



Sharp threshold for the FA-2f kinetically constrained model

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Abstract

The Fredrickson-Andersen 2-spin facilitated model on \mathbb{Z}^d (FA-2f) is a paradigmatic interacting particle system with kinetic constraints (KCM) featuring *dynamical facilitation*, an important mechanism in condensed matter physics. In FA-2f a site may change its state only if at least two of its nearest neighbours are empty. Although the process is reversible w.r.t. a product Bernoulli measure, it is not *attractive* and features degenerate jump rates and anomalous divergence of characteristic time scales as the density q of empty sites tends to 0. A natural random variable encoding the above features is τ_0 , the first time at which the origin becomes empty for the stationary process. Our main result is the sharp threshold

$$\tau_0 = \exp\left(\frac{d \cdot \lambda(d, 2) + o(1)}{q^{1/(d-1)}}\right) \text{ w.h.p.}$$

with $\lambda(d, 2)$ the sharp threshold constant for 2-neighbour bootstrap percolation on \mathbb{Z}^d , the monotone deterministic automaton counterpart of FA-2f. This is the first sharp result for a critical KCM and it compares with Holroyd’s 2003 result on bootstrap percolation and its subsequent improvements. It also settles various controversies accumulated in the physics literature over the last four decades. Furthermore, our novel techniques enable completing the recent ambitious program on the universality phe-

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nomenon for critical KCM and establishing sharp thresholds for other two-dimensional KCM.

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1 Introduction

Fredrickson–Andersen j -spin facilitated models (FA- j f) are a class of interacting particle systems that were introduced by physicists in the 1980s [14] to model the liquid/glass transition, a major and still largely open problem in condensed matter physics [2, 7]. Later on, several models with different update rules were introduced, and this larger class has been dubbed *kinetically constrained models* (KCM) (see e.g. [16] and references therein). The key feature of KCM is that an update at a given vertex x can occur only if a suitable neighbourhood of x contains only holes, the facilitating vertices. The presence of this dynamical constraint gives rise to a mechanism dubbed *dynamical facilitation* [39] in condensed matter physics: motion on smaller scales begets motion on larger scales. Extensive numerical simulations indicate that indeed KCM can display a remarkable glassy behaviour, featuring in particular an anomalous divergence of characteristic time scales. As a good representative of a random variable whose law encodes the above behaviour one could take τ_0 , the first time the origin becomes a hole (or infected, in the jargon of the sequel). In the last forty years physicists have put forward several different conjectures on the scaling of τ_0 as the equilibrium density of the holes goes to zero for FA- j f models. However, to date a clear cut answer on the form of this scaling has proved elusive due to the very slow dynamics and large finite size effects intrinsic to its glassy dynamics.

From the mathematical point of view, the study of FA- j f and KCM in general poses very challenging problems. This is largely due to the fact that these models do *not* feature an *attractive dynamics* (in the sense of [30, Chapter III]), and therefore many of the powerful tools developed to study attractive stochastic spin dynamics, e.g. monotone coupling or censoring, cannot be used. A central issue has been therefore that of developing novel mathematical tools to determine the long time behaviour of the stationary process and, more specifically, to find the scaling of the associated infection time of the origin, τ_0 in the sequel, as the density q of the empty sites (the facilitating ones) shrinks to zero.

With this motivation, an ambitious program was recently initiated in [33] to determine as accurately as possible the divergence of the infection time for the stationary process, as $q \rightarrow 0$ for the FA- j f models in any dimension and for general KCM in two dimensions. This program mirrors in some aspects the analogous program for general \mathcal{U} -bootstrap percolation cellular automata (\mathcal{U} -BP) launched by [10] and carried out in [4, 9] and for j -neighbour bootstrap percolation [5, 18, 27, 28]. Indeed \mathcal{U} -BP models and j -neighbour bootstrap percolation can be viewed as the monotone deterministic counterparts of generic KCM and FA- j f models respectively. Despite the above anal-

ogy, the lack of monotonicity for KCM induces a much more complex behaviour and richer universality classes than BP [22–25, 31, 32].

In spite of several important advances [12, 24, 25, 31–33], the *sharp* estimates of the divergence of τ_0 for stationary KCM still remained a milestone open problem. Solving it requires discovering the optimal infection/healing mechanism to reach the origin and crafting the mathematical tools to transform the knowledge of this mechanism into tight upper and lower bounds for τ_0 for the stationary process. In this paper we solve this problem (see Sect. 1.5 for an account of our most prominent innovations) for the first time and we establish the sharp scaling for FA-2f models in any dimension (Theorem 1.3). In doing so, we also settle various unresolved controversies in the physics literature (see Sect. 1.4 for a detailed account).

Our novel approach not only leads to deeper results, but also extends in breadth. Indeed, it opens the way for accomplishing the final step [22] to complete the program of [33] for establishing KCM universality.

1.1 Bootstrap percolation background

Let us start by recalling some background on j -neighbour bootstrap percolation. Let $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ and call a site $x \in \mathbb{Z}^d$ *infected* (or *empty*) for $\omega \in \Omega$ if $\omega_x = 0$ and *healthy* (or *filled*) otherwise. For fixed $0 < q < 1$, we denote by μ_q the product Bernoulli probability measure with parameter q under which each site is infected with probability q . When confusion does not arise, we write $\mu = \mu_q$. Given two integers $1 \leq j \leq d$ the j -neighbour BP model (j -BP for short) on the d -dimensional lattice \mathbb{Z}^d is the monotone cellular automaton on Ω evolving as follows. Let $A_0 \subset \mathbb{Z}^d$ be the set of *initially infected sites* distributed according to μ . Then for any integer *time* $t \geq 0$ we recursively define

$$A_{t+1} = A_t \cup \{x \in \mathbb{Z}^d, |N_x \cap A_t| \geq j\},$$

where N_x denotes the set of neighbours of x in the usual graph structure of \mathbb{Z}^d . In other words, a site becomes infected *forever* as soon as its constraint becomes satisfied, namely as soon as it has at least j already infected neighbours.

Remark 1.1 The j -BP is clearly monotone in the initial set of infection *i.e.* $A_t \subset A'_t$ for all $t \geq 1$ if $A_0 \subset A'_0$. Such a monotonicity will, however, be missing in the KCM models analysed in this work.

A key quantity for bootstrap percolation is the *infection time of the origin* defined as $\tau_0^{\text{BP}} = \inf\{t \geq 0, 0 \in A_t\}$. For $j=1$, trivially, τ_0^{BP} scales as the distance to the origin of the nearest infected site and thus behaves w.h.p. as $q^{-1/d}$. For $j > 1$, the typical value of τ_0^{BP} w.r.t. μ_q has been investigated in a series of works, starting with the seminal paper of Aizenman and Lebowitz [1] and Holroyd’s breakthrough [28] determining a sharp threshold for $d = j = 2$. We refer to [34] for an account of the field and only recall the more recent results that include second order corrections to the sharp threshold. Here and throughout the paper, when using asymptotic notation

we refer to $q \rightarrow 0$.¹ For 2-BP in $d = 2$, w.h.p. it holds [18, 27] that

$$\tau_0^{\text{BP}} = \exp\left(\frac{\pi^2}{18q}(1 - \sqrt{q} \cdot \Theta(1))\right). \tag{1.1}$$

For j -BP for all $d \geq j \geq 2$, w.h.p. it holds [5, 42]

$$\tau_0^{\text{BP}} \geq \exp^{j-1}\left(\frac{\lambda(d, j)}{q^{1/(d-j+1)}}(1 - o(1))\right), \tag{1.2}$$

$$\tau_0^{\text{BP}} \leq \exp^{j-1}\left(\frac{\lambda(d, j)}{q^{1/(d-j+1)}}(1 - \Omega(q^{1/(2(d-j+1))}))\right), \tag{1.3}$$

where \exp^k denotes the exponential iterated k times and $\lambda(d, j)$ are the positive constants defined explicitly in [6, (1–3)]. We recall that $\lambda(2, 2) = \pi^2/18$ [28, Proposition 5] and we refer the interested reader to [6, Table 1 and Proposition 4] for other values of d, j .

We are now ready to introduce the Fredrickson–Andersen model, a natural stochastic counterpart of j -BP and the main focus of this work.

1.2 The Fredrickson–Andersen model and main result

For integers $1 \leq j \leq d$ the *Fredrickson–Andersen j -spin facilitated model* (FA- j f) is the interacting particle system on $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ constructed as follows. Each site is endowed with an independent Poisson clock with rate 1. At each clock ring the state of the site is updated to an independent Bernoulli random variable with parameter $1 - q$ subject to the crucial constraint that if the site has fewer than j infected (nearest) neighbours currently, then the update is rejected. We refer to updates occurring at sites with at least j infected neighbours at the time of the update as *legal*.

Remark 1.2 Contrary to the j -BP model, the FA- j f process is clearly non-monotone because of the possible recovery of infected sites with at least j infected neighbours. This feature is one of the major obstacles in the analysis of the process.

It is standard to show (see [30]) that the FA- j f process is well defined and it is reversible w.r.t. μ_q . When the initial distribution at time $t = 0$ is a measure ν , the law and expectation of the process on the Skorokhod space $D([0, \infty), \Omega)$ will be denoted by \mathbb{P}_ν and \mathbb{E}_ν respectively. As for j -BP let

$$\tau_0 = \inf\{t \geq 0, \omega_0(t) = 0\}$$

be the first time the origin becomes infected. Our main goal is to quantify precisely $\mathbb{E}_{\mu_q}[\tau_0]$, the average of τ_0 w.r.t. the stationary process as $q \rightarrow 0$. In order to keep the

¹ If f and g are real-valued functions of q with g positive, we write $f = O(g)$ if there exists a (deterministic absolute) constant $C > 0$ such that $|f(q)| \leq Cg(q)$ for every sufficiently small $q > 0$. We also write $f = \Omega(g)$ if f is positive and $g = O(f)$. We further write $f = \Theta(g)$ if both $f = O(g)$ and $f = \Omega(g)$. Finally, we write $f = o(g)$ if for all $c > 0$ for sufficiently small $q > 0$ we have $|f(q)| \leq cg(q)$.

setting simple and the results more transparent, we will focus on the FA-2f model. Other models, including FA- j f for all values of $3 \leq j \leq d$, are discussed in Sect. 1.3. Recall the constants $\lambda(d, 2)$ from (1.2), (1.3), so that $\lambda(2, 2) = \pi^2/18$.

Theorem 1.3 *As $q \rightarrow 0$ the stationary FA-2f model on \mathbb{Z}^d satisfies:*

$$\mathbb{E}_{\mu_q}[\tau_0] \geq \exp\left(\frac{\pi^2}{9q}(1 - \sqrt{q} \cdot O(1))\right), \tag{1.4}$$

$$\mathbb{E}_{\mu_q}[\tau_0] \leq \exp\left(\frac{\pi^2}{9q}(1 + \sqrt{q} \cdot \log^{O(1)}(1/q))\right), \tag{1.5}$$

if $d = 2$, and

$$\mathbb{E}_{\mu_q}[\tau_0] \geq \exp\left(\frac{d \cdot \lambda(d, 2)}{q^{1/(d-1)}}(1 - o(1))\right), \tag{1.6}$$

$$\mathbb{E}_{\mu_q}[\tau_0] \leq \exp\left(\frac{d \cdot \lambda(d, 2)}{q^{1/(d-1)}}(1 + q^{1/(2(d-1))}(\log(1/q))^{O(1)})\right), \tag{1.7}$$

if $d \geq 3$. Moreover, (1.4–1.7) also hold for τ_0 w.h.p.

In particular, recalling (1.2), (1.3), we have the following.

Corollary 1.4 *W.h.p. $\tau_0 = (\tau_0^{\text{BP}})^{d+o(1)}$.*

The above are the first results that establish the sharp asymptotics of $\log \mathbb{E}_{\mu_q}[\tau_0]$ within the whole class of “critical” KCM.

Remark 1.5 We will not provide an explicit proof of the case $d \geq 3$ as it does not require any additional effort with respect to the case $d = 2$. The only significant difference is that the lower bound from (1.1) is not available in higher dimensions, leading to a corresponding weakening of the lower bound (1.6) as compared to (1.4).

Remark 1.6 Despite the resemblance, our results are by no means a corollary of their 2-BP counterpart (1.1). While the lower bounds (1.4) and (1.6) do indeed follow rather easily from (1.1) and (1.2) together with an improvement of the “automatic” lower bound from [12, Theorem 6.9], the proof of (1.5) and (1.7) is much more involved. In particular, it requires guessing an efficient infection/healing mechanism to infect the origin, which has no counterpart in the monotone j -BP dynamics (see Sect. 1.5).

1.3 Extensions

1.3.1 FA- j f with $j \neq 2$

For the sake of completeness, let us briefly discuss the FA- j f model with other values of j . The case $j = 1$ is the simplest to analyse and behaves very differently: relaxation is dominated by the motion of single infected sites and time scales diverge as $1/q^{\Theta(1)}$ (see [12, 38] for the values of the exponent). For $d \geq j \geq 4$ we believe that minor

modifications of the treatment of [33] along the lines provided by [6] should be sufficient to prove that $\mathbb{E}_{\mu_q}[\tau_0]$ scales as τ_0^{BP} (see (1.2), (1.3)). The only remaining case, $d \geq j = 3$, should require some more work, still following the approach of [33]. Let us emphasise that it should be possible to treat all $d \geq j \geq 3$, using the techniques of the present paper. However, the much faster divergence of the scaling involved should allow the less refined technique of [33] to work, as there is a much larger margin for error, making those results easier. We leave the above considerations to future work.

1.3.2 More general update rules: \mathcal{U} -KCM

The full power of the method developed in the present work is required to treat two-dimensional \mathcal{U} -KCM, a very general class of interacting particle systems with kinetic constraints on \mathbb{Z}^2 . These models and their bootstrap percolation counterpart, \mathcal{U} -BP, are defined similarly to FA- j f and j -BP but with arbitrary local monotone constraints (or update rules) \mathcal{U} [12, 33]. There exist several very symmetric constraints, including the so-called modified 2-BP, requiring two *non-opposite* neighbours to be infected, for which the exact asymptotics of $\log \tau_0^{\text{BP}}$, and sometimes even the higher order corrections, are known [11]. Our methods should adapt to this setting to yield equally sharp results for $\mathbb{E}_{\mu_q}[\tau_0]$ of the corresponding \mathcal{U} -KCM. In this general setting the outcome would again be of the form $\mathbb{E}_{\mu_q}[\tau_0] \simeq (\tau_0^{\text{BP}})^2$ as for FA-2f.

We warn the reader that the exponent 2 in two dimensions relating $\mathbb{E}_{\mu_q}[\tau_0]$ to τ_0^{BP} is not general [23, 24] and only applies to ‘isotropic’ models [22]. Nevertheless, developing the approach of the present work further, in [22] $\log \mathbb{E}_{\mu_q}[\tau_0]$ is determined up to a constant factor for all so-called “critical” KCM in two dimensions, matching the lower bounds established in [23] and establishing a richer KCM analogue of the BP universality result of [9].

1.4 Settling a controversy in the physics literature

Soon after the FA- j f models were introduced, some conjectures in the physics literature predicted the divergence of $\mathbb{E}_{\mu_q}[\tau_0]$ at a *positive* critical density q_c ([14, 15, 17]). These conjectures were subsequently ruled out in [12], the first contribution analysing rigorously FA- j f. After [12] and prior to the present work, the best known bounds on the infection time were

$$\begin{aligned} \exp\left(\frac{\Omega(1)}{q^{1/(d-1)}}\right) &\leq \mathbb{E}_{\mu_q}[\tau_0] \leq \exp\left(\frac{\log^{O(1)}(1/q)}{q^{1/(d-1)}}\right), & j = 2, \\ \exp^{j-1}\left(\frac{\lambda(d, j) - o(1)}{q^{1/(d-j+1)}}\right) &\leq \mathbb{E}_{\mu_q}[\tau_0] \leq \exp^{j-1}\left(\frac{O(1)}{q^{1/(d-j+1)}}\right), & j \geq 3. \end{aligned} \tag{1.8}$$

The lower bounds follow from the general lower bound [33, Lemma 4.3] $\mathbb{E}_{\mu_q}[\tau_0] = \Omega(\text{median of } \tau_0^{\text{BP}})$ together with the j -nBP lower bounds (see Sect. 1.1) while the upper bounds were recently obtained by the second and third author in [33]. As such, the above results do not settle a controversy between several conjectures that were put forward in the physics literature.

The first quantitative prediction for the scaling of $E_{\mu_q}[\tau_0]$ appeared in [35] where, based on numerical simulations, a faster than exponential divergence in $1/q$ was conjectured for FA-2f in $d = 2$. For the latter, the first to claim an exponential scaling $\exp(\Theta(1)/q)$ was Reiter [36]. He argued that the infection process of the origin is dominated by the motion of *macro-defects*, *i.e.* rare regions having probability $\exp(-\Theta(1)/q)$ and size $\text{poly}(1/q)$ that move at an exponentially small rate $\exp(-\Theta(1)/q)$. Later Biroli, Fisher and the last author [41] considerably refined the above picture. They argued that macro-defects should coincide with the critical droplets of 2-BP having probability $\exp(-\pi^2/(9q))$ and that the time scale of the relaxation process inside a macro-defect should be $\exp(c/\sqrt{q})$, *i.e.* sub-dominant with respect to the inverse of their density, in sharp contrast with the prediction of [36]. Based on this and on the idea that macro-defects move diffusively, the relaxation time scale of FA-2f in $d = 2$ was conjectured to diverge as $\exp(\pi^2/(9q))$ in $d = 2$ [41, Section 6.3]. Yet, a different prediction was later made in [40] implying a different scaling of the form $\exp(2\pi^2/(9q))$. Concerning the behaviour of FA-2f in higher dimensions, in [41] the relaxation time was predicted to diverge as $(\tau_0^{\text{BP}})^d$, though the prediction was less precise than for the two dimensional case since the sharp results for 2-BP in dimension $d > 2$ proved in [5] were yet to be established.

Theorem 1.3 settles the above controversy by confirming the scaling prediction of [36, 41] and by disproving those of [35, 40]. Moreover, our result on the characteristic time scale of the relaxation process *inside* a macro-defect (see Proposition 4.7) agrees with the prediction of [41] and disproves the one of [36].

1.5 Behind Theorem 1.3: high-level ideas

The main intuition behind Theorem 1.3 is that for $q \ll 1$ the relaxation to equilibrium of the stationary FA-2f process is dominated by the slow motion of patches of infection dubbed *mobile droplets* or just *droplets* with very small probability of occurrence, roughly $\exp(-\pi^2/(9q))$. In analogy with the *critical droplets* of bootstrap percolation (see [28]), mobile droplets have a linear size which is polynomially increasing in q (with some arbitrariness), *i.e.* they live on a much smaller scale than the metastable length scale $e^{\Theta(1/q^{1/(d-1)})}$ arising in 2-BP percolation model. One of the main requirements dictating the choice of the scale of mobile droplets is the requirement that the typical infection environment around a droplet is w.h.p. such that the droplet is able to move under the FA-2f dynamics in any direction. Within this scenario the main contribution to the infection time of the origin for the stationary FA-2f process should come from the time it takes for a droplet to reach the origin.

In order to translate the above intuition into a mathematically rigorous proof, one is faced with two different fundamental problems:

- (1) a precise, yet workable, definition of mobile droplets;
- (2) an efficient model for their “effective” random evolution.

In [25, 32, 33] mobile droplets (dubbed “super-good” regions there) have been defined rather rigidly as fully infected regions of suitable shape and size and their motion has been modelled as a *generalised FA-1f process* on \mathbb{Z}^2 [32, Section 3.1]. In the latter process mobile droplets are freely created or destroyed with the correct heat-bath

equilibrium rates but *only at locations which are adjacent to an already existing droplet*. The main outcome of these papers have been (upper) bounds on the infection time of the origin of the form $\tau_0 \leq 1/\rho_D^{\log \log(1/\rho_D)^{O(1)}}$ w.h.p., where ρ_D is the density of mobile droplets.

While rather powerful and robust, the solution proposed in [25, 32, 33] to (1) and (2) above has no chance to get the *exact* asymptotics of the infection time because of the rigidity in the definition of the mobile droplets *and* of the chosen model for their effective dynamics. Indeed, a mobile droplet should be allowed to deform itself and move to a nearby position like an amoeba, by rearranging its infection using the FA-2f moves. This “amoeba motion” between nearby locations should occur on a time scale much smaller than the global time scale necessary to bring a droplet from far away to the origin. In particular, it should not require to first create a new droplet from the initial one and only later destroy the original one (the main mechanism of the droplet dynamics under the generalised FA-1f process).

With this in mind we offer a new solution to (1) and (2) above which indeed leads to determining the exact asymptotics of the infection time. Concerning (1), our treatment in Sect. 4 consists of two steps. We first propose a sophisticated multiscale definition of mobile droplets which, in particular, introduces a crucial degree of *softness* in their microscopic infection’s configuration². The second and much more technically involved step is developing the tools necessary to analyse the FA-2f dynamics inside a mobile droplet. In particular, we then prove two key features (see Propositions 4.6 and 4.7 for the case $d = 2$):

(1.a) to the leading order the probability ρ_D of mobile droplets satisfies

$$\rho_D \geq \exp\left(-\frac{d\lambda(d, 2)}{q^{1/(d-1)}} - \frac{O(\log^2(1/q))}{q^{1/(2d-2)}}\right),$$

(1.b) the “amoeba motion” of mobile droplets between nearby locations occurs on a time scale $\exp(O(\log(1/q)^3)/q^{1/(2d-2)})$ which is sub-leading w.r.t. the main time scale of the problem and only manifests in the second term of (1.5).

Property (1.a) follows rather easily from well known facts from bootstrap percolation theory, while proving property (1.b), one of the most innovative steps of the paper, requires a substantial amount of new ideas.

While properties (1.a) and (1.b) above are essential, they are not sufficient on their own for solving problem (2) above. In Sect. 5 we propose to model (admittedly only at the level of a Poincaré inequality, which however suffices for our purposes) the random evolution of mobile droplets as a symmetric simple exclusion process with two additional crucial add-ons: a *coalescence* part (when two mobile droplets meet they are allowed to merge) and a *branching* part (a single droplet can create a new one nearby as in the generalised FA-1f process). This model, which we call g -CBSEP, was studied for the purpose of its present application in the preparatory work [26]. Finally, the fact that g -CBSEP relaxes on a time scale proportional to the

² This construction is inspired by one suggested by P. Balister in 2017, which he conjectured would remove the spurious log-corrections in the bound (1.8) available at that time.

inverse density of mobile droplets (modulo logarithmic corrections) (see Proposition 5.2) yields the scaling of Theorem 1.3. We emphasise that modelling the large-scale motion of droplets by g -CBSEP instead of a generalised FA-1f process is an absolute novelty, also with respect to the physics literature.

2 Proof of Theorem 1.3: lower bound

In this section we establish the lower bounds (1.4) and (1.6) of Theorem 1.3. Our proof is actually a procedure to establish a general lower bound for $\mathbb{E}_{\mu_q}[\tau_0]$ based on bootstrap percolation. This approach improves upon a previous general result [33] Lemma 4.3 which lower bounds $\mathbb{E}_{\mu_q}[\tau_0]$ with the mean infection time for the corresponding bootstrap percolation model.

Before spelling the details out, let us explain the proof idea. In BP it is known that the origin typically gets infected by a rare “critical droplet” of size roughly $1/q$ which can be infected only using internal infections. This droplet, initially at distance $\approx (\text{density of critical droplets})^{-1/d}$ from the origin, grows linearly until hitting the origin. Hence $\tau_0^{\text{BP}} \approx (\text{density of critical droplets})^{-1/d}$. On the contrary, the leading behaviour of τ_0 is governed by the inverse probability of a critical droplet, because one needs to wait for a critical droplet to reach the origin under the FA-2f dynamics. Thus, we expect $\tau_0 \approx (\tau_0^{\text{BP}})^d$.

In order to turn this idea into a proof we need a little notation. We call any cuboid of \mathbb{Z}^d with faces perpendicular to the lattice directions simply *cuboid*. For a cuboid $R \subset \mathbb{Z}^d$ and $\eta \in \Omega_{\mathbb{Z}^d}$ we denote by $[\eta]_R$ the set of sites $x \in R$ which can become infected by legal updates (recall Sect. 1.2) only using infections in R . Equivalently, $[\eta]_R$ can be viewed as the set of sites eventually infected by 2-BP with initial condition the set $\{x \in R : \eta_x = 0\}$. Note that $[\eta]_R$ is a union of disjoint cuboids. For $x, y \in R$ we write $\{x \xrightarrow{R} y\}$ for the event that $[\eta]_R$ contains a cuboid containing x and y .

The next proposition essentially states that the infection time is at least the inverse density of critical droplets.

Proposition 2.1 *Let $V = [-\ell, \ell]^d$ with $\ell = \ell(q)$ be such that*

$$\mu_q(0 \in [\eta]_V) = o(1) \tag{2.1}$$

and let

$$\rho := \sup_{x \in V: d(x, V^c)=1} \mu_q(x \xleftrightarrow{V} 0). \tag{2.2}$$

Then

$$\mathbb{E}_{\mu_q}[\tau_0] \geq \frac{\Omega(1)}{\rho|V|^2}$$

and $\tau_0 \geq q/(\rho|V|^2)$ w.h.p.

Proof Let $(\eta(t))_{t \geq 0}$ denote the stationary KCM on \mathbb{Z}^d , let $\mathcal{I} = \{\omega : 0 \in [\omega]_V\}$ and let $\tau = \inf\{t \geq 0, \eta(t) \in \mathcal{I}\}$. Given a configuration $\eta \in \Omega$, we say that the origin

is infectable inside V iff $\eta \in \mathcal{I}$. The key observation here is that, by construction, $\tau_0 \geq \tau$.

Suppose that at time $t = 0$ the origin is not infectable inside V or, equivalently, that $\tau > 0$. Then we claim that at time $\tau > 0$ there exists a site x at the boundary of V such that $\eta(\tau) \in \{x \overset{V}{\longleftrightarrow} 0\}$. In other words, at time τ a suitable very unlikely infection has appeared in V . To prove the claim assume $\tau > 0$ and consider the site $x \in \mathbb{Z}^d$ which is updated at time τ . Necessarily $x \in V$ and $d(x, V^c) = 1$, since otherwise $[\eta^x(\tau)]_V = [\eta(\tau)]_V$, where $\eta^x(\tau)$ is the configuration equal to $\eta(\tau)$ except at site x . Furthermore, by definition of τ , $\eta(\tau) \in \mathcal{I}$ and $\eta^x(\tau) \notin \mathcal{I}$. But this implies $\eta(\tau) \in \{x \overset{V}{\longleftrightarrow} 0\}$, since otherwise a change of the state at x could not change the infectability of the origin inside V .

Recall now the rate one Poisson clocks discussed at the beginning of Sect. 1.2 and let $N_V(s)$ denote the random number of clock rings (legal or not) at sites in V up to time s . Let also $\eta^{(j)}$ denote the configuration right after the j -th clock ring. By the above we have

$$\mathbb{P}_{\mu_q}(0 < \tau \leq s \mid N_V(s)) \leq \sum_{j=1}^{N_V(s)} \sum_{\substack{x \in V \\ d(x, V^c)=1}} \mathbb{P}_{\mu_q}(\eta^{(j)} \in \{x \overset{V}{\longleftrightarrow} 0\} \mid N_V(s)).$$

Yet, conditionally on the clock rings, $\eta^{(j)}$ is distributed according to μ_q for the stationary FA-2f process (see e.g. [23] Claim 3.11). Hence, recalling (2.2), we get

$$\mathbb{P}_{\mu_q}(0 < \tau \leq s \mid N_V(s)) \leq N_V(s) |V| \rho. \tag{2.3}$$

Using $\mathbb{E}(N_V(s)) = s|V|$, (2.3) gives

$$\begin{aligned} \mathbb{P}_{\mu_q}(\tau \leq s) &= \mathbb{P}_{\mu_q}(\tau = 0) + \mathbb{E}(\mathbb{P}_{\mu_q}(0 < \tau \leq s \mid N_V(s))) \\ &\leq o(1) + s|V|^2 \rho, \end{aligned}$$

where \mathbb{E} denotes the average w.r.t. $N_V(s)$ and we used (2.1) to get $\mathbb{P}_{\mu_q}(\tau = 0) = \mu_q(\mathcal{I}) = o(1)$. In conclusion, for all $\varepsilon > 0$ we have

$$\limsup_{q \rightarrow 0} \mathbb{P}_{\mu_q}(\tau_0 \leq \varepsilon / (|V|^2 \rho)) \leq \limsup_{q \rightarrow 0} \mathbb{P}_{\mu_q}(\tau \leq \varepsilon / (|V|^2 \rho)) \leq \varepsilon,$$

which concludes the proof by Markov’s inequality. □

We can now easily deduce the lower bounds of Theorem 1.3 from Proposition 2.1 and the following bootstrap percolation results.

Theorem 2.2 (Eq. (5.11) of [1]). *For any $d \geq 2$ there exists $c = c(d) > 0$ such that (2.1) holds for any $d \geq 2$ and $\ell \leq \exp(cq^{-1/(d-1)})$.*

Theorem 2.3 (Theorem 6.1, Lemma 3.9 and Eq. (4) of [27]) . Let $d = 2$ and $\ell = \frac{1}{4q} \log(1/q')$, where $q' = -\log(1 - q)$. Fix a cuboid (i.e. rectangle) $R \subset \mathbb{Z}^2$ with side lengths a, b such that $1 \leq a \leq b \leq 2\ell$ and $b \geq \ell$. Then

$$\mu_q([\eta]_R = R) \leq \exp\left(-\frac{\pi^2}{9q} + \frac{O(1)}{\sqrt{q}}\right).$$

Theorem 2.4 (Theorem 17 of [5]) . Let $d \geq 2$ and $\varepsilon > 0$. Let C_0 be sufficiently large depending on d and ε . Then for any q small enough, $C > C_0$ not depending on q and cuboid R with longest edge of length $\ell = C/q^{1/(d-1)}$ we have

$$\mu_q([\eta]_R = R) \leq \exp\left(-\frac{d \cdot \lambda(d, 2) - \varepsilon}{q^{1/(d-1)}}\right).$$

Proof of the lower bounds (1.4) and (1.6) in Theorem 1.3 Fix d and ℓ as in Theorem 2.3. Theorem 2.2 implies (2.1). Then Theorem 2.3 and a union bound over all possible cuboids $R \subset V = [-\ell, \ell]^2$ containing both 0 and some x with $d(x, V^c) = 1$ give $\rho \leq \exp(-\frac{\pi^2}{9q} + \frac{O(1)}{\sqrt{q}})$. Thus, (1.4) follows from Proposition 2.1.

Fix d and ℓ as in Theorem 2.4. Theorem 2.2 implies (2.1). The upper bound on ρ leading to (1.6) follows from Theorem 2.4 together with a union bound as above, so we may conclude by Proposition 2.1. □

3 Constrained Poincaré inequalities

In this section we state and prove various Poincaré inequalities for the auxiliary chains that will be instrumental in Sect. 4 (see Lemmas 4.9 and 4.10).

3.1 Notation

Given $\Lambda \subset \mathbb{Z}^2$ and $\omega \in \Omega$, we write $\omega_\Lambda \in \Omega_\Lambda := \{0, 1\}^\Lambda$ for the restriction of ω to Λ and we denote by μ_Λ the marginal of μ on Ω_Λ . The configuration (in Ω or Ω_Λ) identically equal to one is denoted by $\mathbf{1}$. Given disjoint $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^2$, $\omega^{(1)} \in \Omega_{\Lambda_1}$ and $\omega^{(2)} \in \Omega_{\Lambda_2}$, we write $\omega^{(1)} \cdot \omega^{(2)} \in \Omega_{\Lambda_1 \cup \Lambda_2}$ for the configuration equal to $\omega^{(1)}$ in Λ_1 and to $\omega^{(2)}$ in Λ_2 . For $f : \Omega \rightarrow \mathbb{R}$ we will denote by $\mu(f)$ its expectation w.r.t. μ and by $\mu_\Lambda(f)$ and $\text{Var}_\Lambda(f)$ the mean and variance w.r.t. μ_Λ , given $\omega_{\mathbb{Z}^2 \setminus \Lambda}$.

For sake of completeness, we recall the classic definitions of Dirichlet form, Poincaré inequality, and relaxation time. Given a measure ν and a Markov process with generator \mathcal{L} reversible w.r.t. ν , the corresponding Dirichlet form $\mathcal{D} : \text{Dom}(\mathcal{L}) \rightarrow \mathbb{R}$ is defined as

$$\mathcal{D}(f) := -\nu(f \cdot \mathcal{L}f). \tag{3.1}$$

For the FA-2f model, the definition of Sect. 1.2 yields the following Dirichlet form

$$\mathcal{D}_{\mathbb{Z}^d}^{\text{FA-}jf}(f) = \sum_{x \in \mathbb{Z}^d} \mu(c_x \text{Var}_x(f)). \tag{3.2}$$

with c_x the indicator function of the event “the constraint at x is satisfied”, namely for $x \in \mathbb{Z}^2$ and $\eta \in \Omega$ we set

$$c_x(\eta) = \begin{cases} 1 & \text{if } \sum_{y \sim x} (1 - \eta_y) \geq 2 \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

where $y \sim x$ if x, y are nearest neighbours.

We say that a *Poincaré inequality* with constant C is satisfied by the Dirichlet form if for any function $f \in \text{Dom}(\mathcal{L})$ it holds

$$\text{Var}_\nu(f) \leq C\mathcal{D}(f). \tag{3.4}$$

Finally, the *relaxation time* is defined as the best constant in the Poincaré inequality, namely

$$T_{\text{rel}} := \sup_{\substack{f \in \text{Dom}(\mathcal{L}) \\ \text{Var}_\nu(f) \neq 0}} \frac{\text{Var}_\nu(f)}{\mathcal{D}(f)}. \tag{3.5}$$

A finite relaxation time implies that the reversible measure is mixing for the semigroup $P_t := e^{t\mathcal{L}}$ with exponentially decaying correlations (see e.g. [37]), namely for all $f \in L^2(\nu)$ it holds

$$\text{Var}_\nu(P_t f) = \nu(f P_t f) - \nu(f)^2 \leq \exp(-2t/T_{\text{rel}}) \text{Var}_\nu(f). \tag{3.6}$$

3.2 FA-1f-type Poincaré inequalities

Fix $\Lambda \subset \mathbb{Z}^2$ a connected set and let $\Omega_\Lambda^+ = \Omega_\Lambda \setminus \mathbf{1}$. Given $x \in \Lambda$ let N_x^Λ be the set of neighbours of x in Λ and let \mathcal{N}_x^Λ be the event that N_x^Λ contains at least one infection. For any $z \in \Lambda$ consider the two Dirichlet forms

$$\begin{aligned} \mathcal{D}_\Lambda^{\text{FA-1f}}(f) &= \mu_\Lambda \left(\sum_{x \in \Lambda} \mathbb{1}_{\mathcal{N}_x^\Lambda} \text{Var}_x(f) \mid \Omega_\Lambda^+ \right), & f : \Omega_\Lambda^+ &\rightarrow \mathbb{R}, \\ \mathcal{D}_\Lambda^{\text{FA-1f},z}(f) &= \mu_\Lambda \left(\sum_{\substack{x \in \Lambda \\ x \neq z}} \mathbb{1}_{\mathcal{N}_x^\Lambda} \text{Var}_x(f) + \text{Var}_z(f) \right), & f : \Omega_\Lambda &\rightarrow \mathbb{R}. \end{aligned}$$

Remark 3.1 The alert reader will recognise the above expressions as the Dirichlet forms of the FA-1f process on Ω_Λ^+ or on Ω_Λ with the site z unconstrained.

Our first tool is a Poincaré inequality for these Dirichlet forms.

Proposition 3.2 *Let Λ be a connected subset of \mathbb{Z}^2 and let $z \in \Lambda$ be an arbitrary site. Then:*

(1) for any $f : \Omega_\Lambda^+ \rightarrow \mathbb{R}$,

$$\text{Var}_\Lambda(f \mid \Omega_\Lambda^+) \leq \frac{1}{q^{O(1)}} \mathcal{D}_\Lambda^{\text{FA-1f}}(f); \tag{3.7}$$

(2) for any $f : \Omega_\Lambda \rightarrow \mathbb{R}$,

$$\text{Var}_\Lambda(f) \leq \frac{1}{q^{O(1)}} \mathcal{D}_\Lambda^{\text{FA-1f},z}(f), \tag{3.8}$$

where the constants in the $O(1)$ do not depend on z or Λ .

Proof Inequality (3.7) is proved in [8, Theorem 6.1]. In order to prove (3.8), consider the auxiliary Dirichlet form

$$\mu_\Lambda(\text{Var}_z(f)) + \mu_\Lambda(\mathbb{1}_{\Omega_\Lambda^+} \text{Var}_\Lambda(f | \Omega_\Lambda^+)).$$

The corresponding ergodic, continuous time Markov chain on Ω_Λ , reversible w.r.t. μ_Λ , updates the state of z at rate 1 and, if $\omega \in \Omega_\Lambda^+$, it updates the entire configuration w.r.t. $\pi(\cdot | \Omega_\Lambda^+)$. Observe that two copies of this chain attempting the same updates simultaneously couple as soon as they update the state of z to state 0 and then change to the same configuration in Ω_Λ^+ . Thus, by [29, Corollary 5.3 and Theorem 12.4] the relaxation time of this chain is $O(1/q)$, as the first step occurs at rate q . Indeed, after time $1/q$ there is probability $\Omega(1)$ that the above sequence of two consecutive updates has been performed.

Hence,

$$\begin{aligned} \text{Var}_\Lambda(f) &\leq O(1/q) \left(\mu_\Lambda(\text{Var}_z(f)) + \mu_\Lambda(\mathbb{1}_{\Omega_\Lambda^+} \text{Var}_\Lambda(f | \Omega_\Lambda^+)) \right) \\ &\leq \frac{1}{q^{O(1)}} \left(\mu_\Lambda(\text{Var}_z(f)) + \mu_\Lambda(\Omega_\Lambda^+) \mathcal{D}_\Lambda^{\text{FA-1f}}(f) \right), \end{aligned}$$

where the second inequality follows from (3.7). We may then conclude by observing that $\mu_\Lambda(\text{Var}_z(f)) + \mu_\Lambda(\Omega_\Lambda^+) \mathcal{D}_\Lambda^{\text{FA-1f}}(f) \leq 2\mathcal{D}_\Lambda^{\text{FA-1f},z}(f)$. \square

Our second tool is a general constrained Poincaré inequality for two independent random variables.

Proposition 3.3 (See [25, Lemma 3.10]). *Let X_1, X_2 be two independent random variable taking values in two finite sets $\mathbb{X}_1, \mathbb{X}_2$ respectively. Let also $\mathcal{H} \subset \mathbb{X}_1$ with $\mathbb{P}(X_1 \in \mathcal{H}) > 0$. Then for any $f : \mathbb{X}_1 \times \mathbb{X}_2 \rightarrow \mathbb{R}$ it holds*

$$\text{Var}(f) \leq 2\mathbb{P}(X_1 \in \mathcal{H})^{-1} \mathbb{E}(\text{Var}_1(f) + \mathbb{1}_{\{X_1 \in \mathcal{H}\}} \text{Var}_2(f)).$$

with $\text{Var}_i(f) = \text{Var}(f(X_1, X_2) | X_i)$.

Roughly speaking, this states that the chain that updates X_1 at rate 1 and X_2 at rate 1 only if \mathcal{H} occurs, has relaxation time given by the inverse probability of \mathcal{H} .

3.3 Constrained block chains

In this section we define two auxiliary constrained reversible Markov chains and give an upper bound for the corresponding Poincaré constants (Propositions 3.5 and 3.7).

Let $(\Omega_i, \pi_i)_{i=1}^3$ be finite probability spaces and let (Ω, π) denote the associated product space. For $\omega \in \Omega$ we write $\omega_i \in \Omega_i$ for its i^{th} coordinate and we assume for simplicity that $\pi_i(\omega_i) > 0$ for each ω_i . Fix $\mathcal{A}_3 \subset \Omega_3$ and for each $\omega_3 \in \mathcal{A}_3$ consider an event $\mathcal{B}_{1,2}^{\omega_3} \subset \Omega_1 \times \Omega_2$. Analogously, fix $\mathcal{A}_1 \subset \Omega_1$ and for each $\omega_1 \in \mathcal{A}_1$ consider an event $\mathcal{B}_{2,3}^{\omega_1} \subset \Omega_2 \times \Omega_3$. We then set

$$\begin{aligned} \mathcal{H} &= \{ \omega : \omega_3 \in \mathcal{A}_3 \text{ and } (\omega_1, \omega_2) \in \mathcal{B}_{1,2}^{\omega_3} \}, \\ \mathcal{K} &= \{ \omega : \omega_1 \in \mathcal{A}_1 \text{ and } (\omega_2, \omega_3) \in \mathcal{B}_{2,3}^{\omega_1} \} \end{aligned}$$

and let for any $f : \mathcal{H} \cup \mathcal{K} \rightarrow \mathbb{R}$

$$\mathcal{D}_{\text{aux}}^{(1)}(f) = \pi(\mathbb{1}_{\mathcal{H}} \text{Var}_{\pi}(f | \mathcal{H}, \omega_3) + \mathbb{1}_{\mathcal{K}} \text{Var}_{\pi}(f | \mathcal{K}, \omega_1) | \mathcal{H} \cup \mathcal{K}).$$

Observation 3.4 *It is easy to check that $\mathcal{D}_{\text{aux}}^{(1)}(f)$ is the Dirichlet form of the continuous time Markov chain on $\mathcal{H} \cup \mathcal{K}$ in which if $\omega \in \mathcal{H}$ the pair (ω_1, ω_2) is resampled with rate one from $\pi_1 \otimes \pi_2(\cdot | \mathcal{B}_{1,2}^{\omega_3})$ and if $\omega \in \mathcal{K}$ the pair (ω_2, ω_3) is resampled with rate one from $\pi_2 \otimes \pi_3(\cdot | \mathcal{B}_{2,3}^{\omega_1})$. This chain is reversible w.r.t. $\pi(\cdot | \mathcal{H} \cup \mathcal{K})$ and its constraints, contrary to what happens for general KCM, depend on the to-be-updated variables.*

Proposition 3.5 *There exists a universal constant c such that the following holds. Suppose that there exist two events $\mathcal{F}_{1,2}, \mathcal{F}_{2,3}$ such that*

$$\{ \omega : \omega_3 \in \mathcal{A}_3 \text{ and } (\omega_1, \omega_2) \in \mathcal{F}_{1,2} \} \subset \mathcal{H} \cap \mathcal{K}, \tag{3.9}$$

$$\{ \omega : \omega_1 \in \mathcal{A}_1 \text{ and } (\omega_2, \omega_3) \in \mathcal{F}_{2,3} \} \subset \mathcal{H} \cap \mathcal{K} \tag{3.10}$$

and let

$$T_{\text{aux}}^{(1)} = \max_{\omega_3 \in \mathcal{A}_3} \left(\frac{\pi(\mathcal{B}_{1,2}^{\omega_3})}{\pi(\mathcal{F}_{1,2})} \right)^2 \max_{\omega_1 \in \mathcal{A}_1} \frac{\pi(\mathcal{B}_{2,3}^{\omega_1})}{\pi(\mathcal{F}_{2,3})}.$$

Then, for all $f : \mathcal{H} \cup \mathcal{K} \rightarrow \mathbb{R}$,

$$\text{Var}_{\pi}(f | \mathcal{H} \cup \mathcal{K}) \leq c T_{\text{aux}}^{(1)} \mathcal{D}_{\text{aux}}^{(1)}(f).$$

Proof Consider the Markov chain $(\omega(t))_{t \geq 0}$ determined by the Dirichlet form $\mathcal{D}_{\text{aux}}^{(1)}$ as described in Observation 3.4. Given two arbitrary initial conditions $\omega(0)$ and $\omega'(0)$ we will construct a coupling of the two chains such that with probability $\Omega(1)$ we have $\omega(t) = \omega'(t)$ for any $t > T_{\text{aux}}^{(1)}$. Standard arguments (see for example [29, Theorem 12.4 and Corollary 5.3]) then prove that for this chain it holds $T_{\text{rel}} = O(T_{\text{aux}}^{(1)})$ and the conclusion of the proposition follows. To construct our coupling, we use the following representation of the Markov chain. We are given two independent Poisson clocks with rate one and the chain transitions occur only at the clock rings. Suppose that the first clock rings. If the current configuration ω does not belong to \mathcal{H} the ring is ignored. Otherwise, a Bernoulli variable ξ with probability of success $\pi(\mathcal{F}_{1,2} | \mathcal{B}_{1,2}^{\omega_3})$ is sampled.

If $\xi = 1$, then the pair (ω_1, ω_2) is resampled w.r.t. the measure $\pi(\cdot | \mathcal{F}_{1,2}, \mathcal{B}_{1,2}^{\omega_3})$, while if $\xi = 0$, then (ω_1, ω_2) is resampled w.r.t. the measure $\pi(\cdot | \mathcal{F}_{1,2}^c, \mathcal{B}_{1,2}^{\omega_3})$. Clearly, in doing so the couple (ω_1, ω_2) is resampled w.r.t. $\pi(\cdot | \mathcal{B}_{1,2}^{\omega_3})$. Similarly if the second clock rings but with $\mathcal{H}, (\omega_1, \omega_2), \mathcal{F}_{1,2}$ and $\mathcal{B}_{1,2}^{\omega_3}$ replaced by $\mathcal{K}, (\omega_2, \omega_3), \mathcal{F}_{2,3}$ and $\mathcal{B}_{2,3}^{\omega_1}$ respectively. It is important to notice that $\pi(\cdot | \mathcal{F}_{1,2}, \mathcal{B}_{1,2}^{\omega_3}) = \pi(\cdot | \mathcal{F}_{1,2})$ for all $\omega_3 \in \mathcal{A}_3$, as, by assumption, $\mathcal{F}_{1,2} \subset \bigcap_{\omega_3 \in \mathcal{A}_3} \mathcal{B}_{1,2}^{\omega_3}$. Similarly, $\pi(\cdot | \mathcal{F}_{2,3}, \mathcal{B}_{2,3}^{\omega_1}) = \pi(\cdot | \mathcal{F}_{2,3})$ for all $\omega_1 \in \mathcal{A}_1$.

In our coupling both chains use the same clocks. Suppose that the first clock rings and that the current pair of configurations is (ω, ω') . Assume also that at least one of them, say ω , is in \mathcal{H} (otherwise, both remain unchanged). In order to construct the coupling update we proceed as follows.

- If $\omega' \notin \mathcal{H}$ then ω is updated as described above, while ω' stays still.
- If $\omega' \in \mathcal{H}$ we first maximally couple the two Bernoulli variables ξ, ξ' corresponding to ω, ω' respectively. Then:
 - if $\xi = \xi' = 1$, we update both (ω_1, ω_2) and (ω'_1, ω'_2) to the *same* couple $(\eta_1, \eta_2) \in \mathcal{F}_{1,2}$ with probability $\pi((\eta_1, \eta_2) | \mathcal{F}_{1,2})$;
 - otherwise we resample (ω_1, ω_2) and (ω'_1, ω'_2) independently from their respective law given ξ, ξ' .

Similarly if the ring comes from the second clock. The final coupling is then equal to the Markov chain on $\Omega \times \Omega$ with the transition rates described above. Suppose now that there are three consecutive rings occurring at times $t_1 < t_2 < t_3$ such that:

- the first and last ring come from the first clock while the second ring comes from the second clock, and
- the sampling of the Bernoulli variables (if any) at times t_1, t_2 and t_3 all produce the value one.

Then we claim that at time t_3 the two copies are coupled.

To prove the claim, we begin by observing that after the first update at t_1 both copies of the coupled chain belong to \mathcal{K} . Here we use (3.9). Indeed, if the first update is successful for ω (i.e. $\omega \in \mathcal{H}$) then the updated configuration belongs to $\mathcal{F}_{1,2} \times \{\omega_3\} \subset \mathcal{K}$, because of our assumption $\xi = 1$. If, on the contrary, the first update fails (i.e. $\omega \notin \mathcal{H}$) then $\omega \in \mathcal{K} \setminus \mathcal{H}$ before and after the update. The same applies to ω' .

Next, using again the assumption on the Bernoulli variables together with the previous observation, we get that after the second ring the new pair of current configurations agree on the second and third coordinate. Moreover both copies belong to \mathcal{H} thanks to (3.10). Finally, after the third ring the two copies couple on the first and second coordinates using again the assumption on the outcome for the Bernoulli variables.

In order to conclude the proof of the proposition it is enough to observe that for any given time interval Δ of length one the probability that there exist $t_1 < t_2 < t_3$ in Δ satisfying the requirements of the claim is bounded from below by

$$c \min_{\omega_3 \in \mathcal{A}_3} \pi(\mathcal{F}_{1,2} | \mathcal{B}_{1,2}^{\omega_3})^2 \min_{\omega_1 \in \mathcal{A}_1} \pi(\mathcal{F}_{2,3} | \mathcal{B}_{2,3}^{\omega_1}),$$

for some constant $c > 0$.

□

In the same setting consider two other events $\mathcal{C}_{1,2} \subset \Omega_1 \otimes \Omega_2, \mathcal{C}_{2,3} \subset \Omega_2 \otimes \Omega_3$ and let

$$\mathcal{M} = \mathcal{A}_3 \cap \mathcal{C}_{1,2}, \quad \mathcal{N} = \mathcal{A}_1 \cap \mathcal{C}_{2,3}.$$

The Dirichlet form of our second Markov chain on $\mathcal{M} \cup \mathcal{N}$ is then

$$\begin{aligned} \mathcal{D}_{\text{aux}}^{(2)}(f) &= \pi \left(\mathbb{1}_{\mathcal{M}} \text{Var}(f \mid \mathcal{C}_{1,2}, \omega_3) + \mathbb{1}_{\mathcal{M}} \text{Var}(f \mid \mathcal{A}_3, \omega_1, \omega_2) \right. \\ &\quad \left. + \mathbb{1}_{\mathcal{N}} \text{Var}(f \mid \mathcal{C}_{2,3}, \omega_1) + \mathbb{1}_{\mathcal{N}} \text{Var}(f \mid \mathcal{A}_1, \omega_2, \omega_3) \mid \mathcal{M} \cup \mathcal{N} \right). \end{aligned} \tag{3.11}$$

Observation 3.6 *Similarly to the first case, the continuous time chain defined by (3.11) is reversible w.r.t. $\pi(\cdot \mid \mathcal{M} \cup \mathcal{N})$ and it can be described as follows. If $\omega \in \mathcal{M}$ then with rate one (ω_1, ω_2) is resampled w.r.t. $\pi_1 \otimes \pi_2(\cdot \mid \mathcal{C}_{1,2})$ and, independently at unit rate, ω_3 is resampled w.r.t. $\pi_3(\cdot \mid \mathcal{A}_3)$. Similarly, independently from the previous updates at rate one, if $\omega \in \mathcal{N}$ then (ω_2, ω_3) is resampled w.r.t. $\pi_2 \otimes \pi_3(\cdot \mid \mathcal{C}_{2,3})$ and, independently, ω_1 is resampled from $\pi_1(\cdot \mid \mathcal{A}_1)$.*

Proposition 3.7 *There exists a universal constant c such that the following holds. Suppose that there exist an event $\hat{\mathcal{C}}_{1,2} \subset \mathcal{C}_{1,2}$ and a collection $(\mathcal{A}_3^{\omega_1, \omega_2})_{(\omega_1, \omega_2) \in \hat{\mathcal{C}}_{1,2}}$ of subsets of \mathcal{A}_3 such that*

$$\{\omega : (\omega_1, \omega_2) \in \hat{\mathcal{C}}_{1,2} \text{ and } \omega_3 \in \mathcal{A}_3^{\omega_1, \omega_2}\} \subset \mathcal{M} \cap \mathcal{N}, \tag{3.12}$$

and let

$$T_{\text{aux}}^{(2)} = \max_{(\omega_1, \omega_2) \in \hat{\mathcal{C}}_{1,2}} \frac{\pi(\mathcal{A}_3)}{\pi(\mathcal{A}_3^{\omega_1, \omega_2})} \times \frac{\pi(\mathcal{C}_{1,2})}{\pi(\hat{\mathcal{C}}_{1,2})}.$$

Then there exists $c > 0$ such that for all $f : \mathcal{M} \cup \mathcal{N} \rightarrow \mathbb{R}$,

$$\text{Var}(f \mid \mathcal{M} \cup \mathcal{N}) \leq c T_{\text{aux}}^{(2)} \mathcal{D}_{\text{aux}}^{(2)}(f).$$

Proof We proceed as in the proof of Proposition 3.5 with the following representation for the Markov chain. We are given four independent Poisson clocks of rate one and each clock comes equipped with a collection of i.i.d. random variables. The four independent collections, the first being for the first clock etc, are

$$((\omega_1^{(i)}, \omega_2^{(i)}))_{i=1}^\infty, \quad (\eta_3^{(i)})_{i=1}^\infty, \quad ((\omega_2^{(i)}, \omega_3^{(i)}))_{i=1}^\infty, \quad (\eta_1^{(i)})_{i=1}^\infty,$$

where the laws of the collections are $\pi_1 \otimes \pi_2(\cdot \mid \mathcal{C}_{1,2}), \pi_3(\cdot \mid \mathcal{A}_3), \pi_2 \otimes \pi_3(\cdot \mid \mathcal{C}_{2,3})$ and $\pi_1(\cdot \mid \mathcal{A}_1)$ respectively.

At each ring of the first and second clocks the configuration is updated with the variables from the corresponding collection iff $\omega \in \mathcal{M}$. Similarly for the third and

fourth clocks with \mathcal{N} . In order to couple different initial conditions, we use the same collections of clock rings and update configurations.

Suppose now that there are four consecutive rings $t_1 < t_2 < t_3 < t_4$, coming from the first, second, third and fourth clocks in that order, such that:

- at t_1 the proposed update (η_1, η_2) of the first two coordinates belongs to $\hat{\mathcal{C}}_{1,2}$, and
- at t_2 the proposed update η_3 of the third coordinate belongs to $\mathcal{A}_3^{(\eta_1, \eta_2)}$.

We then claim that after t_4 all initial conditions ω are coupled. To prove this, we first observe that after the second ring each chain belongs to \mathcal{N} . Indeed, if $\omega \notin \mathcal{M}$, then the first two proposed updates are ignored and the configuration $\omega \in \mathcal{N} \setminus \mathcal{M}$. If, on the contrary, $\omega \in \mathcal{M}$, then both updates are successful and the configuration is updated to $(\eta_1, \eta_2, \eta_3) \in \hat{\mathcal{C}}_{1,2} \times \mathcal{A}_3^{\eta_1, \eta_2} \subset \mathcal{M} \cap \mathcal{N}$ by (3.12).

Since after t_2 the state of the chain is necessarily in \mathcal{N} , the third and fourth updates to states (η'_2, η'_3) and η'_1 respectively are both successful and thus any initial condition leads to the state $(\eta'_1, \eta'_2, \eta'_3)$ after t_4 , which proves the claim. The proof is then completed as in Proposition 3.5. □

4 Mobile droplets

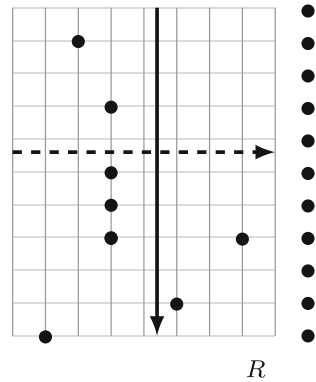
This section, which represents the core of the paper, is split into two parts:

- the definition of mobile droplets together with the choice of the mesoscopic critical length scale L_D characterising their linear size;
- the analysis of two key properties of mobile droplets namely:
 - their equilibrium probability ρ_D ;
 - the relaxation time of FA-2f in a box of linear size $\Theta(L_D)$ *conditionally* on the presence of a mobile droplet.

Mobile droplets are defined as boxes of suitable linear size in which the configuration of infection is *super-good* (see Definition 4.5). In turn, the super-good event (see Sect. 4.2) is constructed recursively via a multi-scale procedure on a sequence of exponentially increasing length scales $(\ell_n)_{n=1}^N$ (see Definition 4.2). While clearly inspired by the classical procedure used in bootstrap percolation [28], an important novelty in our construction is the freedom that we allow for the position of the super-good core of scale ℓ_n inside the super-good region of scale ℓ_{n+1} . The final scale ℓ_N corresponds to the critical scale L_D mentioned above and a convenient choice is $L_D \sim q^{-17/2}$ (see (4.4)). There is nothing special in the exponent 17/2: as long as we choose a sufficiently large exponent our results would not change. The choice of L_D is in fact only dictated by the requirement that w.h.p. there exist no L_D consecutive lattice sites at distance $\exp(\log^{O(1)}(1/q)/q)$ from the origin which are healthy and $L_D = e^{o(1/q)}$. Finally, similarly to their bootstrap percolation counterparts, the probability ρ_D of mobile droplets crucially satisfies $\rho_D \simeq (\tau_0^{\text{BP}})^{-2}$ (see Proposition 4.6) and in general for FA-2f in dimension d it satisfies $\rho_D \simeq (\tau_0^{\text{BP}})^{-d}$.

The extra degree of freedom in the construction of the super-good event provides a much more flexible structure that can be moved around using the FA-2f moves *without* going through the bottleneck corresponding to the creation of a brand new additional

Fig. 1 Black circles denote infected sites. The boundary condition ω in the figure is fully infected on $\partial_r R$ and fully healthy elsewhere. The rectangle R is ω -right-traversable (i.e. $\mathcal{T}_{\rightarrow}^{\omega}(R)$ occurs) but it is neither ω -up-traversable, nor ω -left-traversable. It is also down-traversable ($\mathcal{T}_{\downarrow}(R)$ occurs) but not traversable in any other direction



droplet nearby. The main consequence of this feature (see Proposition 4.7) is that the relaxation time of the FA-2f dynamics in a box of side L_D conditioned on being super-good is sub-leading w.r.t. ρ_D^{-1} as $q \rightarrow 0$ and it contributes only to the second order term in Theorem 1.3.

4.1 Notation

For any integer n , we write $[n]$ for the set $\{1, \dots, n\}$. We denote by $\mathbf{e}_1, \mathbf{e}_2$ the standard basis of \mathbb{Z}^2 , and write $d(x, y)$ for the Euclidean distance between $x, y \in \mathbb{Z}^2$. Given a set $\Lambda \subset \mathbb{Z}^2$, we set $\partial \Lambda := \{y \in \mathbb{Z}^d \setminus \Lambda, d(y, \Lambda) = 1\}$. Given two positive integers a, b , we write $R(a, b) \subset \mathbb{Z}^2$ for the rectangle $[a] \times [b]$ and we refer to a, b as the *width* and *height* of R respectively. We also write $\partial_r R$ ($\partial_l R$) for the column $\{a + 1\} \times [b]$ (the column $\{0\} \times [b]$), and $\partial_u R$ ($\partial_d R$) for the the row $[a] \times \{b + 1\}$ (the row $[a] \times \{0\}$). Similarly for any rectangle of the form $R + x, x \in \mathbb{Z}^2$.

Given $\Lambda \subset \mathbb{Z}^2$ and $\omega \in \Omega$, we write $\omega_{\Lambda} \in \Omega_{\Lambda} := \{0, 1\}^{\Lambda}$ for the restriction of ω to Λ . The configuration (in Ω or Ω_{Λ}) identically equal to one is denoted by $\mathbf{1}$. Given disjoint $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^2, \omega^{(1)} \in \Omega_{\Lambda_1}$ and $\omega^{(2)} \in \Omega_{\Lambda_2}$, we write $\omega^{(1)} \cdot \omega^{(2)} \in \Omega_{\Lambda_1 \cup \Lambda_2}$ for the configuration equal to $\omega^{(1)}$ in Λ_1 and to $\omega^{(2)}$ in Λ_2 . We write μ_{Λ} for the marginal of μ_q on Ω_{Λ} and $\text{Var}_{\Lambda}(f)$ for the variance of f w.r.t. μ_{Λ} , given the variables $(\omega_x)_{x \notin \Lambda}$.

4.2 Super-good event and mobile droplets

As anticipated, mobile droplets will be square regions of a certain side length in which the infection configuration satisfies a specific condition dubbed *super-good*. The latter requires in turn the definition of a key event for rectangles— ω -traversability (see also [28])—together with a sequence of exponentially increasing length scales.

Definition 4.1 (ω -Traversability). Fix a rectangle $R = R(a_1, a_2) + x$ together with $\eta \in \Omega_R$ and a boundary configuration $\omega \in \Omega_{\partial R}$. We say that R is ω -right-traversable for η if each pair of adjacent columns of $R \cup \partial_r R$ contains at least one infection in $\eta \cdot \omega$ (see Fig. 1). We denote this event by $\mathcal{T}_{\rightarrow}^{\omega}(R) \subset \Omega_R$.

We say that R is *right-traversable* for η if it is **1**-right-traversable or, equivalently, if it is ω -right-traversable for all ω . We denote this event by $\mathcal{T}_{\rightarrow}(R) \equiv \mathcal{T}_{\rightarrow}^1(R) \subset \Omega_R$.

Up/left/down-traversability and ω -up/left/down-traversability is defined identically up to rotating η and ω appropriately (see Fig. 1).

In figures we depict traversability by solid arrows and ω -traversability by dashed arrows (see Fig. 1). Notice that right-traversability requires that the rightmost column contains an infection. Similarly for the other directions.

Definition 4.2 (*Length scales and nested rectangles*). For all integer n we set³

$$\ell_m = \begin{cases} 1 & \text{if } m = 0, \\ \lfloor \frac{\exp(m\sqrt{q})}{\sqrt{q}} \rfloor & \text{if } m \geq 1 \end{cases} \tag{4.1}$$

and

$$\Lambda^{(n)} = \begin{cases} R(\ell_{n/2}, \ell_{n/2}) & \text{if } n \text{ is even,} \\ R(\ell_{(n+1)/2}, \ell_{(n-1)/2}) & \text{if } n \text{ is odd,} \end{cases} \tag{4.2}$$

(see Fig. 2). We say that a rectangle R is of *class* n if there exist $w, z \in \mathbb{Z}^2$ such that $\Lambda^{(n-1)} + w \subsetneq R \subset \Lambda^{(n)} + z$. We refer to single sites as rectangles of *class* 0.

Note that $(\Lambda^{(2m)})_{m \geq 0}$ is a sequence of squares, while $(\Lambda^{(2m+1)})_{m \geq 0}$ is a sequence of rectangles elongated horizontally and $\Lambda^{(n_1)} \subset \Lambda^{(n_2)}$ if $n_1 < n_2$. Moreover, for $n = 2m > 0$, a rectangle of class n is a rectangle of width ℓ_m and height $a_2 \in (\ell_{m-1}, \ell_m]$ and for $n = 2m + 1$ it is a rectangle of height ℓ_m and width $a_1 \in (\ell_m, \ell_{m+1}]$.

We are now ready to introduce the key notion of the ω -super-good event on different scales. This event is defined recursively on n and it has a hierarchical structure. Roughly speaking, a rectangle R of the form $R = \Lambda^{(n)} + x, x \in \mathbb{Z}^2$, is ω -super-good if it contains a **1**-super-good rectangle R' of the form $R' = \Lambda^{(n-1)} + x'$ called the *core* and outside the core it satisfies certain ω -traversability conditions (see Fig. 2).

Definition 4.3 (*ω -Super-good rectangles*). Let us fix an integer $n \geq 0$, a rectangle $R = R(a_1, a_2) + x$ of class n and $\omega \in \Omega_{\partial R}$. We say that R is *ω -super-good* for $\eta \in \Omega_R$ and denote the corresponding event by $\mathcal{SG}^\omega(R)$ if the following occurs in $\eta \cdot \omega$.

- $n = 0$. In this case R consists of a single site and $\mathcal{SG}^\omega(R)$ is the event that this site is infected.
- $n = 2m$. For any $s \in [0, \ell_m - \ell_{m-1}]$ write $R = C_s \cup (\Lambda^{(n-1)} + x + s\mathbf{e}_2) \cup D_s$, where C_s (D_s) is the part of R below (above) $\Lambda^{(n-1)} + x + s\mathbf{e}_2$. With this notation we set

$$\mathcal{SG}_s^\omega(R) := \mathcal{T}_\downarrow^\omega(C_s) \cap \mathcal{SG}^1(\Lambda^{(n-1)} + x + s\mathbf{e}_2) \cap \mathcal{T}_\uparrow^\omega(D_s)$$

and let $\mathcal{SG}^\omega(R) = \bigcup_{s \in [0, \ell_m - \ell_{m-1}]} \mathcal{SG}_s^\omega(R)$.

³ This choice of geometrically increasing length scales is inspired by [19].

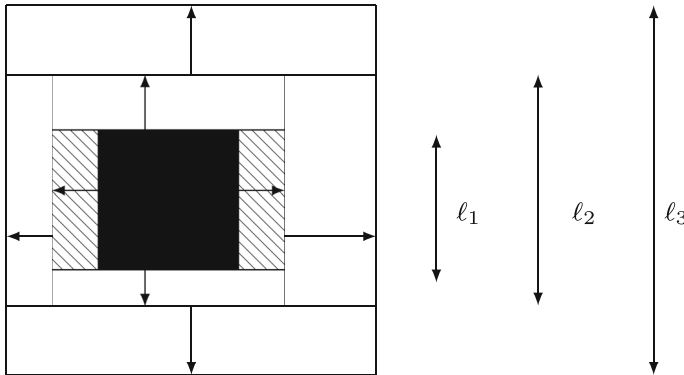


Fig. 2 An example of super-good configuration in the square $\Lambda^{(6)}$. The black square, of the form $\Lambda^{(2)} + x$, is completely infected and it is a super-good core for the rectangle of the form $\Lambda^{(3)} + x$ formed by it together with the two hatched rectangles. This rectangle of the form $\Lambda^{(3)} + x$ is also super-good because of the right/left-traversability of the hatched parts (arrows) and it is a super-good core for the square containing it and so on

- $n = 2m + 1$. In this case $\mathcal{SG}^\omega(R)$ requires that there is a core in R of the form $\Lambda^{(n-1)} + x + s\mathbf{e}_1$, $s \in [0, \ell_{m+1} - \ell_m]$, which is **1**-super-good, and the two remaining rectangles forming R to the left and to the right of the core are ω -left-traversable and ω -right-traversable respectively.

We will say that R is *super-good* if it is **1**-super-good and denote the corresponding event by $\mathcal{SG}(R)$.

Note that $\mathcal{SG}^\omega(R)$ is monotone in the boundary condition in the sense that if R is super-good then R is ω -super-good for all $\omega \in \Omega_{\partial R}$. In order to make notation more concise, whenever a \mathcal{SG} event appears in an average or a variance with respect to a rectangle R , we leave out the argument R of the \mathcal{SG} event, unless confusion arises. For example, $\mu_R(\mathcal{SG})$ will stand for $\mu_R(\mathcal{SG}(R))$.

Remark 4.4 (Irreducibility of the FA-2f chain in $\mathcal{SG}^\omega(R)$) It is not difficult to verify that for all $\eta \in \mathcal{SG}^\omega(R)$, there exists a sequence of legal updates that transforms η into the fully infected configuration. Since the FA-2f dynamics is reversible, the above property implies that the FA-2f chain in R restricted to $\mathcal{SG}^\omega(R)$ is irreducible.

Now let

$$N := \left\lceil \frac{8 \log(1/q)}{\sqrt{q}} \right\rceil \tag{4.3}$$

and observe that

$$\ell_N = q^{-17/2+o(1)}. \tag{4.4}$$

Definition 4.5 (*Mobile droplets*). Given $\omega \in \Omega$, a *mobile droplet* for ω is any square R of the form $R = \Lambda^{(2N)} + x$ for which $\omega_R \in \mathcal{SG}(R)$. We set $\rho_D = \mu_{\Lambda^{(2N)}}(\mathcal{SG})$ to be the probability of a mobile droplet.

The first key property of mobile droplets we will need is the following.

Proposition 4.6 (Probability of mobile droplets). *For all $n \leq 2N$,*

$$\mu_{\Lambda^{(n)}}(\mathcal{SG}) \geq \exp\left(-\frac{\pi^2}{9q}\left(1 + O(\sqrt{q} \log^2(1/q))\right)\right).$$

In particular, this lower bound holds for ρ_D .

The proof of Proposition 4.6 follows from standard 2-BP techniques and it is deferred to ‘‘Appendix A’’. The second property of mobile droplets requires a bit of preparation.

For $\Lambda \subset \mathbb{Z}^2$, $\omega \in \Omega_{\mathbb{Z}^2 \setminus \Lambda}$, $\eta \in \Omega$ and $x \in \Lambda$ we denote by

$$c_x^{\Lambda, \omega}(\eta) = c_x(\eta_{\Lambda} \cdot \omega)$$

with c_x defined in (3.3), so that $c_x^{\Lambda, \omega}$ encodes the constraint at x in Λ with boundary condition ω . Given a rectangle R of class n and $\omega \in \Omega_{\mathbb{Z}^2 \setminus R}$, let $\gamma^\omega(R)$ be the smallest constant⁴ $C \geq 1$ such that the Poincaré inequality (recall Sect. 3.1)

$$\text{Var}_R(f \mid \mathcal{SG}^\omega) \leq C \sum_{x \in R} \mu_R(c_x^{R, \omega} \text{Var}_x(f) \mid \mathcal{SG}^\omega), \tag{4.5}$$

holds for every $f : \Omega_R \rightarrow \mathbb{R}$. In the sequel we will sometimes refer to $\gamma^\omega(R)$ as the *relaxation time of $\mathcal{SG}^\omega(R)$* . The fact that FA-2f restricted to $\mathcal{SG}^\omega(R)$ is irreducible (see Remark 4.4) implies that $\gamma^\omega(R)$ is finite. However, proving a good upper bound on $\gamma^\omega(R)$ is quite hard.

Proposition 4.7 (Relaxation time of mobile droplets). *For all $n \leq 2N$*

$$\max_{\omega} \gamma^\omega(\Lambda^{(n)}) \leq \exp\left(O(\log^2(1/q)n)\right).$$

In particular, recalling (4.3), on the final scale this yields

$$\max_{\omega} \gamma^\omega(\Lambda^{(2N)}) \leq \exp\left(O(\log^3(1/q)/\sqrt{q})\right).$$

Remark 4.8 We stress an important difference in the definition of $\gamma^\omega(\Lambda^{(n)})$ w.r.t. a similar definition in [25, (12)]. Indeed, in (4.5) the conditioning w.r.t. the super-good event $\mathcal{SG}^\omega(R)$ appears in the l.h.s. and in the r.h.s. of the inequality, while in [25, (12)] the conditioning was absent in the r.h.s. Keeping the conditioning also in the r.h.s. is a delicate and important point if one wants to get a Poincaré constant which is *sub-leading* w.r.t. ρ_D^{-1} . Theorem 4.6 of [25] in the context of FA-2f would give a Poincaré constant bounded from above by $\exp(\log(1/q)^3/q)$, much bigger than ρ_D^{-1} .

⁴ The non-standard convention that relaxation times are at least 1 is made for convenience.

4.3 Proof of Proposition 4.7

The proof of the constrained Poincaré inequality of Proposition 4.7 is unfortunately rather long and technical but the main idea and technical ingredients can be explained as follows.

Given the recursive definition of the super-good event $\mathcal{SG}^\omega(\Lambda^{(n)})$ it is quite natural to try to bound from above its relaxation time in progressively larger and larger volumes. A high-level “dynamical intuition” here goes as follows. After every time interval of length $\Theta(\gamma^1(\Lambda^{(n-1)}))$ the core of $\Lambda^{(n)}$, namely a super-good translate of $\Lambda^{(n-1)}$ inside $\Lambda^{(n)}$, will equilibrate under the FA-2f dynamics. Therefore, the relaxation time of $\mathcal{SG}(\Lambda^{(n)})$ should be at most $T_{\text{eff}}^{(n)} \times \gamma^1(\Lambda^{(n-1)})$, where $T_{\text{eff}}^{(n)}$ is the time that it takes for the core to equilibrate its position inside $\Lambda^{(n)}$, assuming that at each time the infections inside it are at equilibrium. The main step necessary to transform this rather vague idea into a proof is as follows.

In order to analyse the characteristic time scale of the effective dynamics of a core, we need to improve and expand a well established mathematical technique for KCM to relate the relaxation times of two ω -super-good regions on different scales. Such a technique introduces various types of *auxiliary constrained block chains* and a large part of our argument is devoted to proving good bounds on their relaxation times (see Sect. 3). The main application of this technique to our concrete problem is summarised in Lemmas 4.9 and 4.10 below which easily imply Proposition 4.7. Let

$$\Lambda^{(n,+)} = \begin{cases} R(\ell_m + 1, \ell_m) & \text{if } n = 2m, \\ R(\ell_{m+1}, \ell_m + 1) & \text{if } n = 2m + 1. \end{cases}$$

The two key steps connecting the relaxation times of super-good rectangles of increasing length scale are as follows.

Lemma 4.9 (From $\ell_{\lfloor n/2 \rfloor} + 1$ to $\ell_{\lfloor n/2 \rfloor + 1}$) For all $0 \leq n \leq 2N - 1$

$$\max_{\omega} \gamma^\omega(\Lambda^{(n+1)}) \leq \exp(O(\log^2(q))) \max_{\omega} \gamma^\omega(\Lambda^{(n,+)}).$$

Lemma 4.10 (From $\ell_{\lfloor n/2 \rfloor}$ to $\ell_{\lfloor n/2 \rfloor} + 1$) For all $0 \leq n \leq 2N - 1$

$$\max_{\omega} \gamma^\omega(\Lambda^{(n,+)}) \leq q^{-O(1)} \max_{\omega} \gamma^\omega(\Lambda^{(n)}).$$

Proof of Proposition 4.7 Lemmas 4.9 and 4.10 combined imply that

$$\max_{\omega} \gamma^\omega(\Lambda^{(n+1)}) \leq \exp(O(\log(q)^2)) \max_{\omega} \gamma^\omega(\Lambda^{(n)}).$$

Thus, Proposition 4.7 follows by induction over n . Indeed, $\gamma^\omega(\Lambda^{(0)}) = 1$ for all $\omega \in \Omega_{\mathbb{Z}^2 \setminus \Lambda^{(0)}}$, since the l.h.s. of (4.5) is zero. \square

Before proving Lemma 4.9 formally, let us provide an informal description of the argument. We seek to apply a bisection technique (see [12, 21]) proceeding by a further

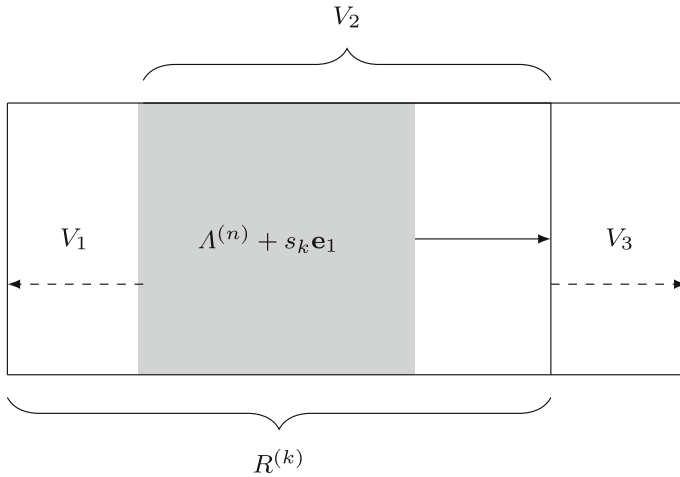


Fig. 3 The partition of $R^{(k+1)}$ into the rectangles V_1, V_2, V_3 . Here we illustrate the event $\mathcal{F}_{1,2} \cap \mathcal{A}_3$. The grey region $\Lambda^{(n)} + s_k \mathbf{e}_1$ at the left boundary of V_2 is $\mathcal{S}\mathcal{G}$ and the *dashed* arrows in V_1 and V_3 indicate their ω -traversability. The *solid* arrow in $V_2 \setminus (\Lambda^{(n)} + s_k \mathbf{e}_1)$ indicates instead the \mathbf{I} -traversability of $V_2 \setminus (\Lambda^{(n)} + s_k \mathbf{e}_1)$. Clearly the entire configuration belongs to the events \mathcal{H} and \mathcal{K} defined in (4.11), (4.12). Indeed, the two (ω -)right-traversability events together imply the ω -right-traversability of $(V_2 \cup V_3) \setminus (\Lambda^{(n)} + s_k \mathbf{e}_1)$

induction. At each step of this bisection, we divide by two the difference of the widths (or heights) between our current rectangle (initially $\Lambda^{(n+1)}$) and $\Lambda^{(n,+)}$. In order to prove a recursive bound on the relaxation times γ^ω of the intermediate rectangles of class $n + 1$ arising in the process, we rely on Proposition 3.5 as follows.

We want to prove a Poincaré inequality on a larger rectangle, given such an inequality on a smaller one. We cover the larger one with two overlapping copies of the smaller one. We then use the relaxation in the smaller one to move the core of shape $\Lambda^{(n)}$, witnessing it being super-good, to the intersection of the two translates. This makes the second copy super-good and allows us to resample it as well, thanks to the lower-scale Poincaré inequality. Thus, the events $\mathcal{F}_{1,2}$ and $\mathcal{F}_{2,3}$ in Proposition 3.5 will roughly correspond to finding the core in the aforementioned overlap region (see Fig. 3).

Proof of Lemma 4.9 Given $0 \leq n \leq 2N - 1$, let K_n be the smallest integer $K > 0$ such that $\lceil (2/3)^K (\ell_{\lfloor n/2 \rfloor + 1} - \ell_{\lfloor n/2 \rfloor}) \rceil = 1$ (if $K = 0$, there is nothing to prove, since $\Lambda^{(n,+)} = \Lambda^{(n+1)}$). Equations (4.1) and (4.3) give $\max_{n \leq 2N-1} K_n \leq O(\log(1/q))$. Consider the (exponentially increasing) sequence

$$d_k = \lceil (2/3)^{K_n - k} (\ell_{\lfloor n/2 \rfloor + 1} - \ell_{\lfloor n/2 \rfloor}) \rceil, \quad k \leq K_n, \tag{4.6}$$

and let $s_k = d_{k+1} - d_k$ for $k \leq K_n - 1$. Next consider the collection $(R^{(k)})_{k=0}^{K_n}$ of rectangles of class $n + 1$ interpolating between $\Lambda^{(n,+)}$ and $\Lambda^{(n+1)}$ defined by

$$R^{(k)} = \begin{cases} R(\ell_m + d_k, \ell_m) & \text{if } n = 2m, \\ R(\ell_{m+1}, \ell_m + d_k) & \text{if } n = 2m + 1. \end{cases}$$

By construction, $R^{(k)} \subset R^{(k+1)}$, $R^{(0)} = \Lambda^{(n,+)}$ and $R^{(K_n)} = \Lambda^{(n+1)}$. Finally, recall the events $\mathcal{SG}^\omega(R)$ and $\mathcal{SG}_s^\omega(R)$ constructed in Definition 4.3 for any rectangle R of class $n + 1 \leq 2N$ and let

$$a_k = \max_{\omega} (\mu_{R^{(k)}}(\mathcal{SG}_{s_k}^1 | \mathcal{SG}^\omega))^{-2} \max_{\omega} (\mu_{R^{(k)}}(\mathcal{SG}_0^\omega | \mathcal{SG}^\omega))^{-1}, \tag{4.7}$$

where \max_{ω} is over all $\omega \in \Omega_{\partial R^{(k)}}$. In Corollary A.3 we prove that

$$\mu_R(\mathcal{SG}_s^\omega | \mathcal{SG}^{\omega'}) \geq q^{O(1)}$$

uniformly over all rectangles R of class $n + 1 \leq 2N$, all possible values of the offset s and all choices of the boundary configurations $\omega, \omega' \in \Omega_{\partial R}$. As a consequence

$$\max_{n \leq 2N-1} \max_{k \leq K_n} a_k \leq (1/q)^{O(1)}. \tag{4.8}$$

With the above notation the key inequality for proving Lemma 4.9 is

$$\max_{\omega} \gamma^\omega(R^{(k+1)}) \leq C a_k \times \max_{\omega} \gamma^\omega(R^{(k)}), \quad k \in [0, K_n - 1], \tag{4.9}$$

for some universal constant $C > 0$. Recalling that $R^{(0)} = \Lambda^{(n,+)}$ and $R^{(K_n)} = \Lambda^{(n+1)}$, from (4.9) it follows that

$$\max_{\omega} \gamma^\omega(\Lambda^{(n+1)}) \leq \left(C^{K_n} \prod_{k=0}^{K_n-1} a_k \right) \times \max_{\omega} \gamma^\omega(\Lambda^{(n,+)}) \tag{4.10}$$

which in turn implies Lemma 4.9 by (4.8) and $K_n \leq O(\log(1/q))$.

The proof of (4.9), which is detailed for simplicity only in the even case $n = 2m$, relies on the Poincaré inequality for a properly chosen auxiliary block chain proved in Proposition 3.5.

In order to exploit that proposition we partition $R^{(k+1)}$ into three disjoint rectangles V_1, V_2, V_3 as follows (see Fig. 3):

$$V_1 = R(s_k, \ell_m), \quad V_2 = R^{(k)} \setminus V_1, \quad V_3 = R^{(k+1)} \setminus R^{(k)}.$$

Then, given a boundary configuration $\omega \in \Omega_{\partial R^{(k+1)}}$, let

$$\mathcal{H} = \{ \eta \in \Omega_R^{(k+1)} : \eta_3 \in \mathcal{T}_{\rightarrow}^\omega(V_3) \text{ and } \eta_1 \cdot \eta_2 \in \mathcal{SG}^{\eta_3 \cdot \omega}(V_1 \cup V_2) \}, \tag{4.11}$$

$$\mathcal{K} = \{ \eta \in \Omega_R^{(k+1)} : \eta_1 \in \mathcal{T}_{\leftarrow}^\omega(V_1) \text{ and } \eta_2 \cdot \eta_3 \in \mathcal{SG}^{\eta_1 \cdot \omega}(V_2 \cup V_3) \}, \tag{4.12}$$

where $\eta_i := \eta_{V_i}$. In words, \mathcal{H} requires that V_3 is ω -right-traversable and $R^{(k)} = V_1 \cup V_2$ is $\omega \cdot \eta_{R^{(k+1)} \setminus R^{(k)}}$ -super good and similarly for \mathcal{K} . Notice that $\mathcal{H} \cup \mathcal{K} = \mathcal{SG}^\omega(R^{(k+1)})$. Indeed, the width of V_2 is in fact $\ell_m + 2d_k - d_{k+1} \geq \ell_m$ and therefore any configuration in $\mathcal{SG}^\omega(R^{(k+1)})$ necessarily contains a super-good core in either $V_1 \cup V_2$ or $V_2 \cup V_3$.

We next introduce two additional events (see Fig. 3)

$$\mathcal{F}_{1,2} = \mathcal{S}\mathcal{G}_{s_k}^{\mathbf{1}_{V_3} \cdot \omega}(V_1 \cup V_2) \quad \mathcal{F}_{2,3} = \mathcal{S}\mathcal{G}_0^{\mathbf{1}_{V_1} \cdot \omega}(V_2 \cup V_3). \tag{4.13}$$

In words, $\mathcal{F}_{1,2}$ ($\mathcal{F}_{2,3}$) consists of super-good configurations in $V_1 \cup V_2$ ($V_2 \cup V_3$) with a super-good core of type $\Lambda^{(n)}$ inside V_2 in the *leftmost* possible position. Monotonicity in the boundary condition easily implies that

$$\{\eta : \eta_3 \in \mathcal{T}_{\rightarrow}^{\omega}(V_3) \text{ and } \eta_1 \cdot \eta_2 \in \mathcal{F}_{1,2}\} \subset \mathcal{H} \cap \mathcal{K},$$

and similarly for $\mathcal{F}_{2,3}$ (see Fig. 3).

We can now apply Proposition 3.5 with parameters $\Omega_i = \Omega_{V_i}$ for $i \in \{1, 2, 3\}$, $\mathcal{A}_1 = \mathcal{T}_{\leftarrow}^{\omega}(V_1)$, $\mathcal{A}_3 = \mathcal{T}_{\rightarrow}^{\omega}(V_3)$, $\mathcal{B}_{1,2}^{\eta_3} = \mathcal{S}\mathcal{G}^{\eta_3 \cdot \omega}(V_1 \cup V_2)$, $\mathcal{B}_{2,3}^{\eta_1} = \mathcal{S}\mathcal{G}^{\eta_1 \cdot \omega}(V_2 \cup V_3)$ and $\mathcal{F}_{1,2}$, $\mathcal{F}_{2,3}$ from (4.13). We claim that

$$\begin{aligned} T_{\text{aux}}^{(1)} &:= \max_{\substack{\eta_1 \in \mathcal{T}_{\leftarrow}^{\omega}(V_1) \\ \eta_3 \in \mathcal{T}_{\rightarrow}^{\omega}(V_3)}} \left(\frac{\mu_{R^{(k+1)}}(\mathcal{S}\mathcal{G}^{\eta_3 \cdot \omega}(V_1 \cup V_2))}{\mu_{R^{(k+1)}}(\mathcal{F}_{1,2})} \right)^2 \times \frac{\mu_{R^{(k+1)}}(\mathcal{S}\mathcal{G}^{\eta_1 \cdot \omega}(V_2 \cup V_3))}{\mu_{R^{(k+1)}}(\mathcal{F}_{2,3})} \\ &= \max_{\eta_1, \eta_3} \mu_{R^{(k)}}^{-2} (\mathcal{S}\mathcal{G}_{s_k}^{\mathbf{1}_{V_3} \cdot \omega} \mid \mathcal{S}\mathcal{G}^{\eta_3 \cdot \omega}) \mu_{s_k \mathbf{e}_1 + R^{(k)}}^{-1} (\mathcal{S}\mathcal{G}_0^{\mathbf{1}_{V_1} \cdot \omega} \mid \mathcal{S}\mathcal{G}^{\eta_1 \cdot \omega}) \leq a_k. \end{aligned}$$

Indeed, the second equality follows from (4.13) together with the fact that $V_1 \cup V_2 = R^{(k)}$ and $V_2 \cup V_3 = R^{(k)} + s_k$, while the inequality follows from (4.7). For the inequality it suffices to use monotonicity in the boundary condition for the first term and observe that $\mathcal{S}\mathcal{G}_0^{\eta_1 \cdot \omega}(s_k \mathbf{e}_1 + R^{(k)})$ does not depend on η_1 for the second one. Thus, Proposition 3.5 yields

$$\begin{aligned} \text{Var}_{R^{(k+1)}}(f \mid \mathcal{S}\mathcal{G}^{\omega}) &= \text{Var}_{R^{(k+1)}}(f \mid \mathcal{H} \cup \mathcal{K}) \leq ca_k \\ &\times \mu_{R^{(k+1)}}(\mathbb{1}_{\mathcal{H}} \text{Var}_{R^{(k+1)}}(f \mid \mathcal{H}, \eta_3) + \mathbb{1}_{\mathcal{K}} \text{Var}_{R^{(k+1)}}(f \mid \mathcal{K}, \eta_1) \mid \mathcal{H} \cup \mathcal{K}), \end{aligned} \tag{4.14}$$

for some universal constant $c > 0$.

In order to conclude the proof of (4.9) we are left with the analysis of the average w.r.t. $\mu_{R^{(k+1)}}(\cdot \mid \mathcal{H} \cup \mathcal{K})$ in the r.h.s. of (4.14). Recalling (4.5) and (4.11), for any $\eta_3 \in \mathcal{T}_{\rightarrow}^{\omega}(V_3)$ we get

$$\begin{aligned} &\text{Var}_{R^{(k+1)}}(f \mid \mathcal{H}, \eta_3) \\ &= \text{Var}_{R^{(k+1)}}(f \mid \eta_3, \mathcal{S}\mathcal{G}^{\eta_3 \cdot \omega}(R^{(k)})) \\ &\leq \max_{\omega' \in \Omega_{\mathbb{Z}^2 \setminus R^{(k)}}} \gamma^{\omega'}(R^{(k)}) \times \sum_{x \in R^{(k)}} \mu_{R^{(k)}}(c_x^{R^{(k)}, \eta_3 \cdot \omega} \text{Var}_x(f) \mid \mathcal{S}\mathcal{G}^{\eta \cdot \omega}). \end{aligned} \tag{4.15}$$

An analogous inequality holds for $\text{Var}(f \mid \mathcal{K}, \eta_1)$ when $\eta_1 \in \mathcal{T}_{\leftarrow}^\omega(V_1)$. Finally, we observe that for any $x \in R^{(k)}$

$$\begin{aligned} & \mu_{R^{(k+1)}}(\mathbb{1}_{\mathcal{H}} \mu_{R^{(k)}}(c_x^{R^{(k)}, \eta_3 \cdot \omega} \text{Var}_x(f) \mid \mathcal{SG}^{\eta_3 \cdot \omega}(R^{(k)})) \mid \mathcal{SG}^\omega(R^{(k+1)})) \\ &= \frac{\mu_{V_3}(\mathbb{1}_{\mathcal{A}_3} \mu_{R^{(k)}}(\mathbb{1}_{\mathcal{SG}^{\eta_3 \cdot \omega}(R^{(k)})} \mu_{R^{(k)}}(c_x^{R^{(k)}, \eta_3 \cdot \omega} \text{Var}_x(f) \mid \mathcal{SG}^{\eta_3 \cdot \omega}(R^{(k)}))))}{\mu_{R^{(k+1)}}(\mathcal{SG}^\omega)} \\ &= \mu_{R^{(k+1)}}(\mathbb{1}_{\mathcal{H}} c_x^{R^{(k+1)}, \omega} \text{Var}_x(f) \mid \mathcal{SG}^\omega(R^{(k+1)})), \end{aligned} \tag{4.16}$$

since $\mathbb{1}_{\mathcal{H}} = \mathbb{1}_{\mathcal{A}_3} \mathbb{1}_{\mathcal{SG}^{\eta_3 \cdot \omega}(R^{(k)})} \leq \mathbb{1}_{\mathcal{SG}^\omega(R^{(k+1)})}$ by (4.11) and $\mu_{R^{(k+1)}} = \mu_{R^{(k)}} \otimes \mu_{V_3}$. A similar result relation holds for \mathcal{K} . Inserting (4.15) and (4.16) into (4.14), we get

$$\begin{aligned} \text{Var}_{R^{(k+1)}}(f \mid \mathcal{SG}^\omega) &\leq O(a_k) \times \max_{\omega'} \gamma^{\omega'}(R^{(k)}) \\ &\times \sum_{x \in R^{(k+1)}} \mu_{R^{(k+1)}}(c_x^{R^{(k+1)}, \omega} \text{Var}_x(f) \mid \mathcal{SG}^\omega), \end{aligned}$$

which proves (4.9) in view of (4.5). □

The proof of Lemma 4.10 is similar to that of Lemma 4.9, but in this case we plan to use Proposition 3.7 instead of Proposition 3.5. The reason why the same proof does not apply is that the intersection of two distinct copies of $\Lambda^{(n)}$ is never large enough to contain another copy of $\Lambda^{(n)}$. Therefore, we are forced to look inside the $\Lambda^{(n)}$ core in order to shrink it by one line (see Fig. 4). Namely, we will position the core of type $\Lambda^{(n-2)}$ so that it is in the middle region corresponding to V_2 in the previous proof. We will then ask for events stronger than traversability on V_1 and V_3 in order to fit the structure in V_2 (see Fig. 4).

Proof of Lemma 4.10 Once again, we provide the details only in the case $n = 2m$. Let us start with the case $m = 0$. Firstly, $\gamma^\omega(\Lambda^{(0)}) = 1$ for all ω by the definition (4.5), as $\text{Var}_{\Lambda^{(0)}}(f \mid \mathcal{SG}^\omega) = 0$ for all f and ω . Moreover, $\mathcal{SG}^\omega(\Lambda^{(0,+)}) \subset \Omega_{\Lambda^{(0,+)}}$ has 1, 2 or 3 elements (depending on ω). If this space has a single point, $\gamma^\omega(\Lambda^{(0,+)}) = 1$ as for $\Lambda^{(0)}$ and we are done. Otherwise, we are dealing with an irreducible reversible Markov process on at most 3 states and transition rates bounded from below by q , so $\max_\omega \gamma^\omega(\Lambda^{(0,+)}) = q^{O(1)}$.

Let $m \geq 1$. We begin by writing $\Lambda^{(n,+)} = R(\ell_m + 1, \ell_m) = V_1 \cup V_2 \cup V_3$, where V_1 denotes the leftmost column, V_3 the rightmost column and V_2 all the remaining columns (see Fig. 4). By construction $V_1 \cup V_2$ and $V_2 \cup V_3$ are translates of $\Lambda^{(n)}$. Then, for any given $\omega \in \Omega_{\partial \Lambda^{(n,+)$, we introduce the events

$$\begin{aligned} \mathcal{M} &= \mathcal{T}_{\rightarrow}^\omega(V_3) \cap \mathcal{SG}(V_1 \cup V_2) = \mathcal{SG}_0^\omega(\Lambda^{(n,+)}), \\ \mathcal{N} &= \mathcal{T}_{\leftarrow}^\omega(V_1) \cap \mathcal{SG}(V_2 \cup V_3) = \mathcal{SG}_1^\omega(\Lambda^{(n,+)}), \end{aligned}$$

and observe that $\mathcal{SG}^\omega(\Lambda^{(n,+)}) = \mathcal{M} \cup \mathcal{N}$, since the only possible values of the offset s in Definition 4.3 in our case are 0 and 1. In order to be able to use Proposition 3.7 we

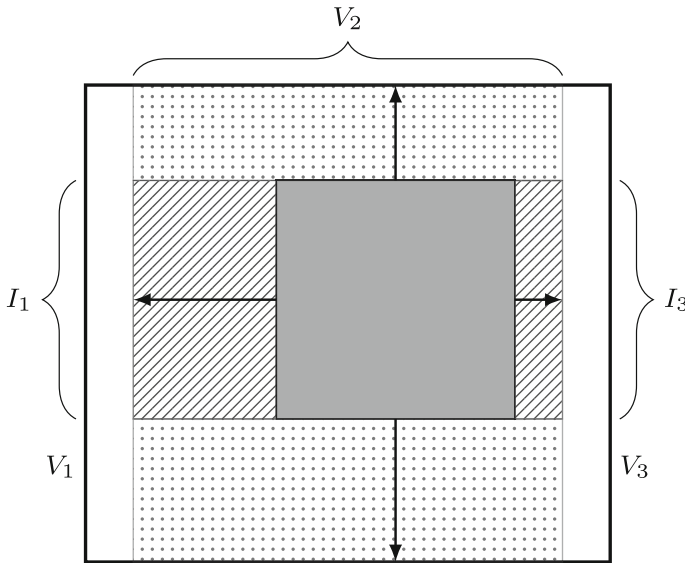


Fig. 4 The partition of $\Lambda^{(n,+)}$ into the rectangle V_2 and the two columns V_1 and V_3 . Here we illustrate the event $\overline{\mathcal{SG}}(V_2)$: the grey region is a super-good rectangle of the type $\Lambda^{(n-2)}$, while the patterned rectangles are **1**-traversable in the arrow directions. If there is at least one infection in I_3 then the rectangle $V_2 \cup V_3$ is super-good. Similarly, an infection in I_1 suffices to make $V_1 \cup V_2$ super good

need some further events. The first one is the event $\overline{\mathcal{SG}}(V_2)$ which is best explained by Fig. 4. It corresponds to requiring that inside the rectangle $V_2 = R(\ell_m - 1, \ell_m) + \mathbf{e}_1$ there exists a **1**-super-good square $R(\ell_{m-1}, \ell_{m-1}) + x$ and the remaining rectangles in $V_2 \setminus R(\ell_{m-1}, \ell_{m-1}) + x$ which sandwich $R(\ell_{m-1}, \ell_{m-1}) + x$ are **1**-traversable. The formal definition is as follows.

Definition 4.11 (*Shrunkten super-good*). Let $R = R(\ell_m - 1, \ell_m) = V_2 - \mathbf{e}_1$. We say that $\overline{\mathcal{SG}}(R)$ occurs if there exist integers $0 \leq s_1 \leq \ell_m - \ell_{m-1} - 1$ and $0 \leq s_2 \leq \ell_m - \ell_{m-1}$ such that the intersection of the following events, in the sequel $\overline{\mathcal{SG}}_{s_1, s_2}(R)$, occurs (see Fig. 4)

$$\begin{aligned} & \mathcal{SG}(\Lambda^{(n-2)} + s_1 \mathbf{e}_1 + s_2 \mathbf{e}_2); \\ & \mathcal{T}_{\leftarrow}(R(s_1, \ell_{m-1}) + s_2 \mathbf{e}_2); \\ & \mathcal{T}_{\rightarrow}(R(\ell_m - \ell_{m-1} - 1 - s_1, \ell_{m-1}) + (s_1 + \ell_{m-1}) \mathbf{e}_1 + s_2 \mathbf{e}_2); \\ & \mathcal{T}_{\downarrow}(R(\ell_m - 1, s_2)); \\ & \mathcal{T}_{\uparrow}(R(\ell_m - 1, \ell_m - \ell_{m-1} - s_2) + (s_2 + \ell_{m-1}) \mathbf{e}_2). \end{aligned}$$

The event $\overline{\mathcal{SG}}(V_2)$ is defined by translation of $\overline{\mathcal{SG}}(R)$. Then for any $\eta_2 \in \overline{\mathcal{SG}}(V_2)$, the segments I_1 and I_3 are given by

$$\begin{aligned} I_1(\eta_2) &= R(1, \ell_{m-1}) + s_2(\eta_2) \mathbf{e}_2 \subset V_1 = R(1, \ell_m), \\ I_3(\eta_2) &= R(1, \ell_{m-1}) + s_2(\eta_2) \mathbf{e}_2 + \ell_m \mathbf{e}_1 \subset V_3 = V_1 + \ell_m \mathbf{e}_1, \end{aligned}$$

where $s_2(\eta_2)$ is an arbitrary one of the choices of s_2 such that $\eta_2 \in \overline{\mathcal{S}\mathcal{G}}_{s_1, s_2}(R)$ for some s_1 .

As before, let $\eta_i := \eta_{V_i}$. Recalling Definition 4.3 and Figs. 2 and 4, it is not hard to check that

$$\hat{\mathcal{C}}_{1,2} := \{\eta \in \Omega_{V_1 \cup V_2} : \eta_2 \in \overline{\mathcal{S}\mathcal{G}}(V_2), \eta_{I_1(\eta_2)} \neq \mathbf{1}\} \subset \mathcal{S}\mathcal{G}(V_1 \cup V_2), \tag{4.17}$$

since I_1 extends the horizontal traversability, while the vertical one and the core of type $\Lambda^{(n-2)}$ are witnessed by $\overline{\mathcal{S}\mathcal{G}}(V_2)$. For $\eta \in \hat{\mathcal{C}}_{1,2}$ we set

$$\mathcal{A}_3^{\eta_1 \cdot \eta_2} = \{\eta_{I_3(\eta_2)} \neq \mathbf{1}\}. \tag{4.18}$$

By (4.17) and its analogue for I_3 we have

$$\{\eta \in \Omega_{\Lambda^{(n,+)}} : \eta_1 \cdot \eta_2 \in \hat{\mathcal{C}}_{1,2} \text{ and } \eta_3 \in \mathcal{A}_3^{\eta_1 \cdot \eta_2}\} \subset \mathcal{M} \cap \mathcal{N}.$$

We can finally apply Proposition 3.7 with parameters $\Omega_i = \Omega_{V_i}$ for $i \in \{1, 2, 3\}$, $\mathcal{C}_{1,2} = \mathcal{S}\mathcal{G}(V_1 \cup V_2)$, $\mathcal{C}_{2,3} = \mathcal{S}\mathcal{G}(V_2 \cup V_3)$, $\mathcal{A}_1 = \mathcal{T}_{\leftarrow}^\omega(V_1)$, $\mathcal{A}_3 = \mathcal{T}_{\rightarrow}^\omega(V_3)$ and $\hat{\mathcal{C}}_{1,2}$ and $\mathcal{A}_3^{\eta_1 \cdot \eta_2}$ as above. Set

$$T_{\text{aux}}^{(2)} = \max_{\eta \in \hat{\mathcal{C}}_{1,2}} \frac{\mu_{\Lambda^{(n,+)}}(\mathcal{A}_3)}{\mu_{\Lambda^{(n,+)}}(\mathcal{A}_3^{\eta_1 \cdot \eta_2})} \times \frac{\mu_{\Lambda^{(n,+)}}(\hat{\mathcal{C}}_{1,2})}{\mu_{\Lambda^{(n,+)}}(\hat{\mathcal{C}}_{1,2})}. \tag{4.19}$$

Then Proposition 3.7 gives that for some $c > 0$ we have

$$\begin{aligned} \text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{S}\mathcal{G}^\omega) &= \text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{M} \cup \mathcal{N}) \\ &\leq c T_{\text{aux}}^{(2)} \times \mu_{\Lambda^{(n,+)}} \left(\mathbb{1}_{\mathcal{M}} \text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{C}_{1,2}, \eta_3) + \mathbb{1}_{\mathcal{N}} \text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{C}_{2,3}, \eta_1) \right. \\ &\quad \left. + \mathbb{1}_{\mathcal{M}} \text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{A}_3, \eta_1, \eta_2) + \mathbb{1}_{\mathcal{N}} \text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{A}_1, \eta_2, \eta_3) \mid \mathcal{M} \cup \mathcal{N} \right). \end{aligned} \tag{4.20}$$

By (4.18), $\min_{\eta \in \hat{\mathcal{C}}_{1,2}} \mu_{\Lambda^{(n,+)}}(\mathcal{A}_3^{\eta_1 \cdot \eta_2}) \geq q$. Furthermore, in Lemma A.4 we will establish that $\mu_{V_1 \cup V_2}(\hat{\mathcal{C}}_{1,2} \mid \mathcal{C}_{1,2}) \geq q^{O(1)}$. Combining these observations with (4.19), we get

$$T_{\text{aux}}^{(2)} \leq q^{-O(1)}. \tag{4.21}$$

We now turn to examine the four averages w.r.t. $\mu_{\Lambda^{(n,+)}}(\cdot \mid \mathcal{M} \cup \mathcal{N})$ appearing in the r.h.s. of (4.20). Recall that $\mathcal{M} \cup \mathcal{N} = \mathcal{S}\mathcal{G}^\omega(\Lambda^{(n,+)})$. Proceeding as for the r.h.s. of (4.14), we obtain that

$$\begin{aligned} &\mu_{\Lambda^{(n,+)}} \left(\mathbb{1}_{\mathcal{M}} \text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{C}_{1,2}, \eta_3) + \mathbb{1}_{\mathcal{N}} \text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{C}_{2,3}, \eta_1) \mid \mathcal{M} \cup \mathcal{N} \right) \\ &\leq O(\gamma^{\mathbf{1}(\Lambda^{(n)})}) \times \sum_{x \in \Lambda^{(n,+)}} \mu_{\Lambda^{(n,+)}}(c_x^{\Lambda^{(n,+)}}, \mathbf{1} \text{Var}_x(f) \mid \mathcal{S}\mathcal{G}^\omega). \end{aligned} \tag{4.22}$$

Indeed, the only difference is that $\mathcal{C}_{1,2} = \mathcal{SG}(\Lambda^{(n)})$, so that we recover a $\mathbf{1}$ boundary condition, and we use that $c_x^{\Lambda^{(n)}, \mathbf{1}} \leq c_x^{\Lambda^{(n,+)}, \mathbf{1}}$ and similarly for $V_2 \cup V_3$ instead of $\Lambda^{(n)}$. We will now explain how to upper bound the third average in (4.20),

$$\mu_{\Lambda^{(n,+)}} \left(\mathbb{1}_{\mathcal{M}} \text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{A}_3, \eta_1, \eta_2) \mid \mathcal{M} \cup \mathcal{N} \right), \tag{4.23}$$

the fourth one being similar. We need to distinguish two cases, according to whether the boundary condition ω has an infection on the column $V_3 + \mathbf{e}_1$ or not. Assume $\omega_{V_3+\mathbf{e}_1} = \mathbf{1}$. In this case $\mathcal{A}_3 = \mathcal{T}_{\rightarrow}(V_3) = \Omega_{V_3} \setminus \{\mathbf{1}\}$ and Proposition 3.2(1), gives that

$$\begin{aligned} \text{Var}_{\Lambda^{(n,+)}}(f \mid \mathcal{A}_3, \eta_1, \eta_2) &= \text{Var}_{V_3}(f \mid \mathcal{T}_{\rightarrow}(V_3)) \\ &\leq q^{-O(1)} \sum_{x \in V_3} \mu_{V_3}(\tilde{c}_x \text{Var}_x(f) \mid \mathcal{T}_{\rightarrow}(V_3)), \end{aligned} \tag{4.24}$$

with $\tilde{c}_x(\eta) = 1$ if x has at least one infected neighbour inside V_3 and $\tilde{c}_x(\eta) = 0$ otherwise. For $x \in V_3$ let

$$A_x = \mu_{\Lambda^{(n,+)}} \left(\mathbb{1}_{\mathcal{M}} \mu_{V_3}(\tilde{c}_x \text{Var}_x(f) \mid \mathcal{T}_{\rightarrow}(V_3)) \mid \mathcal{M} \cup \mathcal{N} \right). \tag{4.25}$$

Recall that $\mathcal{SG}^\omega(\Lambda^{(n,+)}) = \mathcal{M} \cup \mathcal{N} \supset \mathcal{M} = \mathcal{SG}(\Lambda^{(n)}) \cap \mathcal{T}_{\rightarrow}(V_3)$ and $\mu_{\Lambda^{(n,+)}} = \mu_{\Lambda^{(n)}} \otimes \mu_{V_3}$. Then we have

$$\begin{aligned} A_x &= \frac{\mu_{\Lambda^{(n,+)}}(\mathcal{M})}{\mu_{\Lambda^{(n,+)}}(\mathcal{SG}^\omega)} \mu_{\Lambda^{(n,+)}}(\tilde{c}_x \text{Var}_x(f) \mid \mathcal{M}) \\ &= \frac{\mu_{\Lambda^{(n,+)}}(\mathcal{M})}{\mu_{\Lambda^{(n,+)}}(\mathcal{SG}^\omega)} \mu_{\Lambda^{(n,+)}}(\tilde{c}_x \mu_{\Lambda^{(n)}}(\text{Var}_x(f) \mid \mathcal{SG}(\Lambda^{(n)})) \mid \mathcal{M}) \\ &\leq q^{-1} \mu_{V_3}(\mu_{\Lambda^{(n)}}(\text{Var}_x(f) \mid \mathcal{SG}(\Lambda^{(n)})) \mid \tilde{c}_x = 1), \end{aligned}$$

the inequality using $\mathcal{M} \subset \mathcal{SG}^\omega(\Lambda^{(n,+)})$, the fact that $\tilde{c}_x = 1$ implies $\mathcal{T}_{\rightarrow}(V_3)$ and $\mu(\tilde{c}_x \mid \mathcal{T}_{\rightarrow}(V_3)) \geq q$ (here we use that V_3 is not a singleton, which follows from $m \geq 1$). Then, by the law of total variance, we get

$$A_x \leq q^{-1} \mu_{V_3}(\text{Var}_{\{x\} \cup \Lambda^{(n)}}(f \mid \mathcal{SG}(\Lambda^{(n)})) \mid \tilde{c}_x = 1). \tag{4.26}$$

Next, we use Proposition 3.3 with parameters $\mathbb{P} = \mu_{\{x\} \cup \Lambda^{(n)}}(\cdot \mid \mathcal{SG}(\Lambda^{(n)}))$, $X_1 = \eta_{\Lambda^{(n)}}$, $X_2 = \eta_x$, $\mathcal{H} = \{\eta \in \Omega_{\Lambda^{(n)}} : \eta_{x-\mathbf{e}_1} = 1\}$, in order to write

$$\begin{aligned} \text{Var}_{\{x\} \cup \Lambda^{(n)}}(f \mid \mathcal{SG}(\Lambda^{(n)})) &\leq \frac{2}{q} \mu_{\{x\} \cup \Lambda^{(n)}}(\text{Var}_{\Lambda^{(n)}}(f \mid \mathcal{SG}(\Lambda^{(n)})) \\ &\quad + \mathbb{1}_{\{\eta_{x-\mathbf{e}_1}=0\}} \text{Var}_x(f) \mid \mathcal{SG}(\Lambda^{(n)})). \end{aligned} \tag{4.27}$$

Recalling (4.5), we get

$$\begin{aligned} \text{Var}_{\Lambda^{(n)}}(f \mid \mathcal{SG}) &\leq \gamma^1(\Lambda^{(n)}) \sum_{y \in \Lambda^{(n)}} \mu_{\Lambda^{(n)}}(c_y^{\Lambda^{(n)}, \mathbf{1}} \text{Var}_y(f) \mid \mathcal{SG}) \\ &\leq \gamma^1(\Lambda^{(n)}) \sum_{y \in \Lambda^{(n)}} \mu_{\Lambda^{(n)}}(c_y^{\Lambda^{(n),+}, \omega} \text{Var}_y(f) \mid \mathcal{SG}), \end{aligned} \tag{4.28}$$

because $c_y^{\Lambda^{(n)}, \mathbf{1}} \leq c_y^{\Lambda^{(n),+}, \omega}$ for any $y \in \Lambda^{(n)}$ and $\omega \in \Omega_{\mathbb{Z}^2 \setminus \Lambda^{(n),+}}$. Finally, observe that $\mathbb{1}_{\{\eta_{x-\mathbf{e}_1}=0\}} \tilde{c}_x \leq c_x^{\Lambda^{(n),+}, \omega}$, because if $x \in V_3$ has an infected neighbour in V_3 (the constraint \tilde{c}_x) and $x - \mathbf{e}_1 \in V_2$ is also infected, then x has two infected neighbours in $\Lambda^{(n,+)}$. Thus, putting (4.27) and (4.28) together, we obtain

$$\begin{aligned} \tilde{c}_x \text{Var}_{\{x\} \cup \Lambda^{(n)}}(f \mid \mathcal{SG}(\Lambda^{(n)})) &\leq \frac{2}{q} \gamma^1(\Lambda^{(n)}) \times \\ &\sum_{y \in \{x\} \cup \Lambda^{(n)}} \mu_{\{x\} \cup \Lambda^{(n)}}(c_y^{\Lambda^{(n),+}, \omega} \text{Var}_y(f) \mid \mathcal{SG}(\Lambda^{(n)})). \end{aligned} \tag{4.29}$$

Combining (4.24–4.26) and (4.29), yields

$$\begin{aligned} \mu_{\Lambda^{(n,+)}} \left(\mathbb{1}_{\mathcal{M}} \text{Var}(f \mid \mathcal{A}_3, \eta_1, \eta_2) \mid \mathcal{M} \cup \mathcal{N} \right) &\leq q^{-O(1)} \sum_{x \in V_3} A_x \\ &\leq \frac{\gamma^1(\Lambda^{(n)})}{q^{O(1)}} \sum_{x \in V_3} \sum_{y \in \Lambda^{(n,+)}} \mu_{\Lambda^{(n,+)}} \left(c_y^{\Lambda^{(n),+}, \omega} \text{Var}_y(f) \mid \mathcal{SG}(\Lambda^{(n)}), \tilde{c}_x = 1 \right). \end{aligned} \tag{4.30}$$

Moreover, $\mathcal{SG}(\Lambda^{(n)})$ and $\tilde{c}_x = 1$ imply $\mathcal{SG}^\omega(\Lambda^{(n,+)})$ and

$$\frac{\mu_{\Lambda^{(n,+)}}(\mathcal{SG}^\omega)}{\mu_{\Lambda^{(n)}}(\mathcal{SG}) \mu_{V_3}(\tilde{c}_x)} \leq \frac{2}{\mu_{V_3}(\tilde{c}_x)} \leq 2/q,$$

since there are only two possible positions for the core of type $\Lambda^{(n)}$ of $\mathcal{SG}^\omega(\Lambda^{(n,+)})$. Thus, (4.30) is at most

$$\frac{\gamma^1(\Lambda^{(n)}) |V_3|}{q^{O(1)}} \sum_{y \in \Lambda^{(n,+)}} \mu_{\Lambda^{(n,+)}} \left(c_y^{\Lambda^{(n),+}, \omega} \text{Var}_y(f) \mid \mathcal{SG}^\omega(\Lambda^{(n,+)}) \right).$$

Moreover, $|V_3| = \ell_m \leq \ell_N = q^{-O(1)}$ by (4.4), so, recalling (4.5), we are done with the case $\omega_{V_3+\mathbf{e}_1} = \mathbf{1}$.

Assume $\omega_{V_3+\mathbf{e}_1} \neq \mathbf{1}$. In this case $\mathcal{T}_{\rightarrow}^\omega(V_3) = \Omega_{V_3}$, so that $\text{Var}_{V_3}(f \mid \mathcal{T}_{\rightarrow}^\omega(V_3)) = \text{Var}_{V_3}(f)$. The proof is then identical to the previous one except for inequality (4.24) which now follows from Proposition 3.2(2) with the unconstrained site $z \in V_3$ chosen arbitrarily so that $\omega_{z+\mathbf{e}_1} = 0$. □

5 Proof of Theorem 1.3: upper bound

As already announced we will only discuss the two dimensional case. The starting point is as in [25, Section 5]. Let κ be a large enough constant, let

$$t_* = \exp\left(\frac{\pi^2}{9q}(1 + \kappa\sqrt{q}\log^3(1/q))\right) \tag{5.1}$$

and let $T = \lfloor \exp(\log^4(1/q)/q) \rfloor$. Then

$$\begin{aligned} \mathbb{E}_\mu(\tau_0) &= \int_0^{+\infty} ds \mathbb{P}_\mu(\tau_0 > s) \\ &= \int_0^{t_*} ds \mathbb{P}_\mu(\tau_0 > s) + \int_{t_*}^T ds \mathbb{P}_\mu(\tau_0 > s) + \int_T^{+\infty} ds \mathbb{P}_\mu(\tau_0 > s) \\ &\leq t_* + T\mathbb{P}_\mu(\tau_0 > t_*) + \int_T^{+\infty} ds \mathbb{P}_\mu(\tau_0 > s). \end{aligned}$$

The term t_* has exactly the form required in (1.5). In order to bound the last term in the r.h.s. above, we use that $\mathbb{P}_\mu(\tau_0 > s) \leq e^{-sq/T_{\text{rel}}}$ for all $s > 0$ (see e.g. [13, proof of Theorem 4.7]) together with $T_{\text{rel}} \leq e^{O((\log(q))^3/q)}$ (see [32, Theorem 2 (b)]) to get that

$$\lim_{q \rightarrow 0} \int_T^{+\infty} ds \mathbb{P}_\mu(\tau_0 > s) \leq \lim_{q \rightarrow 0} \frac{T_{\text{rel}}}{q} e^{-qT/T_{\text{rel}}} = 0.$$

In conclusion, the proof of (1.5) for $\mathbb{E}_\mu(\tau_0)$ boils down to proving

$$\lim_{q \rightarrow 0} T\mathbb{P}_\mu(\tau_0 > t_*) = 0. \tag{5.2}$$

Similarly, (1.5) for τ w.h.p. follows, since (5.2) gives

$$\mathbb{P}_\mu(\tau_0 > t_*) \leq o(1/T) \leq o(1).$$

The key ingredients to prove (5.2) are Propositions 4.6 and 4.7 and Proposition 5.2 below. The latter is a Poincaré inequality for an auxiliary process, the *generalised coalescing and branching symmetric exclusion process* (g -CBSEP), preliminarily studied in [26]. Once we have these key ingredients, the strategy to prove (5.2) is similar to the one in [25, Section 5]. In particular, for the first part of the proof (Sect. 5.2) we will omit most of the details and refer to [25, Section 5] for a more detailed explanation.

5.1 The g -CBSEP process

Given a finite connected graph $G = (V, E)$ and a finite probability space (\mathcal{S}, π) , assign a variable $\sigma_x \in \mathcal{S}$ to each vertex $x \in V$ and write $\sigma = (\sigma_x)_{x \in V}$ and $\pi_G(\sigma) =$

$\prod_x \pi(\sigma_x)$. Fix also a bipartition $\mathcal{S}_1 \sqcup \mathcal{S}_0 = \mathcal{S}$ such that $\pi(\mathcal{S}_1) > 0$ and define the projection $\varphi : \mathcal{S}^V \rightarrow \{0, 1\}^V$ by $\varphi(\sigma) = (\mathbb{1}_{\{\sigma_x \in \mathcal{S}_1\}})_{x \in V}$. We will say that a vertex x is occupied by a *particle* if $\sigma_x \in \mathcal{S}_1$ and we will write $\Omega_G^+ \subset \Omega_G = \mathcal{S}^V$ for the set of configurations σ with at least one particle. Finally, for any edge $e = \{x, y\} \in E$ let \mathcal{E}_e be the event that there exists a particle at x or at y .

The g -CBSEP continuous time Markov chain on Ω_G^+ with parameters $(\mathcal{S}, \mathcal{S}_1, \pi)$ runs as follows. The state $\{\sigma_x, \sigma_y\}$ of every edge $e = \{x, y\}$ for which \mathcal{E}_e holds is resampled with rate one (independently of all the other edges) w.r.t. $\pi_x \otimes \pi_y(\cdot | \mathcal{E}_e)$. Thus, an edge containing exactly one particle can swap the position of the particle between its endpoints or can create a new particle at the empty endpoint (a branching transition). An edge with two particles can kill one of them (a coalescing transition) with equal probability or keep them untouched. Notice also that the state of an edge can change completely even when the particles are untouched.

Remark 5.1 When the parameters $(\mathcal{S}, \mathcal{S}_1, \pi)$ are the two point space $\{0, 1\}$, the set $\{1\}$ and the Bernoulli(p) measure on \mathcal{S} respectively, the g -CBSEP chain is called the CBSEP chain on G with parameter p . It is easy to verify that the projection of the g -CBSEP chain under the mapping φ defined above coincides with the CBSEP chain with parameter $p = \pi(\mathcal{S}_1)$. This observation will be used in the proof of Proposition 5.2 below. We will use g -CBSEP rather than plain CBSEP, because the space \mathcal{S} will correspond to the state of the chain in a mesoscopic box. The event \mathcal{S}_1 will correspond to the presence of a mobile droplet in this box.

It is immediate to check that g -CBSEP is ergodic on Ω_G^+ with reversible stationary measure $\pi_G^+ := \pi_G(\cdot | \Omega_G^+)$ and that its Dirichlet form $\mathcal{D}^{g\text{-CBSEP}}(f)$ for $f : \Omega_G^+ \rightarrow \mathbb{R}$, takes the form

$$\mathcal{D}^{g\text{-CBSEP}}(f) = \sum_{e \in E} \pi_G^+(\mathbb{1}_{\mathcal{E}_e} \text{Var}_e(f | \mathcal{E}_e)),$$

where $\text{Var}_e(f | \mathcal{E}_e)$ is the variance w.r.t. σ_x, σ_y conditioned on \mathcal{E}_e if $e = \{x, y\}$. Let now $T_{\text{rel}}^{g\text{-CBSEP}}$ be the *relaxation time* of g -CBSEP on Ω_G^+ defined as the best constant C in the Poincaré inequality

$$\text{Var}_{\pi_G^+}(f) \leq C \mathcal{D}^{g\text{-CBSEP}}(f).$$

In the above setting the main result needed to prove (5.2) is as follows. For any positive integers d and L set $n = L^d$ and let $\mathbb{Z}_L = \{0, 1, \dots, L - 1\}$ be the set of remainders modulo L . The d -dimensional discrete torus with n vertices, \mathbb{T}_n^d in the sequel, is the set \mathbb{Z}_L^d endowed with the graph structure inherited from \mathbb{Z}^d . In what follows we will allow $\mathcal{S}, \mathcal{S}_1$ and π to depend on n .

Proposition 5.2 *Let $d \geq 2$, $G = \mathbb{T}_n^d$ and assume that $\lim_{n \rightarrow \infty} n\pi(\mathcal{S}_1) = +\infty$. Then, as $n \rightarrow \infty$, for any function $f : \Omega_G^+ \mapsto \mathbb{R}$*

$$\text{Var}_{\pi_G^+}(f) \leq O(\pi(\mathcal{S}_1)^{-1} \max(1, \log(\pi(\mathcal{S}_1)^{-1}))) \cdot \mathcal{D}^{g\text{-CBSEP}}(f).$$

In particular,

$$T_{\text{rel}}^{\text{g-CBSEP}} \leq O(\pi(\mathcal{S}_1)^{-1} \max(1, \log(\pi(\mathcal{S}_1)^{-1}))).$$

This is proved in ‘‘Appendix B’’.

5.2 Transforming (5.2) into a Poincaré inequality

Using standard ‘‘finite speed of propagation’’ bounds (see [25, Section 5.2.1] (5.2) follows if we prove

$$\lim_{q \rightarrow 0} T \mathbb{P}_\mu(\tau_0^{(n)} > t_*) = 0 \tag{5.3}$$

where $\tau_0^{(n)}$ is the infection time of the origin τ_0 for FA-2f on the discrete torus \mathbb{T}_n^2 with linear size $\sqrt{n} = 2T$. For this purpose we fix a small positive constant $\delta < 1/2$ and choose $N_\delta = N - \lfloor \log(1/\delta)/\sqrt{q} \rfloor$ where $N = \lceil \frac{8 \log(1/q)}{\sqrt{q}} \rceil$ is the final scale in the droplet construction (see (4.3)). With this choice $\ell_{N_\delta} \simeq \delta \ell_N = \delta/q^{17/2+o(1)}$ (cf. (4.1) and w.l.o.g. we assume that ℓ_{N_δ} divides $2T$. We then partition the torus \mathbb{T}_n^2 into $M = n/\ell_{N_\delta}^2$ equal mesoscopic disjoint boxes $(Q_j)_{j=1}^M$, where each Q_j is a suitable lattice translation by a vector in \mathbb{T}_n^2 of the box $Q = [\ell_{N_\delta}]^2 = \Lambda^{(2N_\delta)}$ (see (4.2)). The labels of the boxes can be thought of as belonging to the new torus \mathbb{T}_M^2 and we assume that Q_i, Q_j are neighbouring boxes in \mathbb{T}_n^2 iff i, j are neighbouring sites in \mathbb{T}_M^2 . In $\Omega_{\mathbb{T}_n^2}$ we consider the event

$$\mathcal{E} = \bigcup_{j \in \mathbb{T}_M^2} S\mathcal{G}_j \cap \bigcap_{i \in \mathbb{T}_M^2} \mathcal{G}_i \tag{5.4}$$

where $S\mathcal{G}_j$ is the event that Q_j is super-good (see Definition 4.5) and \mathcal{G}_i is the event that any row and any column (of lattice sites) of Q_i contains an infected site.

In order to apply the same strategy as [25, Section 5] it is crucial to have that the ‘‘environment’’ characterised by \mathcal{E} is so likely that (cf. [25, (28)])

$$\lim_{q \rightarrow 0} \mu(\mathcal{E}^c) T^3 t_* = 0. \tag{5.5}$$

Using $t_* = e^{\frac{\pi^2}{9q}(1+o(1))}$, $T = \lfloor \exp(\log^4(1/q)/q) \rfloor$, $M = 4T^2/\ell_{N_\delta}^2$, $\ell_{N_\delta} = O(1/q^9)$, together with Proposition 4.6, it follows that

$$\lim_{q \rightarrow 0} T^3 t_* \mu\left(\bigcap_{j \in \mathbb{T}_M^2} S\mathcal{G}_j^c\right) \leq \lim_{q \rightarrow 0} T^3 t_* \left(1 - e^{-\frac{\pi^2}{9q}(1+O(\sqrt{q} \log^2(1/q)))}\right)^M = 0.$$

Similarly, using $\mu(\mathcal{G}_i^c) \leq 2\ell_{N_\delta}(1 - q)^{\ell_{N_\delta}} \leq 2\ell_{N_\delta}e^{-q\ell_{N_\delta}}$ together with $\ell_{N_\delta} = \Omega(1/q^8)$, we get

$$\lim_{q \rightarrow 0} T^3 t_* \mu\left(\bigcup_{i \in \mathbb{T}_M^2} \mathcal{G}_i^c\right) \leq T^3 t_* M 2\ell_{N_\delta} e^{-q\ell_{N_\delta}} = 0,$$

and (5.5) follows.

An easy consequence of (5.5) (cf. [25, Eq. (29)]) is that as $q \rightarrow 0$

$$T\mathbb{P}_\mu(\tau_0^{(n)} \geq t_*) \leq T\mathbb{P}_\mu(\tau_{\mathcal{F}}^{(n)} \geq t_*) + o(1), \tag{5.6}$$

where $\tau_{\mathcal{F}}^{(n)}$ denotes the hitting time of the set $\mathcal{F} = \{\omega : \omega_0 = 0\} \cup \mathcal{E}^c$ for the FA-2f chain on \mathbb{T}_n^2 . In order to bound from above the term $\mathbb{P}_\mu(\tau_{\mathcal{F}}^{(n)} \geq t_*)$ we follow the standard ‘‘variational’’ approach (see [25, Eq. (30)] and [3, Theorem 2]).

Let $\mathcal{D}_{\mathbb{T}_n^2}(f) = \sum_{x \in \mathbb{T}_n^2} \mu_{\mathbb{T}_n^2}(c_x^{\mathbb{T}_n^2} \text{Var}_x(f))$, where $c_x^{\mathbb{T}_n^2}$ is the FA-2f constraint at x for the torus \mathbb{T}_n^2 (see (3.3)), be the Dirichlet form of the FA-2f chain on the torus \mathbb{T}_n^2 . Then

$$T\mathbb{P}_\mu(\tau_{\mathcal{F}}^{(n)} \geq t_*) \leq T e^{-t_* \lambda_{\mathcal{F}}^{(n)}}, \quad \lambda_{\mathcal{F}}^{(n)} = \inf \left\{ \frac{\mathcal{D}_{\mathbb{T}_n^2}(f)}{\mu_{\mathbb{T}_n^2}(f^2)} : f|_{\mathcal{F}} = 0 \right\}. \tag{5.7}$$

It remains to prove a precise lower bound on the coefficient $\lambda_{\mathcal{F}}^{(n)}$.

5.3 Bounding $\lambda_{\mathcal{F}}^{(n)}$ from below

The last and most important step is to prove that

$$\lambda_{\mathcal{F}}^{(n)} \geq e^{-O(\log^3(1/q)/\sqrt{q})} \rho_D \geq e^{-\frac{\pi^2}{9q}(1+O(\sqrt{q} \log^3(1/q)))}, \tag{5.8}$$

where $\rho_D \geq \exp(-\frac{\pi^2}{9q}(1 + O(\sqrt{q} \log^2(1/q))))$ is the probability that a box $[\ell_N]^2$ is super-good (cf. Proposition 4.6). Once (5.8) is established, recalling (5.6) and (5.7), the proof of (5.3) is complete because $t_* \lambda_{\mathcal{F}}^{(n)}$ diverges rapidly enough as $q \rightarrow 0$ if the constant κ in the definition (5.1) of t_* is chosen large enough.

It was proved in [25, Eq. (31)] that $\lambda_{\mathcal{F}}^{(n)} \geq q \inf_f \frac{\mathcal{D}_{\mathbb{T}_n^2}(f)}{\text{Var}_{\mathbb{T}_n^2}(f|_{\mathcal{E}})}$, where the infimum is over $f : \Omega_{\mathbb{T}_n^2} \rightarrow \mathbb{R}$ such that $f|_{\mathcal{E}^c} = 0$ and $f|_{\mathcal{E}}$ is not constant. In what follows f will denote an arbitrary such function and the various constants involved in the estimates will be uniform in f . Hence, (5.8) follows, once we prove that

$$\frac{\mathcal{D}_{\mathbb{T}_n^2}(f)}{\text{Var}_{\mathbb{T}_n^2}(f|_{\mathcal{E}})} \geq \exp\left(-\frac{\pi^2}{9q}(1 + O(\log^3(1/q)\sqrt{q}))\right). \tag{5.9}$$

Bounding $\text{Var}_{\mathbb{T}_n^2}(f | \mathcal{E})$, using Proposition 5.2. Write G for the graph \mathbb{T}_M^2 , \mathcal{S} for the state space $\mathcal{G}_i \subset \Omega_{Q_i}$ with $i \in \mathbb{T}_M^2$, π for $\mu_{Q_i}(\cdot | \mathcal{G}_i)$ and $\mathcal{S}_1 \subset \mathcal{S}$ for the event $\mathcal{S}\mathcal{G}_i$. Since both \mathcal{G}_i and $\mathcal{S}\mathcal{G}_i$ are increasing in the set of infections,

$$\pi(\mathcal{S}_1) \geq \mu(\mathcal{S}\mathcal{G}_i) \geq \exp\left(-\frac{\pi^2}{9q}(1 + O(\sqrt{q} \log^2(1/q)))\right), \tag{5.10}$$

where we used the Harris inequality [20] for the first inequality and Proposition 4.6 for the second one. Recalling that $M = n/\ell_{N_\delta}^2$ with $n = 4T^2$, $\ell_{N_\delta} = O(1/q^9)$ and $T = \lfloor \exp(\log^4(1/q)/q) \rfloor$, the above bound implies $\lim_{q \rightarrow 0} M\pi(\mathcal{S}_1) = +\infty$ so that the requirement of Proposition 5.2 is fulfilled.

With this notation we consider the g -CBSEP on Ω_G^+ with parameters $(\mathcal{S}, \mathcal{S}_1, \pi)$. Recalling (5.4), we identify f with a function $f_G : \Omega_G^+ \rightarrow \mathbb{R}$ via the natural bijection between \mathcal{E} and Ω_G^+ : $f(\omega) = f_G(\omega_{Q_1}, \dots, \omega_{Q_M})$. Under this bijection

$$\begin{aligned} \text{Var}_{\pi_G^+}(f_G) &= \text{Var}_{\mathbb{T}_n^2}(f | \mathcal{E}), \\ \mathcal{D}^{g\text{-CBSEP}}(f_G) &= \sum_{i \sim j} \mu_{\mathbb{T}_n^2}(\mathbb{1}_{\mathcal{S}\mathcal{G}_{i,j}} \text{Var}_{Q_i \cup Q_j}(f | \mathcal{S}\mathcal{G}_{i,j}) | \mathcal{E}), \end{aligned}$$

where $\mathcal{S}\mathcal{G}_{i,j}$ is a shorthand notation for the event $(\mathcal{S}\mathcal{G}_i \cup \mathcal{S}\mathcal{G}_j) \cap \mathcal{G}_i \cap \mathcal{G}_j$ and $\sum_{i \sim j}$ denotes the sum over pairs, each counted once, of adjacent boxes. Using Proposition 5.2 and (5.10) we conclude that

$$\begin{aligned} \text{Var}_{\mathbb{T}_n^2}(f | \mathcal{E}) &= \text{Var}_{\pi_G^+}(f_G) \leq O(\pi(\mathcal{S}_1)^{-1} \log(1/\pi(\mathcal{S}_1))) \mathcal{D}^{g\text{-CBSEP}}(f_G) \\ &\leq \exp\left(\frac{\pi^2}{9q}(1 + O(\sqrt{q} \log^2(1/q)))\right) \\ &\quad \times \sum_{i \sim j} \mu_{\mathbb{T}_n^2}(\mathbb{1}_{\mathcal{S}\mathcal{G}_{i,j}} \text{Var}_{Q_i \cup Q_j}(f | \mathcal{S}\mathcal{G}_{i,j}) | \mathcal{E}). \end{aligned} \tag{5.11}$$

Bounding $\mathcal{D}_{\mathbb{T}_n^2}(f)$, using Proposition 4.7. We next compare the sum appearing in the r.h.s. of (5.11) to the Dirichlet form $\mathcal{D}_{\mathbb{T}_n^2}(f)$ and prove that the ‘‘comparison cost’’ is at most $\exp(O(\log^3(1/q)/\sqrt{q}))$, so sub-leading w.r.t. the main term $\exp(\frac{\pi^2}{9q})$ in (5.11).

Lemma 5.3

$$\sum_{i \sim j} \mu_{\mathbb{T}_n^2}(\mathbb{1}_{\mathcal{S}\mathcal{G}_{i,j}} \text{Var}_{Q_i \cup Q_j}(f | \mathcal{S}\mathcal{G}_{i,j}) | \mathcal{E}) \leq e^{O(\log^3(1/q)/\sqrt{q})} \mathcal{D}_{\mathbb{T}_n^2}(f).$$

Remark 5.4 As it will be clear from the proof, we actually prove a stronger statement, namely the constraint $c_x^{\mathbb{T}_n^2}$ in the expression of $\mathcal{D}_{\mathbb{T}_n^2}(f)$ will appear multiplied by the indicator that x belongs to a droplet. While for many choices of f the presence of

this additional constraint may completely change the average $\mu_{\mathbb{T}_n^2}(c_x^{\mathbb{T}_n^2} \text{Var}_x(f))$, it is possible to exhibit choices of f , for which

$$\mathbb{1}_{\{x \text{ belongs to a "droplet"}\}} c_x^{\mathbb{T}_n^2} \text{Var}_x(f) \simeq c_x^{\mathbb{T}_n^2} \text{Var}_x(f).$$

Before proving Lemma 5.3, let us observe that, together with (5.11), it implies the desired (5.9). Lemma 5.3 itself follows by summing the bound from Claim 5.5 below.

Claim 5.5 Fix two adjacent boxes Q_i, Q_j and let $\Lambda_{i,j} \supset Q_i \cup Q_j$ be a translate of the box $\Lambda^{(2N)}$. Then

$$\begin{aligned} &\mu_{\mathbb{T}_n^2}(\mathbb{1}_{\mathcal{SG}_{i,j}} \text{Var}_{Q_i \cup Q_j}(f | \mathcal{SG}_{i,j}) | \mathcal{E}) \\ &\leq e^{O(\log^3(1/q)/\sqrt{q})} \sum_{x \in \Lambda_{i,j}} \mu_{\mathbb{T}_n^2}(\mathbb{1}_{\mathcal{SG}(\Lambda_{i,j})} c_x^{\mathbb{T}_n^2} \text{Var}_x(f)). \end{aligned}$$

Proof of Claim 5.5 Let $\mathcal{G} = \bigcap_{k \in \mathbb{T}_M^2} \mathcal{G}_k \supset \mathcal{E}$ and recall that $\mu(\mathcal{E}) = 1 - o(1)$. Let $\rho_{i,j} = \mu(\mathcal{SG}_{i,j} | \mathcal{G})$ and observe that the term $\text{Var}_{Q_i \cup Q_j}(\cdot)$ does not depend on the variables $\omega_{Q_i}, \omega_{Q_j}$. Thus,

$$\begin{aligned} &\mu_{\mathbb{T}_n^2}(\mathbb{1}_{\mathcal{SG}_{i,j}} \text{Var}_{Q_i \cup Q_j}(f | \mathcal{SG}_{i,j}) | \mathcal{E}) \\ &\leq (1 + o(1)) \mu_{\mathbb{T}_n^2}(\mathbb{1}_{\mathcal{SG}_{i,j}} \text{Var}_{Q_i \cup Q_j}(f | \mathcal{SG}_{i,j}) | \mathcal{G}) \\ &= (1 + o(1)) \rho_{i,j} \mu_{\mathbb{T}_n^2}(\text{Var}_{Q_i \cup Q_j}(f | \mathcal{SG}_{i,j}) | \mathcal{G}). \end{aligned} \tag{5.12}$$

Let $\mathcal{G}(\Lambda_{i,j})$ be the event that any ℓ_{N_δ} lattice sites contained in $\Lambda_{i,j}$ forming either a row or a column of some Q_k contain an infection. With reference to Fig. 5, we emphasise that the event $\mathcal{G}(\Lambda_{i,j})$ does not require anything about columns/rows which go out of $\Lambda_{i,j}$. We define the event $\mathcal{G}(\Lambda_{i,j}^c)$ similarly with $\Lambda_{i,j}$ replaced by $\mathbb{T}_n^2 \setminus \Lambda_{i,j}$. Clearly $\mathcal{G} \subset \mathcal{G}(\Lambda_{i,j}) \cap \mathcal{G}(\Lambda_{i,j}^c)$ and

$$\begin{aligned} &\mu_{\mathbb{T}_n^2}(\text{Var}_{Q_i \cup Q_j}(f | \mathcal{SG}_{i,j}) | \mathcal{G}) \\ &\leq (1 + o(1)) \mu_{\mathbb{T}_n^2}(\text{Var}_{Q_i \cup Q_j}(f | \mathcal{SG}_{i,j}) | \mathcal{G}(\Lambda_{i,j}) \cap \mathcal{G}(\Lambda_{i,j}^c)) \\ &= (1 + o(1)) \mu_{\mathbb{T}_n^2}(\mu_{\Lambda_{i,j}}(\text{Var}_{Q_i \cup Q_j}(f | \mathcal{SG}_{i,j}) | \mathcal{G}(\Lambda_{i,j})) | \mathcal{G}(\Lambda_{i,j}^c)) \end{aligned} \tag{5.13}$$

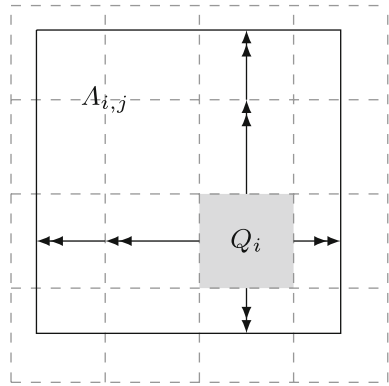
In turn, the law of total variance implies that

$$\mu_{\Lambda_{i,j}}(\text{Var}_{Q_i \cup Q_j}(f | \mathcal{SG}_{i,j}) | \mathcal{G}(\Lambda_{i,j})) \leq \text{Var}_{\Lambda_{i,j}}(f | \mathcal{SG}_{i,j} \cap \mathcal{G}(\Lambda_{i,j})). \tag{5.14}$$

Next comes a simple but key observation illustrated in Fig. 5, whose formal proof is left to the reader.

Observation 5.6 The event $\mathcal{SG}_{i,j} \cap \mathcal{G}(\Lambda_{i,j})$ implies the event $\mathcal{SG}(\Lambda_{i,j})$.

Fig. 5 Illustration of Observation 5.6. The shaded square of shape $\Lambda^{(2N_s)}$ is \mathcal{SG} and the arrows indicate the presence of an infection in each row/column, as guaranteed by $\mathcal{G}(\Lambda_{i,j})$ with $\Lambda_{i,j}$ being the larger square of shape $\Lambda^{(2N)}$. Observation 5.6 asserts that these events combined imply $\mathcal{SG}(\Lambda_{i,j})$ (see Fig. 2). The dashed lines delimit the boxes Q_k



Taking Observation 5.6 into account together with the inequality $\text{Var}(X | \mathcal{A}) \leq \frac{\mathbb{P}(\mathcal{B})}{\mathbb{P}(\mathcal{A})} \text{Var}(X | \mathcal{B})$, valid for any random variable X and events $\mathcal{A} \subset \mathcal{B}$ with $\mathbb{P}(\mathcal{A}) > 0$, we conclude that

$$\text{Var}_{\Lambda_{i,j}}(f | \mathcal{SG}_{i,j} \cap \mathcal{G}(\Lambda_{i,j})) \leq \frac{\mu_{\Lambda_{i,j}}(\mathcal{SG}(\Lambda_{i,j}))}{\mu_{\Lambda_{i,j}}(\mathcal{SG}_{i,j} \cap \mathcal{G}(\Lambda_{i,j}))} \text{Var}_{\Lambda_{i,j}}(f | \mathcal{SG}(\Lambda_{i,j})). \tag{5.15}$$

From (5.12–5.15) we finally get

$$\begin{aligned} & \mu_{\mathbb{T}_n^2}(\mathbb{1}_{\mathcal{SG}_{i,j}} \text{Var}_{Q_i \cup Q_j}(f | \mathcal{SG}_{i,j}) | \mathcal{G}) \\ & \leq \frac{(1 + o(1))\rho_{i,j}\mu_{\Lambda_{i,j}}(\mathcal{SG}(\Lambda_{i,j}))}{\mu_{\Lambda_{i,j}}(\mathcal{SG}_{i,j} \cap \mathcal{G}(\Lambda_{i,j}))} \mu_{\mathbb{T}_n^2}(\text{Var}_{\Lambda_{i,j}}(f | \mathcal{SG}(\Lambda_{i,j})) | \mathcal{G}(\Lambda_{i,j}^c)) \\ & \leq (1 + o(1))\mu_{\Lambda_{i,j}}(\mathcal{SG}(\Lambda_{i,j}))\mu_{\mathbb{T}_n^2}(\text{Var}_{\Lambda_{i,j}}(f | \mathcal{SG}(\Lambda_{i,j})) | \mathcal{G}(\Lambda_{i,j}^c)), \end{aligned} \tag{5.16}$$

where we used

$$\begin{aligned} \rho_{i,j} &= \mu(\mathcal{SG}_{i,j} | \mathcal{G}) \leq (1 + o(1))\mu_{\Lambda_{i,j}}(\mathcal{SG}_{i,j} | \mathcal{G}(\Lambda_{i,j})) \\ & \leq (1 + o(1))\mu_{\Lambda_{i,j}}(\mathcal{SG}_{i,j} \cap \mathcal{G}(\Lambda_{i,j})) \end{aligned}$$

to get the last inequality.

By applying Proposition 4.7 to the term $\text{Var}_{\Lambda_{i,j}}(f | \mathcal{SG}(\Lambda_{i,j}))$ and using that $c_x^{\Lambda_{i,j}} \leq c_x^{\mathbb{T}_n^2}$, we conclude that

$$\begin{aligned} & \mu_{\mathbb{T}_n^2}(\mathbb{1}_{\mathcal{SG}_{i,j}} \text{Var}_{Q_i \cup Q_j}(f \mid \mathcal{SG}_{i,j}) \mid \mathcal{E}) \\ & \leq e^{O(\log^3(1/q)/\sqrt{q})} \mu_{\Lambda_{i,j}}(\mathcal{SG}(\Lambda_{i,j})) \\ & \times \sum_{x \in \Lambda_{i,j}} \mu_{\mathbb{T}_n^2}(\mu_{\Lambda_{i,j}}(c_x^{A_{i,j}} \text{Var}_x(f) \mid \mathcal{SG}(\Lambda_{i,j})) \mid \mathcal{G}(\Lambda_{i,j}^c)) \\ & \leq e^{O(\log^3(1/q)/\sqrt{q})} \sum_{x \in \Lambda_{i,j}} \mu_{\mathbb{T}_n^2}(\mathbb{1}_{\mathcal{SG}(\Lambda_{i,j})} c_x^{\mathbb{T}_n^2} \text{Var}_x(f)), \end{aligned}$$

where we used $\mu_{\mathbb{T}_n^2}(\mathcal{G}(\Lambda_{i,j}^c)) = 1 - o(1)$ in the last inequality. □

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A Probability of super-good events

In this appendix we prove Proposition 4.6 and we gather several more technical and relatively standard bootstrap percolation estimates on the probability of super-good events used in Sect. 4.

For $z > 0$ we define

$$g(z) = -\log(\beta(1 - e^{-z})),$$

where $\beta(u) = (u + \sqrt{u(4 - 3u)})/2$. It is known [28, Proposition 5(ii)] that $\int_0^\infty g(z) dz = \pi^2/18$. We next recall some straightforward properties of g .

Fact A.1 *The function g is positive, decreasing, differentiable and convex on $(0, \infty)$. Moreover, the following asymptotic behaviour holds:*

$$\begin{aligned} g(z) &\sim \frac{1}{2} \log(1/z), & g'(z) &\sim \frac{-1}{2z}, & \text{as } z \rightarrow 0, \\ g(z) &\sim e^{-2z}, & g'(z) &\sim -2e^{-2z}, & \text{as } z \rightarrow \infty, \end{aligned}$$

where $x \sim y$ stands for $x = (1 + o(1))y$.

The relevance of this function comes from its link to the probability of traversability. Recalling Definition 4.1, for any positive integers a and b we set

$$T^1(a, b) = \mu(\mathcal{T}_\rightarrow^1(R(a, b))), \quad T^0(a, b) = \mu(\mathcal{T}_\rightarrow^0(R(a, b))),$$

where $\mathbf{0}$ stands for the fully infected configuration. Note that these probabilities are the same for left-traversability, while for up or down-traversability a and b are inverted in the r.h.s. The next lemma follows easily from [28, Lemma 8]. Let $q' = -\log(1 - q) = q + O(q^2)$.

Lemma A.2 *For any positive integers a and b and $\omega \in \{\mathbf{0}, \mathbf{1}\}$ we have*

$$T^\omega(a, b) = q^{O(1)} e^{-ag(bq')}.$$

Corollary A.3 *For any positive integers a and b we have*

$$\max_{0 \leq s, s' \leq a} \frac{T^\mathbf{0}(s, b)T^\mathbf{0}(a - s, b)}{T^\mathbf{1}(s', b)T^\mathbf{1}(a - s', b)} \leq q^{-O(1)}. \tag{A.1}$$

Furthermore, for any boundary conditions ω, ω' and rectangle R of class $1 \leq n \leq 2N$ (recall Definitions 4.2 and 4.3), we have

$$\mu_R(\mathcal{SG}_s^\omega(R) \mid \mathcal{SG}_{s'}^{\omega'}(R)) \geq q^{O(1)} \tag{A.2}$$

uniformly over all possible values of s and boundary conditions ω, ω' .

Proof Equation (A.1) follows immediately from Lemma A.2. To obtain (A.2) with n odd (the even case is treated identically), recall that

$$\mathcal{SG}^{\omega'}(R) = \bigcup_{s'} \mathcal{SG}_{s'}^{\omega'}(R);$$

there are $q^{-O(1)}$ possible values of s' ; by (A.1), for all s, s', ω and ω' ,

$$\mu_R(\mathcal{SG}_s^\omega) / \mu_R(\mathcal{SG}_{s'}^{\omega'}) \geq q^{O(1)}. \tag{A.2}$$

□

We are now ready for the main result of this appendix.

Proof of Proposition 4.6 We will prove the same bound for the super-good event occurring with all $s = 0$ in Definition 4.3 on all scales, i.e. the initial infection $\Lambda^{(0)}$ being in the bottom-left corner of $\Lambda^{(n)}$. Once the offsets are fixed, it suffices to prove the bound on this probability for $n = 2N$, in which case it reads

$$\begin{aligned} & q \prod_{m=1}^N T^\mathbf{1}(\ell_m - \ell_{m-1}, \ell_m) T^\mathbf{1}(\ell_m - \ell_{m-1}, \ell_{m-1}) \\ &= q^{O(N)} \exp\left(-\sum_{m=1}^N (\ell_m - \ell_{m-1})(g(q'\ell_m) + g(q'\ell_{m-1}))\right), \end{aligned} \tag{A.3}$$

by Lemma A.2 and symmetry. Since g is decreasing, the last sum is at most

$$2 \sum_{m=1}^{\infty} (\ell_m - \ell_{m-1})g(q'\ell_{m-1}).$$

The term for $m = 1$ is $O(\log(1/q)/\sqrt{q})$ by Fact A.1. For the other terms we use that by convexity for any $0 < a < b$

$$(b - a)g(a) \leq \int_a^b g(z) dz - O((b - a)^2g'(a)).$$

Using Fact A.1, we get

$$-(b - a)^2g'(a) \leq O((b - a))^2 \times \begin{cases} 1/a & \text{if } a = O(1) \\ e^{-a} & \text{if } a = \Omega(1). \end{cases}$$

Finally, for $m \geq 2$ we have $\ell_m - \ell_{m-1} \leq 2\sqrt{q}\ell_{m-1}$ by (4.1), so

$$q' \sum_{m=2}^{m_0} \frac{(\ell_m - \ell_{m-1})^2}{\ell_{m-1}} \leq O(q'\sqrt{q}) \sum_{m=2}^{m_0} (\ell_m - \ell_{m-1}) = O(q^{3/2}\ell_{m_0}) = O(\sqrt{q})$$

$$(q')^2 \sum_{m=m_0+1}^{\infty} (\ell_m - \ell_{m-1})^2 e^{-q'\ell_{m-1}} \leq O(q^3) \sum_{m=m_0+1}^{\infty} \ell_{m-1}^2 e^{-q'\ell_{m-1}} = O(\sqrt{q}),$$

setting $m_0 = \max\{m, \ell_m \leq 1/q\}$. Putting these bounds together and recalling (4.3), we obtain that the r.h.s. of (A.3) is at least

$$\exp\left(\frac{-2}{q'}\left(\int_0^{\infty} g(z) dz + O(\sqrt{q}) \log(1/q)\right)\right) - \frac{O(\log^2(1/q))}{\sqrt{q}}$$

$$= \exp\left(-\frac{\pi^2}{9q} - \frac{O(\log^2(1/q))}{\sqrt{q}}\right).$$

This concludes the proof of Proposition 4.6. □

We next turn to the event $\overline{\mathcal{SG}}(V_2)$ from Definition 4.11 required in the proof of Lemma 4.10, so we fix $n = 2m \in [2, 2N)$.

Lemma A.4 *Recalling (4.17), we have*

$$\mu_{\Lambda^{(n)}}(\hat{C}_{1,2} | \mathcal{SG}) \geq q^{-O(1)}.$$

Proof Recall that $V_1 \cup V_2 = \Lambda^{(n)}$ and assume $\mathcal{SG}(\Lambda^{(n)})$ occurs. For any $0 \leq s_1, s_2 \leq \ell_m - \ell_{m-1}$ we write

$$\mathcal{SG}_{s_1, s_2}(\Lambda^{(n)}) = \mathcal{SG}_{s_2}(\Lambda^{(n)}) \cap \mathcal{SG}_{s_1}(\Lambda^{(n-1)} + s_2\mathbf{e}_2).$$

Then by Corollary A.3 for any such s_1, s_2 we have

$$\mu_{\Lambda^{(n)}}(\mathcal{SG}_{s_1, s_2}(\Lambda^{(n)})) = \mu_{\Lambda^{(n)}}(\mathcal{SG}(\Lambda^{(n)}))q^{O(1)},$$

so it suffices to show that

$$\mu_{V_2}(\overline{\mathcal{SG}}_{0,0}(V_2)) \geq \mu_{\Lambda^{(n)}}(\mathcal{SG}_{1,0}(\Lambda^{(n)}))q^{O(1)},$$

since $\mu(T_{\leftarrow}(I_1(\eta_{V_2}))) \geq q$ for any $\eta_{V_2} \in \overline{\mathcal{SG}}(V_2)$.

However, by Definitions 4.3 and 4.11 and symmetry we have

$$\begin{aligned} \frac{\mu_{V_2}(\overline{\mathcal{SG}}_{0,0}(V_2))}{\mu_{\Lambda^{(n)}}(\mathcal{SG}_{1,0}(\Lambda^{(n)}))} &= \frac{T^1(\ell_m - \ell_{m-1} - 1, \ell_{m-1})T^1(\ell_m - \ell_{m-1}, \ell_m - 1)}{T^1(\ell_m - \ell_{m-1} - 1, \ell_{m-1})T^1(\ell_m - \ell_{m-1}, \ell_m)T^1(1, \ell_{m-1})} \\ &\geq \frac{T^1(\ell_m - \ell_{m-1}, \ell_m - 1)}{T^1(\ell_m - \ell_{m-1}, \ell_m)} = q^{O(1)}e^{-(\ell_m - \ell_{m-1})(g((\ell_m - 1)q') - g(\ell_m q'))}, \end{aligned}$$

the last equality following from Lemma A.2.

By convexity of g we get

$$g((\ell_m - 1)q') - g(\ell_m q') \leq -q'g'((\ell_m - 1)q'). \tag{A.4}$$

By Fact A.1 we have that the r.h.s. of (A.4) is $O(1/\ell_m)$. Putting this together we obtain

$$\frac{\mu_{V_2}(\overline{\mathcal{SG}}_{0,0}(V_2))}{\mu_{\Lambda^{(n)}}(\mathcal{SG}_{1,0}(\Lambda^{(n)}))} \geq q^{O(1)}e^{-O(\ell_m - \ell_{m-1})/\ell_m} \geq q^{O(1)}e^{-O(\sqrt{q})} = q^{O(1)}, \tag{A.5}$$

as desired, the second inequality coming from (4.1) as in the proof of Proposition 4.6. □

B Proof of Proposition 5.2

Let (\mathcal{S}, S_1, π) be the parameters of g -CBSEP on \mathbb{T}_n^d and let $\ell = \lceil \pi(S_1)^{-1/d} \rceil \geq 2$. For simplicity we assume that $n^{1/d}/\ell \in \mathbb{N}$ and we partition the torus \mathbb{T}_n^d into $M = (n/\ell)^d$ equal boxes $(B_j)_{j=1}^M$, where each B_j is a suitable lattice translation by a vector in \mathbb{T}_n^d of the box $B = [\ell]^d$. The labels of the boxes can be thought of as belonging to \mathbb{T}_M^d and we say that B_i, B_j are neighbouring boxes in \mathbb{T}_n^d iff i, j are nearest neighbours in \mathbb{T}_M^d .

We then set $\hat{\mathcal{S}} = \mathcal{S}^B, \hat{\pi}((\sigma_x)_{x \in B}) = \bigotimes_{x \in B} \pi(\sigma_x), \hat{S}_1 = \bigcup_{x \in B} \{\sigma_x \in S_1\}$ and we consider the auxiliary renormalised g -CBSEP (in the sequel \hat{g} -CBSEP) on the graph $\hat{G} = \mathbb{T}_M^d$ with parameters $(\hat{\mathcal{S}}, \hat{S}_1, \hat{\pi})$. Using the assumption $\lim_{n \rightarrow \infty} \pi(S_1) = 0$, we have that

$$\lim_{n \rightarrow \infty} \hat{\pi}(\hat{S}_1) = \lim_{n \rightarrow \infty} 1 - (1 - \pi(S_1))^{\ell^d} = 1 - e^{-1}.$$

Lemma B.1 *Let $T_{\text{rel}}^{\hat{g}\text{-CBSEP}}$ be the relaxation time of \hat{g} -CBSEP on \hat{G} . Then there exists a constant $C = C(d) > 0$ such that $T_{\text{rel}}^{\hat{g}\text{-CBSEP}} \leq C$.*

Proof We closely follow [32, Appendix A]. Write $\hat{\Omega}_+$ for the space of \hat{g} -CBSEP configurations with at least one particle and consider the projection $\varphi : \hat{\Omega}_+ \mapsto \Omega_+$ given by $\varphi(\hat{\omega}) := \{\mathbb{1}_{\{\hat{\omega}_j \in \hat{\mathcal{S}}_1\}}\}_{j \in \hat{G}}$. As discussed in Remark 5.1, the projection of the \hat{g} -CBSEP chain is the CBSEP chain on \hat{G} reversible w.r.t. π^+ , the product Bernoulli measure with parameter $p = \hat{\pi}(\hat{\mathcal{S}}_1)$ conditioned on Ω_+ . For the latter, using $p = \Theta(1)$ as $n \rightarrow \infty$, it was proved in [26, Theorem 1] that its relaxation time $T_{\text{rel}}^{\text{CBSEP}} = O(1)$. Hence, it is enough to prove that $T_{\text{rel}}^{\hat{g}\text{-CBSEP}} \leq C' T_{\text{rel}}^{\text{CBSEP}}$ for some constant C' .

Let $\hat{\mathbb{P}}_{\hat{\omega}}(\cdot)$, $\hat{\mathbb{E}}_{\hat{\omega}}(\cdot)$ be the law and associated expectation of the \hat{g} -CBSEP chain with initial condition $\hat{\omega} \in \hat{\Omega}_+$ and let $\mathbb{P}_\eta(\cdot)$, $\mathbb{E}_\eta(\cdot)$ be the same objects for the projected chain (the CBSEP chain) with initial condition $\eta \in \Omega_+$.

In order to prove the lemma, it is sufficient to prove that for any function $f : \hat{\Omega}_+ \mapsto \mathbb{R}$ with zero mean w.r.t. $\hat{\pi}^+$ and for any $\hat{\omega} \in \hat{\Omega}_+$ the rate of exponential decay as $t \rightarrow +\infty$ of $|\hat{\mathbb{E}}_{\hat{\omega}}(f(\hat{\omega}(t)))|$ is at least $c/T_{\text{rel}}^{\text{CBSEP}}$ for some $c = c(p) > 0$ independent of f and $\hat{\omega}$.

More formally,

$$\liminf_{t \rightarrow +\infty} -\frac{1}{t} \log (|\hat{\mathbb{E}}_{\hat{\omega}}(f(\hat{\omega}(t)))|) \geq c/T_{\text{rel}}^{\text{CBSEP}}.$$

For any such f write

$$|\hat{\mathbb{E}}_{\hat{\omega}}(f(\hat{\omega}(t)))| \leq \left| \hat{\mathbb{E}}_{\hat{\omega}}(f(\hat{\omega}(t))\mathbb{1}_{\{\forall j \in \hat{G}, \tau_j < t\}}) \right| + \|f\|_\infty M \max_j \hat{\mathbb{P}}_{\hat{\omega}}(\tau_j \geq t), \tag{B.1}$$

where τ_j is the first time such that $\varphi(\hat{\omega}(t))_j \neq \varphi(\hat{\omega}(0))_j$, which is measurable w.r.t. the projected chain.

It follows from standard tools for finite reversible Markov chains (see e.g. [3, Section 5]) that there exists $K = K(\hat{\omega}) < +\infty$ such that $\hat{\mathbb{P}}_{\hat{\omega}}(\tau_j \geq t) \leq K e^{-\lambda(j, \hat{\omega})t}$ with

$$\lambda(j, \hat{\omega}) = \hat{\pi}^+(\hat{\omega}' : \varphi(\hat{\omega}')_j \neq \varphi(\hat{\omega})_j)/T_{\text{rel}}^{\text{CBSEP}} \geq (p \wedge (1 - p))/T_{\text{rel}}^{\text{CBSEP}}.$$

In particular, the rate of exponential decay as $t \rightarrow +\infty$ of the second term of the r.h.s. of (B.1) satisfies our requirement.

In order to prove a similar result for the first term in the r.h.s. of (B.1), we observe that, conditionally on the event $\bigcap_j \{\tau_j < t\}$ and on $\varphi(\hat{\omega}(t))$, the variables $(\hat{\omega}_j(t))_{j \in \hat{G}}$ become independent with $\hat{\omega}_j(t) \sim \hat{\pi}(\cdot | \varphi(\hat{\omega}(t))_j)$. Hence, if we set $g(\eta) = \hat{\pi}(f(\hat{\omega}) | \varphi(\hat{\omega}) = \eta)$, we get

$$\hat{\mathbb{E}}_{\hat{\omega}}(f(\hat{\omega}(t))\mathbb{1}_{\{\forall j \in \hat{G}, \tau_j < t\}}) = \mathbb{E}_{\varphi(\hat{\omega})}(g(\eta(t))) - \hat{\mathbb{E}}_{\varphi(\hat{\omega})}(g(\eta(t))\mathbb{1}_{\{\exists j \in \hat{G}, \tau_j \geq t\}}),$$

so that

$$\max_{\hat{\omega}} \left| \mathbb{E}_{\hat{\omega}} \left(f(\hat{\omega}(t)) \mathbb{1}_{\{\forall j \in \hat{G}, \tau_j < t\}} \right) \right| \leq \max_{\eta} |\mathbb{E}_{\eta}(g(\eta(t)))| + \|f\|_{\infty} M \max_{j, \eta} \mathbb{P}_{\eta}(\tau_j \geq t).$$

The rate of exponential decay as $t \rightarrow +\infty$ of both terms in the r.h.s. above is again at least $c/T_{\text{rel}}^{\text{CBSEP}}$ for some $c > 0$, since $\pi^+(g) = \hat{\pi}^+(f) = 0$. \square

Proof of Proposition 5.2 For any pair of neighbouring boxes B_i and B_j we write $\hat{\mathcal{E}}_{i,j}$ for the event $\bigcup_{x \in B_i \cup B_j} \{\sigma_x \in \mathcal{S}_1\}$. Using Lemma B.1 and the definition of $T_{\text{rel}}^{\hat{g}\text{-CBSEP}}$ we get that

$$\text{Var}_{\pi_{\mathbb{T}_n^d}^+}(f) \leq C \sum_{i \sim j} \pi_{\mathbb{T}_n^d}^+(\mathbb{1}_{\hat{\mathcal{E}}_{i,j}} \text{Var}_{B_i \cup B_j}(f | \hat{\mathcal{E}}_{i,j})),$$

where the sum in the r.h.s. is an equivalent way to express the Dirichlet form of \hat{g} -CBSEP. Now fix a pair of adjacent boxes B_i, B_j and let $T_{\text{rel}}^{g\text{-CBSEP}}(i, j)$ be the relaxation time of our original g -CBSEP with parameters $(\mathcal{S}, \mathcal{S}_1, \pi)$ on $B_i \cup B_j$. By symmetry $T_{\text{rel}}^{g\text{-CBSEP}}(i, j)$ does not depend on i, j and the common value will be denoted by \tilde{T}_{rel} . If we plug the Poincaré inequality for g -CBSEP on $B_i \cup B_j$

$$\text{Var}_{B_i \cup B_j}(f | \hat{\mathcal{E}}_{i,j}) \leq \tilde{T}_{\text{rel}} \sum_{x \sim y \in B_i \cup B_j} \pi_{B_i \cup B_j}^+(\mathbb{1}_{\mathcal{E}_{x,y}} \text{Var}_{x,y}(f | \mathcal{E}_{x,y})).$$

into the r.h.s. above, we get

$$\begin{aligned} \text{Var}_{\pi_{\mathbb{T}_n^d}^+}(f) &\leq C \tilde{T}_{\text{rel}} \sum_{i \sim j} \sum_{x \sim y \in B_i \cup B_j} \pi_{\mathbb{T}_n^d}^+(\mathbb{1}_{\hat{\mathcal{E}}_{i,j}} \pi_{B_i \cup B_j}^+(\mathbb{1}_{\mathcal{E}_{x,y}} \text{Var}_{x,y}(f | \mathcal{E}_{x,y}))) \\ &\leq 2dC \tilde{T}_{\text{rel}} \sum_{x \sim y \in \mathbb{T}_n^d} \pi_{\mathbb{T}_n^d}^+(\mathbb{1}_{\mathcal{E}_{x,y}} \text{Var}_{x,y}(f | \mathcal{E}_{x,y})) \\ &= 2dC \tilde{T}_{\text{rel}} \mathcal{D}^{g\text{-CBSEP}}(f), \end{aligned}$$

where the second inequality uses $\mathbb{1}_{\hat{\mathcal{E}}_{i,j}} \mathbb{1}_{\mathcal{E}_{x,y}} = \mathbb{1}_{\mathcal{E}_{x,y}}$ and

$$\pi_{\mathbb{T}_n^d}^+(\mathbb{1}_{\hat{\mathcal{E}}_{i,j}} \cdot) = \pi_{\mathbb{T}_n^d}^+(\hat{\mathcal{E}}_{i,j}) \pi_{\mathbb{T}_n^d \setminus (B_i \cup B_j)} \otimes \pi_{B_i \cup B_j}^+.$$

Thus, $T_{\text{rel}}^{g\text{-CBSEP}} \leq O(\tilde{T}_{\text{rel}})$. It remains to bound \tilde{T}_{rel} from above.

Let $T_{\text{mix}}^{\text{CBSEP}}$ denote the mixing time of g -CBSEP on $B_i \cup B_j$ with parameters $S' = \{0, 1\}$, $S'_1 = \{1\}$ and $\pi'(1) = \pi(S_1) = 1 - \pi'(0)$. Let $T_{\text{cov}}^{\text{rw}}$ be the cover time of the continuous-time random walk on $B_i \cup B_j$. Theorem 2 of [26] implies $\tilde{T}_{\text{rel}} \leq O(T_{\text{mix}}^{\text{CBSEP}} + T_{\text{cov}}^{\text{rw}})$. Moreover, it is well known (see e.g. [29]) that $T_{\text{cov}}^{\text{rw}}$ is at most $O(\ell^d \log(\ell)) = O(\pi(S_1)^{-1} \max(1, \log(1/\pi(S_1))))$ and [26, Corollary 3.1] proves⁵ the same bound for $T_{\text{mix}}^{\text{CBSEP}}$. In conclusion,

$$\tilde{T}_{\text{rel}} \leq O(\pi(S_1)^{-1} \max(1, \log(1/\pi(S_1)))) \quad \square$$

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⁵ Strictly speaking [26, Corollary 3.1] deals with the torus of cardinality $\pi(S_1)^{-1}$ but the same proof extends to our case of the box $B_i \cup B_j$.

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