ) started at η and ξ.*

Recall the definition of ergodicity given in Definition 3.3.1, the definitions of upper and lower invariant measures for attractive spin IPS (see Theorem3.5.7) and the result of Corollary 3.5.8. Let $\lambda_c(d)$ be the critical threshold for ergodicity, namely

$$\lambda_c(d) := \sup\{\lambda \geqslant 0 : \underline{\nu} = \bar{\nu}\}$$

Exercise 19. *Prove that for any λ_1, λ_2 with $\lambda_1 < \lambda_2$*

- *if CP(d, λ_2) is ergodic, then CP(d, λ_1) is also ergodic;*
- *if CP(d, λ_1) is not ergodic, then CP(d, λ_2) is also not ergodic.*

In other words it holds

$$\lambda_c(d) := \inf\{\lambda \geqslant 0 : \underline{\nu} \neq \bar{\nu}\}.$$

The main results for CP on $\mathbb{Z}^d$ are the two following theorems that we will prove in the remainder of this chapter:

Theorem 4.1.3. *For any $d \in \mathbb{Z}_+$, the critical threshold for survival coincides with the critical threshold for ergodicity, namely $\lambda_c(d) = \bar{\lambda}_c(d)$.*

Theorem 4.1.4. *For any $d \in \mathbb{Z}_+$ it holds $0 < \lambda_c(d) < \infty$.*

Before presenting the proofs of Theorems 4.1.3 and 4.1.4, we sketch an approximated argument supporting the occurrence of a phase transition and present other results for CP whose proofs go beyond the scope of these lectures.

Remark 4.1.5 (A mean field argument supporting the occurrence of a phase transition)**.** *Let*

$$\rho_t(x) := \mathbb{E}^{\mu}(\eta_t(x)).$$

Then, using $\frac{d}{dt}P_t f = P_t \mathcal{L} f$ *with* $f(\eta) = \eta(x)$ *we get*

$$\frac{d}{dt}\rho_t(x) = -\rho_t(x) + \lambda \sum_{y:y\sim x} \mathbb{E}^\mu(\eta_t(y)(1-\eta_t(x)) \tag{4.1.3}$$

where we used

$$\mathcal{L}\eta(x) = r(x,\eta)(1-2\eta(x)) = -\eta(x) + \lambda(1-\eta(x))\sum_{y\sim x}\eta(y).$$

By the translation invariance of the dynamics, if μ *is translation invariant,* $\rho_t(x)$ *does not depend on* x*. The mean field approximation consist in neglecting higher order corrections in* (4.1.3) *namely letting*

$$\mathbb{E}^\mu(\eta_t(y)(1-\eta_t(x)) \sim \mathbb{E}^\mu(\eta_t(y))\mathbb{E}^\mu(1-\eta_t(x)) = \rho_t(1-\rho_t).$$

With this approximation we get

$$\frac{d}{dt}\rho_t = -\rho_t + 2d\lambda\rho_t(1-\rho_t) \tag{4.1.4}$$

which has a single stationary solution ($\rho = 0$*) for* $\lambda \leqslant 1/(2d)$ *and an additional stationary solution* $\rho = 1 - \frac{1}{2d\lambda}$ *for* $\lambda > 1/(2d)$*.*

Theorem 4.1.6 (Complete convergence)**.** *For any* $\pi \in \mathcal{P}$ *it holds*

$$\lim_{t\to\infty} \pi P_t = \rho(A)\bar{\nu}_\lambda + (1-\rho(A))\delta_0$$

where

$$\rho(A) := \int \mathbb{P}^\eta(\eta_t \neq \emptyset,\ \forall t \geqslant 0)d\pi(\eta).$$

This implies in particular that $\bar{\nu}$ *and* δ_0 *are the only extremal invariant measures for CP.*

We emphasise that complete convergence does not follow from monotonicity. A counterexample is the case of CP on regular trees, where despite monotonicity it has been proven that there exists $\tilde{\lambda_c}$ s.t. $\tilde{\lambda_c} > \lambda_c$ and for $\lambda \in (\lambda_c, \tilde{\lambda}_c)$ complete convergence does not hold (see [Lig85]).

Another issue which has been studied is the following: for $\lambda > \lambda_c$, how do infected areas look like at large time when we start from a single infection and we condition on survival? the answer is that the growth of the infected regions is linear with time, a result that goes under the name of *shape theorem for CP*.

Let us conclude by discussing the behavior at criticality and stating a major result, which we do not have time to prove in this course and a conjecture that despite very clear numerical confirmation and non rigorous analytical results in the physics community, still lacks a full rigorous proof.

Theorem 4.1.7 (Continuity)**.** *For any $d \in \mathbb{Z}_+$ it holds $\theta_d(\lambda_c) = 0$*

Conjecture 1. *$\exists \beta = \beta(d) > 0$ s.t. for CP on $\mathbb{Z}^d$ it holds*

$$\theta(\lambda) \sim (\lambda - \lambda_c)^\beta \quad \textit{for } \lambda \downarrow \lambda_c,$$

namely $\lim_{\lambda \to \lambda_c} \log \theta(\lambda)/\log(\lambda - \lambda_c)^\beta = 1$. Furthermore

- *β is universal once the spatial dimension has been fixed, namely should not change by varying some details in the definition of CP (while λ_c is certainly not universal),*
- *$\beta(d) = 1$ for $d \geqslant 4$*

4.2 Survival vs ergodicity: proof of Theorem 4.1.3

The key ingredients of this proof are: (i) the self duality of the contact process (see Exercise 13) and (ii) Theorem 3.6.7 that connects survival and ergodicity for some special couples of dual processes, those defined in Theorem 3.6.2.

Proof. Fix $\lambda > 0$. Theorem 3.6.7 and Exercise 13 imply that CP(λ) is ergodic iff for any A a finite subset of $\mathbb{Z}^d$ it holds $\mathbb{P}^A(\tau < \infty) = 1$, with $\mathbb{P}^A$ the evolution of the infected sets of the CP(λ) when at time 0 the set of infected sites coincides with A. On the other hand the event $\{\tau < \infty\}$ is the complementary of the survival event, thus

$$\mathbb{P}^A(\tau < \infty) = 1 - \mathbb{P}^A(\eta_t \neq 0 \, \forall t \geqslant 0).$$

Thus Remark 4.1.2 guarantees that $\mathbb{P}^A(\tau < \infty) = 1$ if $\lambda < \bar{\lambda}_c$ and $\mathbb{P}^A(\tau < \infty) < 1$ if $\lambda > \bar{\lambda}_c$, yielding $\lambda_c = \bar{\lambda}_c$. □

4.3 $\lambda_c \in (0,1)$: proof of Theorem 4.1.4

Let us start by stating three Lemmas that will be proven in the following

Lemma 4.3.1. $\lambda_c(d) \geqslant \frac{1}{2d}$

Lemma 4.3.2. $\lambda_c(d) \leqslant \lambda_c(1)/d$

Lemma 4.3.3. $\lambda_c(1) < \infty$

Proof of Theorem 4.1.4. The proof follows immediately gathering the results of the three Lemmas above. □

Exercise 20. *In order to prove Theorem 4.1.4 it is enough to prove a milder version of the inequality in Lemma 4.3.2, namely $\lambda_c(d) \leqslant \lambda_c(1)$. As an exercice, find a proof of this result (shorter than the proof of Lemma 4.3.2 provided below).*

Prof of Lemma 4.3.1. Notice that the cardinality of the set of infections, $|A_t|$, decreases by 1 at rate $|A_t|$ and increases by 1 at a rate which is upper bounded by $2d\lambda|A_t|$ (since a site can create a new infection only on an empty nearest neighbour). Thus if $2d\lambda < 1$, $|A_t|$ has a drift to decrease and will eventually hit 0 (since at time 0 we start from $|A_0| = 0$). Therefore CP(λ) certainly dies out if $\lambda < 1/(2d)$, which yields $\lambda_c \geqslant 1/(2d)$. □

Proof of Lemma 4.3.2. The idea here is to couple versions of CP on different dimensions and with different parameters λ. More precisely we consider

- $(A_t)_{t\geqslant 0}$ the CP on $\mathbb{Z}^d$ with infection rate λ and started with a single infected site at the origin, namely $A_0 = \{(0,\dots,0)\}$
- $(\tilde{A}_t)_{t\geqslant 0}$ the CP on $\mathbb{Z}$ with infection rate $d\lambda$ and started with $\tilde{A}_0 = \{0\}$.

Now we will prove that it holds

$$\mathbb{P}^{A_0}_{d,\lambda}(A_t \neq \emptyset) \geqslant \mathbb{P}^{\tilde{A}_0}_{1,d\lambda}(\tilde{A}_t \neq \emptyset) \qquad \forall t \geqslant 0. \tag{4.3.1}$$

This implies in particular that if $\tilde{A}_t$ survives also A_t survives , namely

$$\text{if } d\lambda > \lambda_c(1) \text{ then necessarily } \lambda > \lambda_c(d)$$

which yields

$$\lambda_c(d) \leqslant \frac{\lambda_c(1)}{d}$$

We are left with proving inequality (4.3.1). Define the projection map $\pi_d : \mathbb{Z}^d \to \mathbb{Z}$ as

$$\pi_d(x_1,\dots,x_d) := \sum_{i=1}^{d} x_i$$

and let for $A \subset \mathbb{Z}^d$

$$\pi_d(A) := \{\pi_d(x) : x \in A\} \subset \mathbb{Z}.$$

Fix $\tilde{A} \subset \mathbb{Z}$ and $A \subset \mathbb{Z}^d$. Suppose that $\tilde{A} \subset \pi_d(A)$, then necessarily for each $x \in \tilde{A}$ the set $\{y \in A : x = \pi_d(y)\}$ is not empty. Choose one of the sites in this set and denote it $y = y^{x,A,\tilde{A}}$.

We will now construct a coupling of the processes $(A_t)_{t \geqslant 0}$ and $(\tilde{A}_t)_{t \geqslant 0}$ s.t.

$$\tilde{A}_t \subset \pi_d(A_t) \quad \forall t \geqslant 0 \tag{4.3.2}$$

This implies that if $\tilde{A}_t \neq \emptyset$ also $A_t \neq \emptyset$ and (4.3.1) follows.

Let us start by noticing that $\tilde{A}_0 = \pi_d(A_0)$ so that (4.3.2) holds at time 0. Then fix the Poisson point processes to construct $(A_t)_{t \geqslant 0}$, namely the realisation of the arrival times of the healing maps for each site, and of the $2d$ infection maps pointing from each site to its nearest neighbours. Now we construct a coupled process $(\tilde{A}_t)_{t \geqslant 0}$ (which in the following we will call the tilde process) as follows. We let $\tilde{A}_0 = 0$ and let it evolve according to the following rules

- whenever an healing event occurs for a site $y \in \mathbb{Z}^d$, if $y = y^{x,A_t\tilde{A}_t}$ for an $x \in \mathbb{Z}$, in the tilde process we heal at this time site $x \in \mathbb{Z}$;
- whenever an infection event from $y \in \mathbb{Z}^d$ to one of the d points $y - \vec{e}_i$ occurs, if there exists $x \in \mathbb{Z}$ s.t. $y = y^{x,A_t\tilde{A}_t}$, in the tilde process we infect at this time site $x-1$ if x was infected;
- analogously, whenever an arrival time of the infection map from $x \in \mathbb{Z}^d$ to one of the d points $x + \vec{e}_i$ occurs, if $y = y^{x,A_t\tilde{A}_t}$ for an $x \in \mathbb{Z}$, in the tilde process we infect at this time site $x+1$ if x was infected;
- on any other time the tilde process does not evolve

It is not difficult to verify that the marginal under this coupling of $(\tilde{A}_t)_{t \geqslant 0}$, is a CP on $\mathbb{Z}$ with infection parameter $d\lambda$ and that this coupling preserves the relation $\tilde{A}_t \subset \pi_d(A_t)$.

□

There are several alternative proves of Lemma 4.3.3 using comparison with different types of oriented percolation. One of these proof uses comparison with *directed (or oriented) edge percolation on* $\mathbb{Z}^2$. Another proof, that we have decided to follow here, uses comparison with *directed (or oriented) site percolation on* $\mathbb{Z}^2$. The interested reader may find the first proof on Section 2.5 and 2.6 of [Swaa] or on Section 7.2, 7.3 and 7.4 of [Swab].

Definition 4.3.4 (Directed (or oriented) site percolation on $\mathbb{Z}^2$)**.** *Let G_0 be the set of sites of the form $x = (2i, 0)$ with $i \in \mathbb{Z}$, and draw from each $x \in G_0$ two arrows directing to $x \pm \vec{e}_1 + \vec{e}_2$. The tips of the arrows points towards sites of the form $x = (2i+1, 1)$, we call this set of sites G_1 and then draw for each $x \in G_1$ two arrows directing to $x \pm \vec{e}_1 + \vec{e}_2$. Continuing in the same way we construct an oriented lattice with vertex set $\mathbb{G}$ containing all sites x of the form $2i, 2j$*

or $2i+1, 2j+1$ with $i \in \mathbb{Z}$, $j \in \mathbb{Z}^+$ and such that from each site of G_i there are two oriented edges pointing towards two sites of G_{i+1}. In other words $\mathbb{G}$ corresponds to a lattice $\mathbb{Z}^2$ rotated of 45 degrees and multiplied by a factor $\sqrt{2}$ and with edges having an upwards orientation. Fix $p \in [0,1]$ and declare each site $x \in G$ to be open with probability p and closed with probability $1-p$, independently from all other. More precisely let μ_p be Bernoulli product measure at density p on $\{0,1\}^{\mathbb{G}}$, and pick a configuration σ distributed with μ_p and declare x open iff $\sigma(x) = 1$. Fix $x, y \in \mathbb{G}$, we say that x is connected to y and write $x \to y$ iff there is a path that

- *connects x to y*
- *traverses only edges along the orientation of the arrows*
- *all sites visited by the path (including x and y) are open.*

Let C_0 be the set of all sites that are connected to the origin

$$C_0 := \{x \in \mathbb{Z}^2 : (0,0) \to x\}$$

and let

$$p_c := \sup\{p \in [0,1] \ \text{ s.t. } \ \mu_p(|C_0| = \infty) = 0\}$$

Exercise 21. *Prove that*

$$p_c = \inf\{p \in [0,1] : \mu_p(|C_0| = \infty) > 0\}.$$

[Hint: use the result of exercise 12 to prove monotonicity in p of $\mu_p(|C_0| = \infty)$]

Exercise 22. *Prove that*

$$p_c \geqslant \frac{1}{2}.$$

[Hint: Let $N_i = |C_0 \cap G_i|$ and show that this is a branching process that dies out if $p < 1/2$.]

Theorem 4.3.5.

$$p_c < \frac{80}{81} \tag{4.3.3}$$

Proof of Theorem 4.3.5. The type of argument used in the above proof, known as Peierls contour argument, is a basic tool that is often used in statistical mechanics/percolation theory. Fix a configuration $\sigma \in \{0,1\}^{\mathbb{G}}$, where we recall that 0 and 1 are closed and open sites respectively.

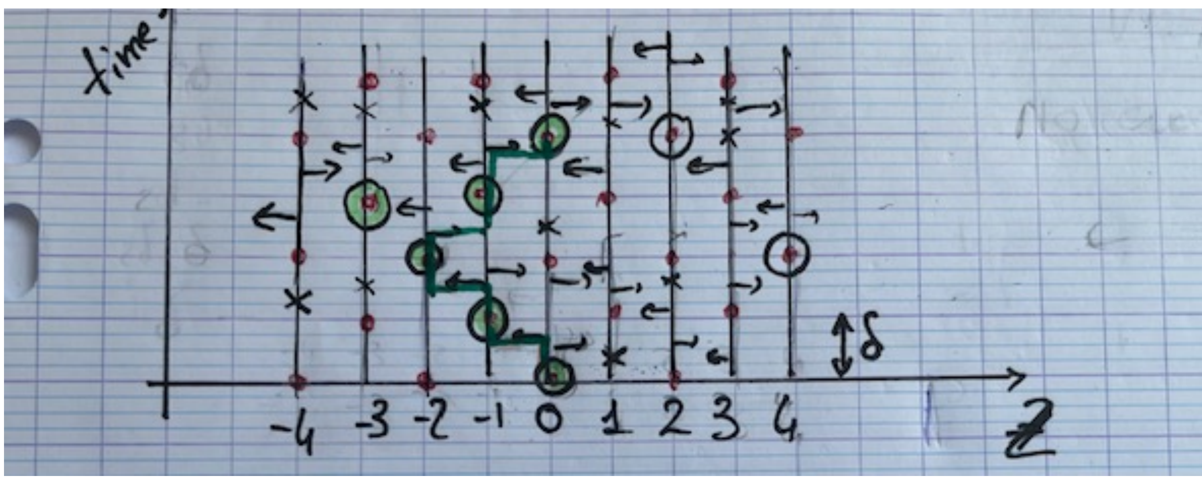

Figure 4.1: Coupling of the graphical construction of CP and coupled oriented percolation. The red dots are sites in $\mathbb{G}$. Red dots encircled in black are open sites, when they are coloured green they are open and belong to the open connected cluster of the origin, C_0. Site $(0, 4\delta)$ belongs to C_0. We highlight in green the influence path connecting it to $(0, 0)$ in the graphical construction.

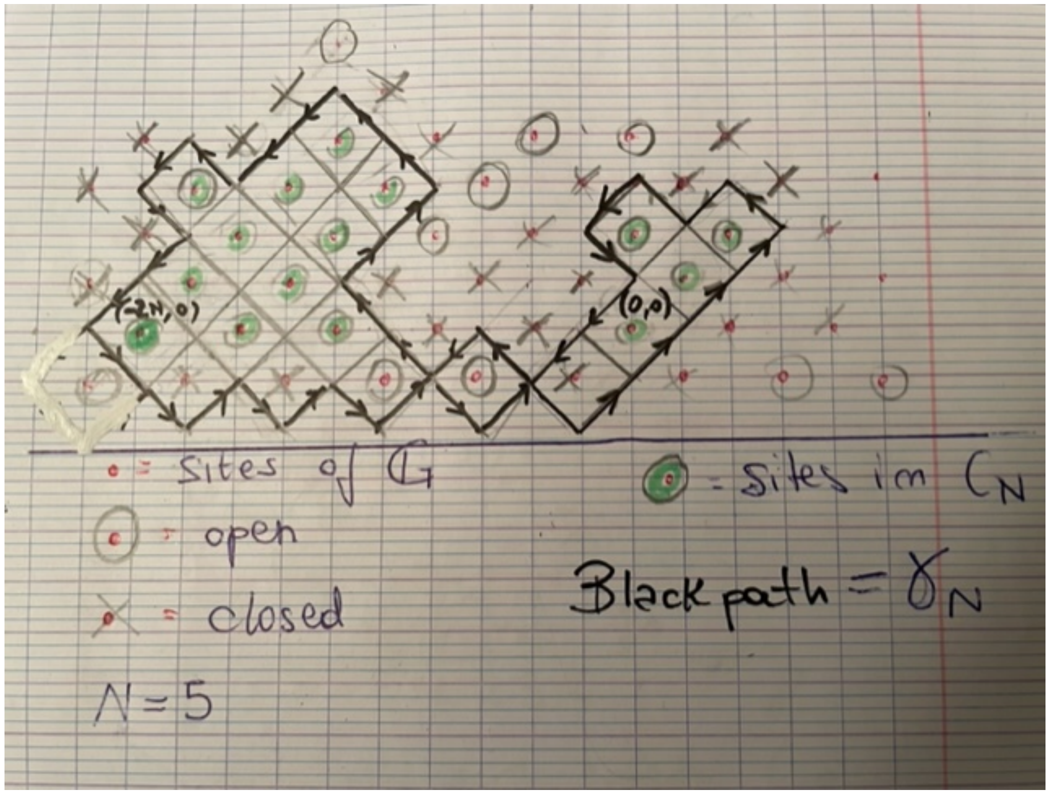

Figure 4.2: The oriented lattice, C_N and its contour Γ_N

Let C_N be the set if sites connected to $O = (0, 0)$ or to one of the N sites of $\mathbb{G}$ lying on the left of O

$$C_N := C_N(\eta) = \cup_{i=1}^{N}(x : (-2i, 0) \to x).$$

We want to lay a contour around C_N. Consider now for each site $x \in \mathbb{G}$ a square of side $\sqrt{2}$ centered around x and tilted of 45 degrees w.r.t. the horizontal and vertical axes, as in Fig. 4.3. Then colour all squares that either contain a site of C_N or contain a site of the form $(-2i+1, -1)$ with $i = 1, \ldots, N$. If $|C_N| < \infty$ we can define an external contour for this coloured region, Γ_N. We let $|\Gamma_N|$ be its length measured in number of diagonal segment of length $\sqrt{2}$ and $\mathcal{G}_{N,n}$ be the set of the different contours of length n obtained when varying $\sigma \in \{0, 1\}^{\mathbb{Z}^2}$. Using Claim 4.3.6 and a union bound we get

$$\mu_p(|C_N| < \infty) \leqslant \sum_{n=2N}^{\infty} \mu_p(\sigma : \Gamma_N(\sigma) \in \mathcal{G}_{N,n}) \leqslant \sum_{n=2N}^{\infty} 3^n (1-p)^{n/4} \tag{4.3.4}$$

For $p > 80/81$ it holds $3(1-p)^{1/4} < 1$ which implies that $\lim_{N\to\infty} \sum_{n=2N}^{\infty} 3^n(1-p)^{n/4} = 0$. This, together with (4.3.4) yields

$$\text{for } p > 80/81 \ \exists\, \bar{N}(p) \text{ s.t. } \forall N > \bar{N}(p) \text{ it holds } \mu_p(|C_N| = \infty) > 0.$$

Furthermore, by translation invariance it holds

$$\mu_p(|C_N| = \infty) \leqslant (N+1)\mu_p(|C_0| = \infty)$$

which, together with the former conclusion implies

$$\mu_p(|C_0| = \infty) > 0 \ \ \forall p > 80/81$$

yielding $p_c \leqslant 80/81$.

□

We are left with stating and proving a technical result on contours that was a key ingredient of the previous proof.

Claim 4.3.6. *The following holds*

(i) $|\Gamma_N| \geqslant 2N$ *for all* $\sigma \in \{0,1\}^{\mathbb{Z}^2}$

(ii) $|\mathcal{G}_{N,n}| \leqslant 3^n$

(iii) if σ *is such that* $\Gamma_N(\sigma) \in \mathcal{G}_{N,n}$, *we can identify at least* $n/4$ *sites adjacent to* Γ_N *that are closed*

Proof. (i) follows trivially by the definition of Γ_N. (ii) is a consequence of the fact that when we travel along the contour at each point we have at most three choices on how to proceed (the contours does not come back to itself). In order to prove (iii) we start by noticing that if we follow Γ_N counter clock wise, each time it turns left there must be a closed site on a specific vertex external to the contour and adjacent to it (see Fig. 4.3). Indeed, if we give to each diagonal edge of Γ_N an orientation which correspond to traveling along the contour counterclockwise, when γ_N contains an edge from x to $x + \vec{e}_2 - \vec{e}_1$, necessarily $x + \vec{e}_2$ is closed (otherwise Γ_N should have moved around it), when it contains the edge from x to $x - \vec{e}_2 - \vec{e}_1$, necessarily $x - \vec{e}_1 + \vec{e}_2$ is closed (otherwise Γ_N should have moved around it). Then letting n_l and n_r be the number of left (right) directed diagonal edges of γ_n and using $n_l + n_r = n$ and $n_l = n_r$, we get $n_l = n/2$. Gathering the above observations and noticing that at most two edges of γ_n directed to the left identify the same closed site, point (iii) is proven. $\square$

We are now ready to prove Lemma 4.3.3 thus concluding the proof of Theorem 4.1.4.

Proof of Lemma 4.3.3. The proof uses a coupling argument that shows that for λ sufficiently large but finite and δ sufficiently small there exists $p > 80/81$ such that the probability that for $\mathrm{CP}(1, \lambda)$ the infection survives up to time $n\delta$ is larger than the probability for oriented site percolation on $\mathbb{Z}^2$ with probability p that there exists $y \in \mathbb{Z}$ such that C_0 contains (y, n). Therefore, thanks to Theorem 4.3.5, CP necessarily survives for λ sufficiently large but finite, namely $\lambda_c < \infty$.
The coupled construction of CP and OP is obtained as follows. Start by fixing Δ a realisation of the Poisson Point processes associated to the healing and infection maps the define CP in $d = 1$ and draw the corresponding graphical representation. Then superimpose to this graphical representation the oriented lattice $\mathbb{G}$ with vertical coordinate shrinked by δ, with $\delta > 0$ a fixed parameter (see Fig. 4.3) so that the vertexes of $\mathbb{G}$ are now of the form $(x, n\delta)$ with $x \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ s.t. $|x|$ and n have the same parity. Given a site $(x, n\delta) \in \mathbb{G}$ we let it be open if and only if in the graphical construction of CP these two conditions are satisfied:

(i) there is no cross in the time interval $(n-1)\delta, (n+1)\delta$ on the vertical line at site x, i.e. no arrival time of the healing map $\mathcal{H}_x$;

(ii) there are a right and a left arrow starting from site x in the time interval $(n\delta, (n+1)\delta)$, namely there is at least one arrival time of the infection map $\mathcal{I}_{x,x+1}$ and of $\mathcal{I}_{x,x-1}$.

We leave as an exercice to prove that with the above definition

(a) sites of $\mathbb{G}$ are open and closed independently and the probability that a site is open is $p(\delta) = e^{-2\delta}(1 - e^{-\lambda\delta})^2$

(b) if $(0, 0) \to (x, n\delta)$ (where connection is in the sense of the open paths for oriented percolation), necessarily there is a path of influence (in the sense of the connection fixed by the graphical representation, see Section 2.1) from the space time point $(0, 0)$ to the space time point $(x, n\delta)$

We exploit now our freedom in the choice of $\delta > 0$ and fix it to $\bar{\delta}(\lambda) = \log(\lambda+1)/\lambda$ to maximise $p(\delta)$. This yields

$$p(\bar{\delta}(\lambda)) = \frac{\lambda}{1+\lambda}\left(\frac{1}{\lambda+1}\right)^{2\lambda}.$$

Since the above expression tends to 1 as $\lambda \to \infty$ we get, using Theorem 4.3.5, that for λ sufficiently large the probability for a site to be open is $> p_c$. From point (b) above it follows that survival of the contact process holds. $\square$

Chapter 5

STOCHASTIC ISING MODEL (SIM)

Fix $\beta \geqslant 0$ and $d \in \mathbb{N}$, we call SIM(d,β) the Stochastic Ising Model (SIM) on $\mathbb{Z}^d$, namely the spin interacting particle system on $X = \{-1,+1\}^{\mathbb{Z}^d}$ with the rate to flip a spin at site x for the configuration η given by $r(x,\eta)$ where for each $x \in \mathbb{Z}^d$ $r_(x,\cdot)$ is a function (that depends on β) from $X \to \mathbb{R}$ defined as

$$r(x,\eta) = e^{-\beta \sum_{y,y\sim x} \eta(x)\eta(y)} = e^{-2d\beta + 2\beta \tilde{N}_x(\eta)} \tag{5.0.1}$$

with $\tilde{N}_x(\eta) = \sum_{y\sim x} 1_{\eta(y)\neq\eta(x)}$. In formulas, the generator of the dynamics acts on local functions $f : X \to \mathbb{R}$ as

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}^d} r(x,\eta)(f(\eta^x) - f('\eta)) \tag{5.0.2}$$

where

$$\eta^x(y) = \begin{cases} -\eta(y) & \text{if } y = x \\ \eta(y) & \text{if } y \neq x \end{cases} \tag{5.0.3}$$

Notice that this can be also reformulated (modulo a time rescaling) by saying that in configuration η we refresh the state of site x by setting it to $+$ at rate $r^+(x,\eta)$ and we refresh it setting it to $-$ at rate $r^-(x,\eta)$ where

$$r^+(x,\eta) = \frac{e^{\beta N^+(x,\sigma)}}{e^{\beta N^+(x,\sigma)} + e^{\beta N^-(x,\sigma)}}$$

$$r^-(x,\eta) = \frac{e^{\beta N^-(x,\sigma)}}{e^{\beta N^+(x,\sigma)} + e^{\beta N^-(x,\sigma)}}$$

with

$$N^+(x,\eta) := \sum_{y\sim x} \mathbb{1}_{\eta(y)=+}$$

and analogous definition for $N^-(x,\eta)$.

Remark 5.0.1. *Recall (see Exercise 11) that SIM(d,β) is monotone. Thus, thanks to Corollary 3.5.8, ergodicity of the process holds iff $\bar{\nu} = \underline{\nu}$ where $\bar{\nu}$ (resp. $\underline{\nu}$) is the measure attained at infinite time evolving from δ_+ (resp. from δ_-). Note also that (as for VM and unlike CP), SIM is ± 1 symmetric. Namely the law of the trajectory of the process starting from any configuration η coincides with the law of the spin-reversed trajectory where by spin-reversed trajectory we mean that at each time all spins are flipped. This implies that, for any function $f : X \to \mathbb{R}$, it holds*

$$\bar{\nu}(f) = \underline{\nu}(\tilde{f})$$

where $\tilde{f} : X \to \mathbb{R}$ is the function defined by letting $\tilde{f}(\eta) = f(\tilde{\eta})$ with $\tilde{\eta}$ the configuration obtained by flipping each spin in η.

Let us provide a possible graphical construction for SIM. Define for each $x \in \mathbb{Z}^d$ and $A \subset N_x := \{y : y \sim x\}$ two maps $m^-_{x,A}$ and $m^+_{x,A}$ from Ω to Ω as follows:

$$m^-_{x,A}\sigma(y) = \begin{cases} -1 & \text{if } y = x \text{ and } \sigma(z) = -1 \, \forall z \in A \\ \sigma(y) & \text{otherwise} \end{cases} \tag{5.0.4}$$

$$m^+_{x,A}\sigma(y) = \begin{cases} 1 & \text{if } y = x \text{ and } \sigma(z) = 1 \, \forall z \in A \\ \sigma(y) & \text{otherwise} \end{cases} \tag{5.0.5}$$

Then it is just a direct calculation to verify that

$$\mathcal{L}f(\sigma) = \sum_x r(x,\eta)(f(\eta^x) - f(\eta)) = \sum_{\xi \in \pm} \sum_x \sum_{A \in N_x} e^{-2d\beta}(1 - e^{-2\beta})^{|A|} e^{2\beta|A|} \left(f(m^{\xi}_{x,A}\sigma) - f(\sigma) \right)$$

where the sum over x runs on all sites of $\mathbb{Z}^d$, the sum over $A \in N_x$ also includes the case $A = \emptyset$. For example if $d = 2$ and $\sigma(x) = +1$ and 3 of its nearest neighbours are -1 we get from the previous formula a spin flip rate

$$e^{-4\beta} + e^{-4\beta} e^{2\beta}(1 - e^{-2\beta}) = e^{-2\beta}$$

which indeed corresponds to (5.0.1).

Recalling the definition of $D(m)$ and $R_x(m)$ given in (2.2.1) and (2.2.2) we can easily check that $D(m^{\pm}_{x,A}) = \{x\}$ and $R_y(m^{\pm}_{x,A}) = A \cup \{i\}$ if $A \neq \emptyset$, otherwise $R_y(m^{\pm}_{x,\emptyset}) = \emptyset$ and verify that the maps satisfy assumptions A1-A2 of Section 2.3. Therefore, thanks to Theorem 2.2.8, SIM is a will defined Markov process. Here the set of maps is $\mathcal{M} = \cup_{\xi \in \pm} \cup_{x \in \mathbb{Z}^d} \cup_{A \in N_x}$, so the points of Poisson Point Process Δ are of the form (ξ, x, A, t). Furthermore, in this setting the paths of influence (see Definition 2.2.2) are paths such that

- $\forall t \in [s,u]$ s.t. $\gamma_{t-} \neq \gamma_t$ there exists $(\xi, x, A, t) \in \Delta$ s.t. $\gamma_{t-} \in A, \gamma_t = x$;

- $\nexists(\xi, x, A, t) \in \Delta$ s.t. $A = \emptyset$, $t \in [s,u]$, $\gamma_t = x$

Pictorially we can draw at each $(\xi, x, A, t) \in \Delta$ a circle at (x,t) containing the sign of ξ and with arrows arriving from the points of A to x and a path of influence is a path which goes vertically in the positive direction, can follows the arrows, never crosses circles which do not have incoming arrows (this are the points at which the spin at the corresponding site flips to either plus or minus, regardless of the state of the system at the prior time)

Lemma 5.0.2. *For any $d \geqslant 1$ there exists $\bar{\beta}(d) > 0$ s.t. for $\beta \leqslant \bar{\beta}(d)$ SIM is ergodic*

Proof. Recall formula (2.2.20) defining the constant K which enters in the exponential bound of Lemma 2.2.7 of influence path, we get

$$K(\beta) = 2 \sum_{A \subset N_x} e^{-2d\beta}(1-e^{-2\beta})^{|A|} e^{2\beta|A|}(|A| - 1_{A=\emptyset}) = 2\left[2d(1-e^{-\beta}) - e^{-2d\beta}\right] \tag{5.0.6}$$

Set $\bar{\beta} := \sup\{\beta > 0 : K(\beta) < 0\}$. From (5.0.6) it follows immediately that $\bar{\beta} > 0$. The result than follows by Lemma 2.2.7. $\square$

The main content of this chapter will be the prove of the following result.

Theorem 5.0.3 (Phase transition for the Ising model)**.**

- *SIM$(1,\beta)$ is ergodic at any $\beta \geqslant 0$, namely in one dimension there is at any temperature a unique invariant measure*

- *there exists a critical value $\beta_c = \beta_c(d)$ satisfying $0 < \beta_c < \infty$ s.t. SIM(d,β) is ergodic for $\beta < \beta_c$ and not ergodic for $\beta > \beta_c$.*

In order to prepare the proof of this result we should first consider the model on finite volume.

Definition 5.0.4 (Finite volume SIM)**.** *Let $\Lambda \subset \mathbb{Z}^d$ be a finite volume and $\tau \in \{\pm\}^{\partial\Lambda}$ with $\partial\Lambda \subset \mathbb{Z}^d \setminus \Lambda$ the set of sites outside Λ that have a nearest neighbour inside Λ. We define SIM on Λ with boundary condition τ by letting its generator act on functions from $f : \{0,1\}^\Lambda \to \mathbb{R}$ as*

$$\mathcal{L}_\Lambda^\tau f(\eta) := \sum_{x \in \Lambda} r^\tau(x,\eta)(f(\eta^x) - f(\eta))$$

with

$$r^\tau(x,\eta) = r(x,\eta^\tau)$$

where η^τ is a configuration that coincides with η inside Λ, with τ on $\partial\Lambda$ and with the completely filled configuration outside (the latter condition is completely arbitrary, since the dynamics uses only flips inside Λ, whose rate depend only on $\Lambda \cup \partial\Lambda$).

Definition 5.0.5 (Finite and Infinite volume Gibbs measure for the Ising model)**.** *Fix $\beta \in \mathbb{R}^+$. Let $\Lambda \subset \mathbb{Z}^d$ be a finite volume and $\tau \in \{\pm\}^{\partial\Lambda}$. The finite volume Gibbs measure for the Ising model on Λ with boundary condition τ and at inverse temperature β is the measure $\mu^\tau_{\Lambda,\beta}$ that associates to a configuration $\eta \in \{\pm\}^\Lambda$ the weight*

$$\mu^\tau_{\Lambda,\beta}(\eta) := \frac{\exp\Big(-\beta \sum_{x\in\Lambda}\sum_{y\sim x}\eta^\tau(x)\eta^\tau(y)\Big)}{\sum_\sigma \exp -\Big(\beta \sum_{x\in\Lambda}\sum_{y\sim x}\sigma^\tau(x)\sigma^\tau(y)\Big)}. \tag{5.0.7}$$

where the configuration η^τ (that depends on η and τ) has been defined in Definition 5.0.4. We say that a measure μ on $\Omega = \{-1,+1\}^{\mathbb{Z}^d}$ is an infinite volume Gibbs measure for the Ising model at inverse temperature β, if $\forall \Lambda \subset \mathbb{Z}^d$ with $|\Lambda| < \infty$ and for μ almost every $\tau \in \{-1,+1\}^{\mathbb{Z}^d\setminus\Lambda}$ it holds

$$\mu\Big(\{\sigma(i)\}_{i\in\Lambda} \in \cdot \Big|\sigma(i) = \tau(i)\,\forall i \in \mathbb{Z}^d \setminus \Lambda\Big) = \mu^\tau_{\Lambda,\beta}(\sigma)$$

Remark 5.0.6. *A similar procedure can be used to define the Gibbs measure of other statistical mechanics models (besides the Ising model). In general for a statistical mechanics model one defines a configuration space (which for the Ising model is $\{\pm\}^{\mathbb{Z}^d}$) an energy function (which for the Ising model is $H(\sigma) = -\sum_{x,y,x\sim y}\sigma(x)\sigma(y)$). Then the finite volume Gibbs measure are defined as*

$$\mu^\tau_{\Lambda,\beta}(\eta) := \frac{\exp\left(-\beta H(\eta^\tau)\right)}{\sum_\sigma \exp -\left(\beta H(\sigma^\tau)\right)} \tag{5.0.8}$$

Then via the same procedure as in Definition 5.0.5 one can define the infinite volume measures.

Exercise 23. *Prove that*

- *the unique invariant mesure of SIM at inverse temperature β on finite volume Λ with boundary condition τ is the finite volume Gibbs measure $\mu^\tau_{\Lambda,\beta}$*
- *this invariant measure is also reversible.*

[Hint: use Remark 3.4.5]

The next result shows that the definition of infinite volume Gibbs measures is not void and furthermore these measure are reversible invariant measure for SIM.

Theorem 5.0.7. *Let $\{\Lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{Z}^d$ be a sequence of finite sets invading $\mathbb{Z}^d$, namely $\Lambda_n \subset \Lambda_m$ for $n \leqslant m$ and $\lim_{n\to\infty} \Lambda_n = \mathbb{Z}^d$. Let $X^{\Lambda_n} \in \Omega$ be a random variable distributed with the measure ν_n which gives occupation value $+1$ to all sites outside Λ_n and satisfies*

$$\nu_n(\{X^{\Lambda_n}(i)\}|_{i\in\Lambda_n} \in \cdot) = \mu_+^{\Lambda_n,\beta}(\cdot).$$

Then the following holds

- *(i)* $\lim_{n\to\infty} \nu_n = \bar{\nu}$, *where $\bar{\nu}$ is the upper invariant measure of SIM on $\mathbb{Z}^d$ at inverse temperature β*
- *(ii) $\bar{\nu}$ is a reversible measure for SIM on $\mathbb{Z}^d$ at inverse temperature β*
- *(iii) $\bar{\nu}$ satisfies the requirement of Definition 5.0.5, namely it is an infinite volume Gibbs measure at temperature β*

Analogous statements hold for -1 boundary conditions, in which case the limiting law is $\underline{\nu}$.

Proof. Let

- $\{X_t\}_{t\,\geqslant\,0}$ the infinite volume SIM started from a completely plus configuration, namely at X_0 satisfying $X_0(i) = +1$ for all $i \in \mathbb{Z}^d$
- $\{X_t^{\Lambda_n}\}_{t\,\geqslant\,0}$ be a process obtained setting $X_t^{\Lambda_n}(i) = +1$ for all $t \geqslant 0$ and $i \in \mathbb{Z}^d \setminus \Lambda_n$ and letting $X_t^{\Lambda_n}|_{\Lambda_n}$ evolve as SIM on Λ_n with boundary condition $\tau = +1$

Using the graphical construction it is not difficult to prove that we can couple the above defined processes so that for all $n \leqslant m$ it holds

$$X_t \leqslant X_t^{\Lambda_m} \leqslant X_t^{\Lambda_n} \qquad \forall t \geqslant 0.$$

Note that if we take the limit $t \to \infty$ $X_t^{\Lambda_n}$ is asymptotically distributed as ν_n and X_t as $\bar{\nu}$. So we get

$$\bar{\nu} \leqslant \nu_m \leqslant \nu_n. \tag{5.0.9}$$

Furthermore the limit $\lim_{n\to\infty} \nu_n$ exists and it is an invariant measure for SIM on infinite volume. Let us call ν this limit law. Combining this observation with the inequality (5.0.9) and the fact that by monotonicity $\bar{\nu}$ is the largest invariant measure yields

$$\nu = \bar{\nu}.$$

This proves (i). Statement (ii) follows by first proving that $\{X_t^{\Lambda_n}\}_{t\,\geqslant\,0}$ is reversible w.r.t. ν_n (see Exercise 23) and then taking the $n \to \infty$ limit. Statement (iii) follows from the fact that ν_n have the correct conditional distributions. □

The above theorem says that SIM provides indeed an "algorithmic way" of sampling configurations distributed w.r.t. Gibbs measures for the Ising model, a particularly difficult task when $\beta > \beta_c$ and correlations becomes long range. Before continuing with the study of SIM, let us mention that there are also other ways of defining a spin IPS so that the Gibbs measure are invariant for this dynamics. One of this choices, which goes under the name of Metropolis dynamics is to choose spin flip rates

$$\tilde{r}(x,\eta) = \min(1, e^{-\beta\Delta H}), \text{ with } \Delta H = (\eta^x) - H(\eta)$$

and H formally defined as $H(\sigma) = -\sum_{x,y\in\mathbb{Z}^d, x\sim y} \sigma(x)\sigma(y)$ (H is not well defined due to the infinite sum, but the rates contain only the difference that is well defined).

Definition 5.0.8 (Spontaneous magnetisation)**.** *We call spontaneous magnetisation for SIM on $\mathbb{Z}^d$ at inverse temperature β the following quantity*

$$m(\beta, d) := \int \eta(0)\bar{\nu}(d\eta)$$

Note that, by translation invariance, for all $x \in \mathbb{Z}^d$ it holds $\int \eta(x)\bar{\nu}(d\eta) = m(\beta, d)$. Furthrmore, thanks to the $\pm$ symmetry (see Remark 5.0.1) it holds

$$\int \eta(i)\underline{\nu}(d\eta) = -m(\beta, d)$$

Lemma 5.0.9. *SIM on infinite volume is ergodic iff $m(\beta, d) = 0$.*

Exercise 24. *Prove the above Lemma.*

In view of Lemma 5.0.9 in order to prove Theorem 5.0.3 it is enough to prove the following

Theorem 5.0.10 (Phase transition for the Ising model)**.**

- *for $d = 1$ it holds $m(\beta, 1) = 0$ for any $\beta \geqslant 0$*
- *for $d \geqslant 2$ there exists a critical value $\beta_c = \beta_c(d)$ s.t. $m(\beta) = 0$ for $\beta < \beta_c$ and $m(\beta) > 0$ for $\beta > \beta_c$.*

In turn, the proof of Theorem 5.0.10 follows immediately from the following three lemma

Lemma 5.0.11. *The function* $m(\beta, d)$ *is*

- *nondecreasing and right-continuous in* β
- *nondecreasing in* d*.*

Lemma 5.0.12 (No phase transition in dimension one)**.** *Fix* $\beta \geqslant 0$*, there exists a unique infinite volume Gibbs measure on* $\{-1,+1\}^{\mathbb{Z}}$ *with inverse temperature* β*.*

Lemma 5.0.13. *In dimension* 2 *there exists* $\bar{\beta} < \infty$ *s.t. for* $\beta > \bar{\beta}$ *it holds* $m(\beta, 2) > 0$*.*

Indeed, Lemma 5.0.12 proves the result of Theorem 5.0.10 in dimension 1, while Lemma 5.0.2 together with Lemma 5.0.11 and Lemma 5.0.13 complete the result in $d \geqslant 2$. We are therefore left with proving Lemmas 5.0.11, 5.0.12 and 5.0.13.

Proof of Lemma 5.0.12. This proof uses as a key ingredients the fact that $\bar{\nu}$ is invariant under translation invariance and under mirror reflections, in particular for any $x, y \in \mathbb{Z}$ it holds

$$\bar{\nu}(\{\sigma(x) = +\}) = \bar{\nu}(\{\sigma(y) = +\})$$

and

$$\bar{\nu}(\{\sigma(x) = +,\, \sigma(y) = -\}) = \bar{\nu}(\{\sigma(y) = +,\; \sigma(x) = -\}).$$

Another key ingredient is the use of Theorem 5.0.7 with together with the definition of infinite volume Gibbs measures (see Definition 5.0.5)

$$\bar{\nu}(\sigma(x-1) = -,\, \sigma(x) = +,\, \sigma(x+1) = -|\sigma(x-1) = -,\, \sigma(x+1) = -) = e^{-2\beta}.$$

The last ingredient is that the upper invariant measure inherits from the finite volume Gibbs measures the following property: for all $s, s', s'' \in \pm$ it holds

$$\bar{\nu}(\sigma(x-1) = s|\sigma(x) = s',\, \sigma(x+1) = s'') = \bar{\nu}(\sigma(x-1) = s|\sigma(x) = s').$$

Putting all this ingredients together and with some algebra we discover that

$$\bar{\nu}(\sigma(x) = -|\sigma(x+1) = +) = \bar{\nu}(\sigma(x) = +|\sigma(x+1) = -)$$

which yields, setting for simplicity of notation

$$A := \bar{\nu}(\sigma(x) = -|\sigma(x+1) = +),\;\; B := \bar{\nu}(\{\sigma(x) = -\})$$

$$B = A\,\bar{\nu}(\{\sigma(x+1)=+\}) + \bar{\nu}(\{\sigma(x)=-\}|\{\sigma(x+1)=-\})\bar{\nu}(\{\sigma(x+1)=-\}) \tag{5.0.10}$$
$$= A(1-B) + (1-A)B \tag{5.0.11}$$

where in the last equality we used translation invariance which yields $\bar{\nu}(\sigma(x+1)=+) = 1-\bar{\nu}(\sigma(x+1)=-) = 1-\bar{\nu}(\sigma(x)=-)$ This implies

$$B = \bar{\nu}(\{\sigma(x+1)=-\}) = \frac{1}{2}$$

thus $m(\beta,1)=0$. (We refer the reader to Lemma 3.16 of [Swaa] for a more formal proof.) $\square$

The proof of Lemma 5.0.11 is based on a result known as Griffiths' inequalities. Let $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$. Let $\mathcal{P}(\Lambda)$ be the set of all subsets of Λ and given $A \in \mathcal{P}(\Lambda)$ and $\sigma \in \{-1,+1\}^{\Lambda}$ let

$$f(\sigma, A) := \prod_{i\in A} \sigma(i)$$

and

$$f(\sigma, \emptyset) := 1$$

and $J : \mathcal{P}(\Lambda) \to \mathbb{R}$. We let μ_J be a measure on $\Omega_\Lambda := \{-1,+1\}^{\Lambda}$ defined as

$$\mu_J(\sigma) := \frac{e^{\sum_{A\subset\Lambda} J_A\sigma_A}}{Z_J}$$

where

$$Z_J := \sum_{\sigma'\in\Omega_\Lambda} e^{\sum_A J_A \sigma'_A}$$

and for simplicity of notation we let $J_A := J(A)$ and $\sigma_A = f(\sigma, A)$. Notice that

- $\frac{\partial}{\partial J_A} \log Z_J = \int \mu_J(d\sigma)\sigma_A$
- $\frac{\partial^2}{\partial J_A \partial J_B} \log Z_J = \int \mu_J(d\sigma)\sigma_A\sigma_B - \int \mu_J(d\sigma)\sigma_A \int \mu_J(d\sigma)\sigma_B$

Lemma 5.0.14 (Griffiths' inequalities)**.** *If for all $A \subset \Lambda$ it holds $J_A \geqslant 0$ then*

- $\frac{\partial}{\partial J_A} \log Z_J \geqslant 0$*, for all* $A \subset \Lambda$
- $\frac{\partial^2}{\partial J_A \partial J_B} \log Z_J \geqslant 0$ *for all* $A, B \subset \Lambda$

Proof. By definition

$$Z_J = \sum_\sigma e^{\sum_A J_A\sigma_A} = \sum_\sigma \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{A_1} \cdots \sum_{A_n} (\prod_{k=1}^{n} J_{A_k}) \sum_\sigma \prod_{k=1}^{n} \sigma_{A_k} = 2^{|\Lambda|} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{A_1} \cdots \sum_{A_n} 1_{A_1\Delta\ldots\Delta A_n=\emptyset} (\prod_{k=1}^{n} J_{A_k})$$

with $A\Delta B$ the symmetric difference of A and B.

Use the fact that

$$\sum_\sigma \prod_{k=1}^n \sigma_{A_k} = \sum_\sigma \sigma_{A_1\Delta\ldots\Delta A_n}$$

and the right hand side above is zero unless $A_1\Delta\ldots\Delta A_n = \emptyset$. Analogously

$$\frac{\partial}{\partial J_A}\log Z_J = \frac{1}{Z_J}2^{|\Lambda|}\frac{1}{n!}\sum_{A_1}\cdots\sum_{A_n} 1_{A\Delta A_1\Delta\ldots\Delta A_n=\emptyset}(\prod_{k=1}^n J_{A_k})$$

which yields immediately the first result. The proof of the second result is analogous (just more lengthy ...) □

Proof of Lemma 5.0.11. Up to an additive constant we can rewrite the finite volume Hamltonian with plus boundary conditions as

$$H_+^\Lambda = -\frac{!}{2}\sum_{i,j\in\Lambda, i\sim j}\sigma(i)\sigma(j) - \frac{1}{2}\sum_{i\in\Lambda, j\in\partial\Lambda}\sigma(i)$$

Thus

$$\mu_+^{\Lambda,\beta} = \mu_J$$

with

$$J_A := \begin{cases} \frac{1}{2}\beta & \text{for } A = \{i,j\} \text{ with } i\sim j, i,j\in\Lambda \\ \frac{1}{2}\beta\,|\{j\in\Lambda^c : i\sim j| & \text{for } A = i \text{ with } i\in\Lambda \\ 0 & \text{otherwise} \end{cases} \tag{5.0.12}$$

Then

$$\frac{\partial}{\partial\beta}\int \mu_+^{\Lambda,\beta}(d\sigma)\sigma(i) \geqslant 0$$

follows by noticing that increasing β corresponds to increasing J. Now using Theorem 5.0.7 it follows that

$$\frac{\partial}{\partial\beta}m(\beta,d) \geqslant 0.$$

The monotonicity in d is proven analogously: for $d < d'$ view $\mathbb{Z}^d$ as a subset of $\mathbb{Z}^{d'}$. If we now "switch on" the interaction among sites in $\mathbb{Z}^d$ and $\mathbb{Z}^{d'}\setminus\mathbb{Z}^d$ the magnetisation will be higher in any point of $\mathbb{Z}^d$.

For the proof continuity of $m(\beta,d)$ in β we start by choosing a sequence $\{\beta_n\}$ with for each $\beta_n\downarrow\beta$. Let ν_n be the upper invariant measure of SIM at temperature β_n, namely $\nu_n := \bar{\nu}_{\beta_n}$. By compactness of the space there exists a probability law ν s.t. $\nu_n\to\nu$ with ν. Then, by using the fact that ν_n are invariant laws for SIM at temperature β_n, one can show that ν is an invariant measure for SIM at inverse temperature β [1] . Furthermore, since we proved that $m(\beta)$ is non decreasing in β, we have $\lim_{\beta \downarrow \beta_n} m(\beta_n) = \nu(\sigma(0)) \geqslant m(\beta)$. But since $\bar{\nu}_\beta$ is the largest invariant law at inverse temperature β we get $\nu = \bar{\nu}_\beta$.

□

Proof of Lemma 5.0.13. The proof of this Lemma uses a contour argument similar in spirit to the one we used for the contact process in the previous chapter to prove that $\lambda_c(1) > \infty$. Let Λ_n be the square of side $2n+1$ centred around the origin $\Lambda_n = [-n, n]^2$. Fix a configuration $\sigma \in \Lambda_n$ and let $\bar{\sigma} \in \Omega_{\Lambda_n \cup \partial \Lambda_n}$ be the configuration that equals $+1$ on $\partial \Lambda_n$ and σ inside Λ_n. Let $\mathcal{E}_n$ be the collection of edges in $\Lambda_n \cup \partial \Lambda_n$, namely the collection of couples of sites $\{i, j\}$ with $i \sim j$. We let $\Gamma(\sigma)$ be all the closed curves of the configuration σ obtained by drawing a "dual edge—" perpendicular to any edge joining two neighbouring disagreeing spins, namely

$$\Gamma(\sigma) := \{\{i, j\} \in \mathcal{E}_n : \bar{\sigma}(i) \neq \bar{\sigma}(j)\}.$$

Note (see Fig. 5.1) that

- $\Gamma(\sigma)$ forms a collection of closed curves (these closed curves are those that surround all the connected components of $-$ spin and of $+1$ spins in the configuration σ);
- given $\Gamma(\sigma)$ we can reconstruct σ in a unique way
- the origin has spin configuration $+1$ iff it is surrounded by an even number of closed curves

We now let $\mathcal{G}_n$ be the collection of all possible closed curves

$$\mathcal{G}_n := \{\Gamma(\sigma) : \sigma \in \Omega_{\Lambda_n}\}$$

and note that for each $\Gamma \in \mathcal{G}_n$ it holds

$$\bar{\nu}\{\sigma : \Gamma(\sigma) = \Gamma\} = \frac{e^{-\beta|\Gamma|}}{\sum_{\Gamma' \in \mathcal{G}_n} e^{-\beta|\Gamma'|}}$$

with $|\Gamma|$ the length of the curve Γ.

Therefore given a collection of nearest neighbour edges which formes a single closed curve γ containing the origin it holds

$$\bar{\nu}\{\sigma : \gamma \subset \Gamma(\sigma)\} = \frac{\sum_{\Gamma \in \mathcal{G}_n : \gamma \subset \Gamma} e^{-\beta|\Gamma|}}{\sum_{\Gamma' \in \mathcal{G}_n} e^{-\beta|\Gamma'|}} \leqslant \frac{\sum_{\Gamma \in \mathcal{G}_n : \gamma \subset \Gamma} e^{-\beta|\Gamma|}}{\sum_{\Gamma \in \mathcal{G}_n : \gamma \subset \Gamma} e^{-\beta|\Gamma|} + \sum_{\Gamma \in \mathcal{G}_n : \gamma \cap \Gamma = \emptyset} e^{-\beta|\Gamma|}}$$

[1] To prove this result let P_t^β denote the semigroup of SIM at inverse temperature β. Then for each function f and each $t \geqslant 0$ write $|\nu P_t^\beta(f) - \nu(f)| \leqslant |\nu P_t^\beta(f) - \nu_n P_t^\beta(f)| + |\nu_n P_t^\beta(f) - \nu_n P_t^{\beta_n}(f)| + |\nu_n P_t^{\beta_n}(f) - \nu_n(f)| + |\nu_n(f) - \nu(f)|$. It is now not difficult to upper bound this terms (sending $n \to \infty$ on the r.h.s.) to discover that $|\nu P_t^\beta(f) - \nu(f)| = 0$

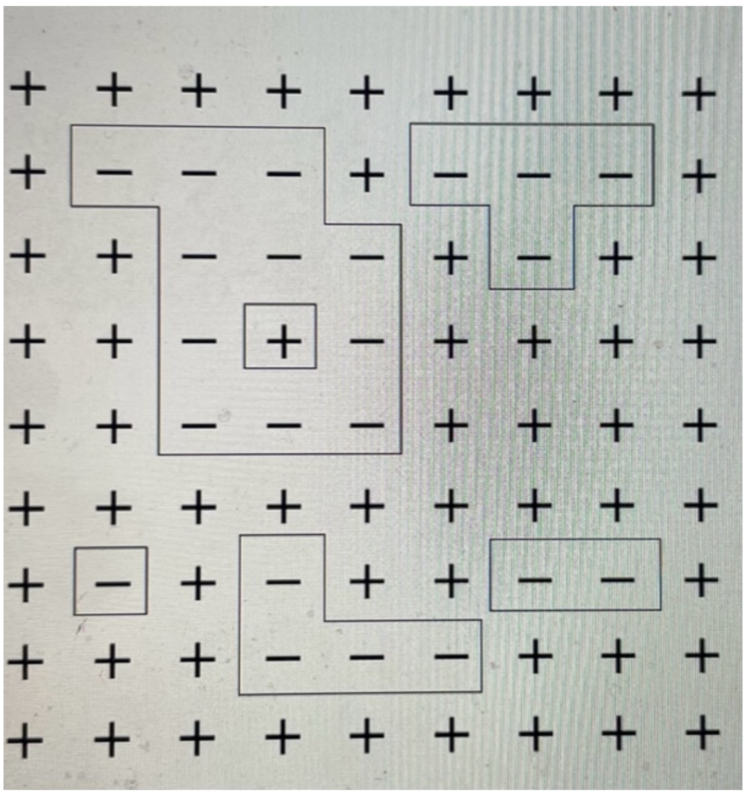

Figure 5.1:

Peierls contours for the Ising model

$$= \frac{\sum_{\Gamma\in\mathcal{G}_n:\gamma\subset\Gamma} e^{-\beta|\Gamma|}}{\sum_{\Gamma\in\mathcal{G}_n:\gamma\subset\Gamma} e^{-\beta|\Gamma|} + e^{\beta|\gamma|}\sum_{\Gamma\in\mathcal{G}_n:\gamma\subset\Gamma} e^{-\beta|\Gamma|}} = \frac{1}{1+e^{\beta|\gamma|}} \leqslant e^{-\beta|\gamma|}$$

Using the fact that there are at most $k3^k$ different curves of length k surrounding the origin we get

$$\mu_+^{\Lambda_n,\beta}(\mathcal{N}_0) \leqslant \sum_{k=4}^{\infty} k3^k e^{-k\beta}$$

with $\mathcal{N}_0$ the number of closed curves surrounding the origin. Since the r.h.s. goes to 0 as $\beta \to \infty$, we can certainly choose β sufficiently large (uniformly in n) so that the probability of the event $\mathcal{N}_0 = 0$, is $> 3/4$. This in turn implies

$$\int \mu_+^{\Lambda_n,\beta}(d\sigma)\sigma(0) = \mu_+(\{\mathcal{N}_0 \text{ is 0 or odd}\}) - \mu_+(\mathcal{N}_0\{\text{ is even}\}) \geqslant \frac{3}{4} - \left(1-\frac{3}{4}\right) = \frac{1}{2}.$$

□

We conclude by discussing some results and conjecture concerning the behavior at criticality. SIM in dimension 2 is exactly solvable, Onsager has shown that $\beta_c(d) = \log(1+\sqrt{2})$ and $m(\beta,2) = (1-(\sinh\beta)^{-4})^{1/8}$, where $\sinh(\beta) = 1/2(e^{\beta} - e^{-\beta})$. This yields

$$m(\beta,2) \sim (\beta-\beta_c)^{1/8} \quad \text{as} \quad \beta \downarrow \beta_c.$$

Furthermore it is possible to prove that $\bar{\nu}$ and $\underline{\nu}$ are the only two extremal invariant measures, namely complete convergence holds as for the contact process. Instead in higher spatial dimensions there exists spatially non homogeneous invariant measures.

In higher dimension the value of the critical exponent is not known, based on numerical simulations and non-rigorous renormalisation group theory it is conjecture that in three dimension the exponent is ~ 0.308.

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