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On the structure of the manuscript

This manuscript presents in a synthetic way the research I have conducted after my PhD thesis and until now. It is divided in four chapters of unequal length. The purpose of the first one is to introduce the various concepts and notations used in the rest of the paper. In Section 1.1 we recall the model of zero-sum stochastic games and briefly discuss certain questions about their asymptotic behavior ; in Section 1.2 we introduce some algebraic objects (semi-algebraic sets, o-minimal structures) that will appear in the next chapters.

Chapter 2 is the bulk of the manuscript and concerns my works on zero-sum repeated games. It is itself divided in four sections. The first one is dedicated to positive results on the asymptotic behavior of zero-sum stochastic games (either as the number of stages goes to infinity, or as the discount factor goes to 0), using different techniques with either an algebraic, analytic or geometric flavor, and applications to different type of games (absorbing, recursive, definable,...) The second one gives "negative" results, meaning examples of games whose values do not converge. The third section concerns a comparison with continuous time processes: a first framework concerns games with varying stage durations ; in a second part we consider the asymptotic properties of payoffs and occupation measures when the game is embedded in the time frame $[0, 1]$. The fourth and last section study some links between values with different evaluations of the stage payoff.

Chapter 3 concerns a totally different topic: the structure of sets of Nash equilibria or equilibrium payoffs of N-players one shot games. We prove that every set that could reasonably be a set of equilibrium payoff of a 3-player game indeed is, and related results for sets of equilibria under projection. This has implications on the complexity and computability of some problems concerning Nash equilibria of a game.

Finally the very short Chapter 4 gives a new minmax theorem with no regularity assumption on the payoff.

Due to the nature of the manuscript there are several bibliographies. A personal bibliography (separated in published papers and preprints) is given before the table of contents; its elements will be referred to with letters, as [Vig10] for example. The global bibliography is at the end of the manuscript; its elements will be cited with numbers.

Some of our results will be given with just a sketch of proof or under some additional assumptions; of course the interested reader is invited to consult full proofs in the cited works.

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[SV17] Sylvain Sorin and Guillaume Vigeral, Zero-sum stochastic games: asymptotic approach and limit game. *Preprint* (2017).

[Vig17] Guillaume Vigeral, A characterization of sets of equilibrium payoffs of finite games with at least 3 players. *Preprint* (2017).

Contents

Chapter 1

Introduction

1.1 Zero-sum stochastic games

1.1.1 Definitions and fundamental properties

A zero-sum stochastic game is a two-person repeated game where the state evolves from stage to stage, depending on the current state and the actions of the players. A stochastic game is determined by

– Three nonempty sets: a set of *states* Ω, and two sets of *actions* I and J for Player 1 and 2 respectively.

– A *payoff function* $g : I \times J \times \Omega \to \mathbb{R}$ and a *transition* probability $\rho : I \times J \times \Omega \to \Delta(\Omega)$, where $\Delta(\Omega)$ is the set of probability measures over Ω.

Such a game is denoted by $\Gamma = (\Omega, X, Y, g, \rho)$. At stage $t = 1$ to infinity, knowing the past history of states and actions $(\omega_1, i_1, j_1, \cdots, \omega_{t-1}, i_{t-1}, j_{t-1}, \omega_t)$, Player 1 and Player 2 chose independently $x_t \in X := \Delta(I)$ and $y_t \in Y := \Delta(J)$. The actions i_t and j_t are drawn according to x_t and y_t respectively, and ω_{t+1} is drawn according to $\rho(\omega_t, i_t, j_t)$. The triplet (i_t, j_t, ω_{t+1}) is publicly announced and the game goes to stage $t + 1$. Denote by $H_t := (\Omega \times I \times J)^t \times \Omega$ the set of histories after stage t, and $H := \cup_{t=0}^{+\infty} H_t$ the set of all finite histories. A (behavioral) strategy of Player 1 (resp. 2) in Γ is thus a mapping $\sigma : H \to \Delta(I)$ (resp. $\tau : H \to \Delta(J)$).

A probability measure θ being given in $\Delta(\mathbb{N}^*)$, the stochastic game $\Gamma_\theta(\omega)$ with evaluation θ and starting state ω is the zero-sum game in which Player 1 (resp. Player 2) maximizes (resp. minimizes) the quantity $\mathbb{E}_{\sigma,\tau} \sum_{t=1}^{+\infty} \theta_t g(i_t, j_t, \omega_t)$. Two important special cases are games Γ_n with finite horizon n, $n \in \mathbb{N}^*$, for which θ is uniform on $\{1, \cdots, n\}$; and λ-discounted games Γ_λ, $\lambda \in]0, 1]$, for which $\theta_t = (1 - \lambda)\lambda^{t-1}$.

Denote $Val_{X \times Y}$ the operator $\max_X \min_Y = \min_Y \max_X$ (when a suitable minmax theorem holds). The following fundamental result is due to Shapley [83].

Proposition 1.1.1 *Assume that Ω, I and J are finite. Then for every starting state ω, $\Gamma_n(\omega)$ has a value $v_n(\omega)$ for all n, and $\Gamma_\lambda(\omega)$ has a value $v_\lambda(\omega)$ for all λ.*
Moreover, define $\Psi : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ and $\Phi :]0, 1] \times \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ by

$$\Psi(f)(\omega) = \text{Val}_{(x,y)\in X\times Y}\left[g(x,y,\omega) + \mathbb{E}_{\rho(x,y,\omega)}f(\cdot)\right] \tag{1.1.1}$$

$$\Phi(\alpha,f)(\omega) = \alpha\Psi\left(\frac{1-\alpha}{\alpha}f\right)(\omega) \tag{1.1.2}$$

$$= \text{Val}_{(x,y)\in X\times Y}\left[\alpha g(x,y,\omega) + (1-\alpha)\mathbb{E}_{\rho(x,y,\omega)}f(\cdot)\right] \tag{1.1.3}$$

where g and ρ have been bilinearly extended to $\Delta(I)$ and $\Delta(J)$. Then the functions $v_n : \Omega \to \mathbb{R}$ satisfy the recursive equation

$$v_n = \Phi\left(\frac{1}{n}, v_{n-1}\right) \tag{1.1.4}$$

which implies

$$v_n = \frac{\Psi^n(0)}{n}. \tag{1.1.5}$$

Similarly, the function $v_\lambda : \Omega \to \mathbb{R}$ is the only solution of the fixed point equation

$$v_\lambda = \Phi\left(\lambda, v_\lambda\right). \tag{1.1.6}$$

An immediate consequence is that in Γ_λ players have optimal strategies that only depend on the current state (called stationary strategies), and that in Γ_n players have optimal strategies that only depend on the current state and current stage m (called Markovian strategies). In particular v_n and v_λ do not depend on the observation of past actions.

The map $\Psi : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ is the Shapley operator of the game and describe its recursive structure. It is immediate that Ψ satisfies the two following properties:

- Monotonicity: $f \leq g \implies \Psi(f) \leq \Psi(g)$.
- Additive Homogeneity: $\Psi(f+c) = \Psi(f) + c$ for every $c \in \mathbb{R}$

In fact, in the finite state case, a reverse holds [47]: any monotone and homogeneous operator $\Psi : \mathbb{R}^k \to \mathbb{R}^k$ is the Shapley operator a game with k states (but with non compact action sets and an unbounded payoff).

Monotonicity and additive homogeneity imply [14] that Ψ is nonexpansive for the uniform norm:

$$\|\Psi(f) - \Psi(g)\|_\infty \leq \|f-g\|_\infty$$

and thus that $\Phi(\alpha,\cdot)$ is $(1-\alpha)-$ contracting:

$$\|\Phi(\alpha,f) - \Phi(\alpha,g)\|_\infty \leq (1-\alpha)\|f-g\|_\infty$$

hence

$$v_\lambda = \Phi^\infty(\lambda,\cdot)(0). \tag{1.1.7}$$

Also remark that Φ can be continuously extended for $\alpha = 0$ to the recession operator $\Phi(0, \cdot)$:

$$\Phi(0, f)(\omega) = \lim_{\alpha \to 0} \Phi(0, \alpha) = \mathrm{Val}_{(x,y)\in X\times Y} \left[\mathbb{E}_{\rho(x,y,\omega)} f(\cdot)\right].$$

Finally, for $\alpha \in [0, 1]$, $f \in \mathbb{R}^\Omega$ and $\omega \in \Omega$ one denotes $X_\alpha(f)(\omega)$ (resp $Y_\alpha(f)(\omega)$) the set of optimal strategies of Player 1 (resp. of Player 2) in (1.1.3). In the particular case where $f = v_\alpha$ we denote simply $X_\alpha(\omega)$ the set $X_\alpha(v_\alpha)(\omega)$ of optimal stationary strategies in the state ω in the α-discounted game.

It turns out that Proposition 1.1.1 can be extended to a much wider settings than just finite games, see for example [23, 33, 56, 61, 68, 88, 94]: the recursive structure of a game is summarized by its nonexpansive Shapley operator Ψ defined from some set $\mathcal{F}$ to itself, where $\mathcal{F}$ is some subset of the set of bounded functions from Ω to $\mathbb{R}$. We will consider in this manuscript two settings in particular, in addition to the finite setting:

- The "finite/compact" setting: Ω is finite, I and J are compact, $g(\cdot, \cdot, \omega)$ and $\rho(\cdot, \cdot, \omega)$ are separately continuous for all ω, and g is bounded. Proposition 1.1.1 then applies word for word.

- The "compact/compact" setting: Ω, I and J are compact metric, g is continuous, and $\int f(\omega')\rho(d\omega'|\omega, i, j)$ is continuous for every bounded and continuous f. Then one considers $\mathcal{F}$ the set of bounded continuous functions from Ω to $\mathbb{R}$, and Proposition 1.1.1 applies with Ψ mapping $\mathcal{F}$ to itself.

In terms of game structure, we will be interested in the following classes of stochastic games.

a) Markov decision processes corresponds to the one-player case, meaning that J is a singleton. In that case Proposition 1.1.1 is due to Bellman [8].

b) The game is controlled by Player 1 (resp. 2) if $\rho(i, j, \omega)$ does not depend on j (resp i).

c) The game is with perfect information if in each state the transition and payoff are controlled by one of the player, but this player depends on the state.

d) The game is with switching control if in each state the transition is controlled by one of the player (depending on the state): the action of the other player has no influence on transitions from this state, but it may alter the payoff.

e) An absorbing game [45] is a game in which only one state is nonabsorbing ; a state ω being absorbing if $\rho(\omega|i, j, \omega) = 1$ for all $(i, j) \in I \times J$.

f) A recursive game [31] is a game in which $g(i, j, \omega) = 0$ for every $(i, j) \in I \times J$ and every nonabsorbing state ω.

g) A game with incomplete information and standard signaling [4] is a finite game in which the starting state $\omega = (k, l)$ in $\Omega = K \times L$ does not evolve under time but is not perfectly known: k and l are chosen with known probabilities $p \in \Delta(K)$ and

$q \in \Delta(L)$ respectively and Player 1 (resp. 2) is only informed on k (resp l). The actions are observed along the play but not the state, hence this does not belong to the class of stochastic games per se. However it turns out that the recursive structure of this game is the same that in an auxiliary stochastic game with state set $\Delta(K) \times \Delta(L)$ representing the belief of each player on the knowledge of the other. Hence to study v_n and v_λ one can consider this auxiliary stochastic game with compact state set. Remark that for games with incomplete information on one side (L is a singleton), this auxiliary stochastic game is controlled by Player 1.

h) More generally, some stochastic games with incomplete information, see for example [74, 76, 78, 79, 81, 86, 87, 89, 92], can be rewritten as stochastic games on a larger set of states.

1.1.2 The asymptotic study of zero-sum stochastic games: vanishing stage weight

A substantial part of this memoir (sections 2.1, 2.2, 2.4 and part of section 2.3) will be concerned with the asymptotic study of the families v_λ and v_n as λ goes to 0 and n goes to infinity. In both cases the relative weight of a stage payoff with respect to the future payoff goes to 0, hence we will call this vanishing stage weight.

The asymptotics of zero-sum games with vanishing stage weight has been studied in many frameworks. The main questions are

- Existence of $\lim v_n$ and $\lim v_\lambda$.
- Equality of the two limits (call then v the common value) ; we then say that the game as v as an asymptotic value.
- Characterization of v.
- Convergence to v for games with general evaluations θ.

In Section 2.1 we will prove existence of an asymptotic value v in several frameworks, with a characterization in some cases. Here we briefly recall (without any claim of exhaustivity) several type of proofs used in the literature. All of these are parts of what is called the *operator approach*, that puts emphasis on the study of the Shapley operator Ψ and its properties rather than on a precise construction of optimal strategies. Given a nonepansive operator Ψ from some (closed convex subset of a) Banach space to itself, we search sufficient conditions for the quantity v_n and v_λ defined in (1.1.5) and (1.1.6) to converge. Observe that, in contrast with a significant portion of the literature (see for example [5, 21, 48, 57]) on iterations of nonexpansive mappings, a Shapley operator Ψ does not preserve any bounded set and has no fixed point (except in trivial cases). As we are interested in the asymptotics of $\frac{\Psi^n}{n}$ and not Ψ^n the focus is more on the existence of invariant or almost invariant half lines [80, 90].

A first type of proofs follows what we will call the *algebraic approach*. One uses algebraic properties of the game (typically, finiteness), or of its Shapley operator, to prove the existence of a common limit [BGV15, 9, 10, 13, 69, 99], its ergodicity [1], to characterize it [50], to compute it [20] or to compute the optimal stationary strategies in Γ_λ [85]. One could also incorporate into this category Everett's asymptotic result on recursive games [31], as the proof is by induction on the number of finite states. This type of proof is particularly adapted to the finite setting. Typical tools are semi algebraic sets (see Section 1.2.1) ; and one sometimes obtains qualitative algebraic property of the family of values: development as Puiseux series, finite length of the curve (λ, v_λ), Blackwell optimality in the case of Markov Decision Processes. We will present our results with this approach in Section 2.1.1.

A second type of proof corresponds to what we will call the *analytic approach*. Here one uses analytical properties of the game (compactness, (equi)continuity of the values) or of its Shapley operator to prove existence of a common limit (see for example [SV13, 19, 45, 49, 59, 75, 78, 80, 90]). A common technique is to prove uniqueness of any accumulation point of v_n or v_λ (by establishing that all such accumulation points are solutions of a common equation with a unique solution, by proving that for two such accumulation points v and w, $v - w$ is nonnegative on $\text{Argmax}(w - v)$,...) One sometimes obtains qualitative analytic properties of the family of values: unique solution of some fixed point equation [59] or of some variational inequalities [SV13, 19, 49, 80]. This type of proof does not rely on the finiteness of the action set and is thus more adapted to the finite/compact or compact/compact settings. We will present our results with this approach in Section 2.1.2.

Finally, a third type of proof is the *geometric approach*. Here the focus is less on the operator Ψ but on its domain $(\mathcal{F}, \|\cdot\|)$. Outside game theory, equation (1.1.5) has been studied for general non expansive operators Ψ ; convergence of v_n has been established when $(\mathcal{F}, \|\cdot\|)$ is a closed convex subset of a Hilbert space [71], more generally under suitable strict convexity assumptions [73], and finally when $(\mathcal{F}, \|\cdot\|)$ is a closed convex subset of a Banach space whose dual has a Fréchet differentiable norm [46]. Unfortunately in the framework of stochastic games, $\|\cdot\|$ is the uniform norm and none of these theorems apply. In fact, in this uniform norm case, without any assumption on Ψ apart from nonexpansiveness v_n fails to converge even for $\mathcal{F} = \mathbb{R}^2$ [46]. If in addition Ψ is assumed to be monotone and additively homogeneous, v_n fails to converge even for $\mathcal{F} = \mathbb{R}^3$ [39]. This approach is useful however to establish some properties weaker than convergence for all starting states but true with very mild assumptions on the game. For example, $\|v_n\|_\infty$ and $\|v_\lambda\|_\infty$ converge [46] as soon as Ψ is well defined ; and in the compact/compact case $v_n(\omega)$ converges for at least two starting states ω [GV12, 64]. We will present our results with this approach in Section 2.1.3.

In recent years several examples of stochastic games for which neither v_n nor v_λ converge have been found in various frameworks. In [Vig13] a finite/compact game with oscillating values is constructed. In [97] examples are given for stochastic games with compact state set and finite action sets, games with perfect informationl in the finite/compact setting, and for other classes of games that do not belong to the framework of Section 1.1.1 (games without full observation of the state variable). In [SV15] a general method is given to construct such examples in a variety of frameworks, recovering the ones of [Vig13, 97] and giving a new example in the finite/compact framework with finitely many actions for Player 2. In Section 2.2 we will present our negative results.

Concerning now the question of the equality of the limits of v_n and v_λ (when they exists), so called Tauberian theorems establish that one family (uniformly) converges as soon as the other does. They were successively established in the zero player case [40], in the one player case in discrete time [53] and then for continuous time with ergodicity conditions [2] or without [OBV13]. In the two player case, the result was obtained for recursive games [55] and then for any stochastic game in the compact/compact framework [98], as well as for very general games played in continuous time [42, 43]. One can also investigate related theorems for more general families of evaluations θ, see for example [Vig10, 63, 100]. In Section 2.4 we will present our results on this subject.

1.1.3 The asymptotic study of zero-sum stochastic games: vanishing duration

As in section 1.1.2, we are interested in the asymptotics of values when the number of interactions during any given fraction of the play goes to infinity. Here however, instead of having a fixed stage length of 1 and a varying evaluation of the payoffs, the time evaluation θ is fixed and the stage duration is varying. A particular case is vanishing stage duration, leading to a continuous time game at the limit. As in the previous section, we are interested in the asymptotics of values when the number of interactions during any given fraction of the play goes to infinity.

One introduces two families of varying stage duration games, see Neyman [65], associated to Γ: linearization via “exact" games, and “discretization" of a continuous time model. In both frameworks let us describe the link with some fractional Shapley operator.

a) **Exact games**

Consider first a zero-sum stochastic game $\Gamma = (\Omega, I, J, g, \rho)$ in the compact/compact framework, with a Shapley operator Ψ. Define $q = \rho - Id$, that is $q(\omega'|i,j,\omega) = \rho(\omega'|i,j,\omega)$ if $\omega' \neq \omega$ and $q(\omega|i,j,\omega) = \rho(\omega|i,j,\omega) - 1$. Given a step size $h \in (0,1]$, define an *exact game* Γ^h with stage duration h, stage payoff hg and stage transition $\rho_h = Id + hq$. That is, $\Gamma^h = (\Omega, I, J, hg, \rho_h)$. Γ^h appears as a linearization of the game Γ: during a stage of duration h both the payoff and the state variation are proportional with factor h to those of a stage of duration one.

Definition 1.1.2 *Given $h \in [0,1]$, let $\Psi_h = (1-h)Id + h\Psi$.*

Then one has:

Proposition 1.1.3 *Ψ_h is the Shapley operator of the game Γ^h.*

Proof.

$$\begin{aligned}
\Psi_h(f)(\omega) &= (1-h)f(\omega) + h\ \mathrm{Val}\left\{g(x,y,\omega) + \int_\Omega f(\omega')\rho(d\omega'|x,y,\omega))\right\} \\
&= (1-h)f(\omega) + \mathrm{Val}\left\{h\, g(x,y,\omega) + \int_\Omega f(\omega')(h(Id+q))(d\omega'|x,y,\omega)\right\} \\
&= \mathrm{Val}\left\{h\, g(x,y,\omega) + \int_\Omega f(\omega')\rho_h(d\omega'|x,y,\omega)\right\}
\end{aligned}$$

with $\rho_h = Id + hq$. Hence Ψ_h is the one stage operator associated to the game Γ^h. ■
One can now consider for example the repeated games with finite horizon and discounted games associated to Γ^h. Natural questions are, in the finite horizon case :

- given a total time length M, what is the asymptotic behavior of the value of the N-stage game with stage duration h, as h vanishes and $Nh = M$.

- what is the asymptotic behavior of the value, as Nh goes to ∞,

and similarly in the discounted framework.

b) **Discretization**

Let now Γ be a stochastic game in the finite/compact framework and define q as in the previous case. We consider here a continuous time jointly controlled Markov process associated to the kernel q.
Explicitly, define $\mathsf{P}^t(i,j)$ as the continuous time homogeneous Markov chain on Ω, indexed by $\mathbb{R}^+$, with generator $Q(i,j)(\omega,\omega') = q(\omega'|i,j,\omega)$:

$$\dot{\mathsf{P}}^t(i,j) = \mathsf{P}^t(i,j)Q(i,j). \tag{1.1.8}$$

Given a stepsize $h \in (0,1]$, $\overline{G}^h$ is the discretization with mesh h of the game in continuous time $\overline{\Gamma}$ where the state variable follows P^t and is controlled by both players, see [37,66,95,96]. More precisely the players act at time $s = kh$ by choosing actions (i_s, j_s) (at random according to some x_s, resp. y_s), knowing the current state. Between time s and $s+h$, the state ω_t evolves with conditional law P^t following (1.1.8) with $Q(i_s, j_s)$ and $\mathsf{P}^s = Id$.

The associated Shapley operator of this stochastic game is $\overline{\Psi}_h$ with

$$\overline{\Psi}_h(f) = \underset{X\times Y}{\mathrm{Val}}\{g^h + \mathsf{P}^h \circ f\} \tag{1.1.9}$$

where $g^h(\omega_0, x, y)$ stands for $\mathbb{E}[\int_0^h g(\omega_t; x, y)dt]$ and $\mathsf{P}^h(x,y) = \int_{I\times J} \mathsf{P}^h(i,j)x(di)y(dj)$.

In section 2.3.1 we investigate both types of game with varying stage duration, based on results from [Vig10,SV16] using the "analytic" operator approach. Some of these results can also be proved using more algebraic or strategic tools [65], or viscosity techniques [91].

1.2 Semi algebraic sets and o-minimal structures

In this section we present several algebraic tools that we will use in sections 2.1.1 and 3

1.2.1 Semi algebraic sets

Let us recall some facts about semi algebraic sets that will be used in the manuscript. The reader interested in proofs of these results is referred to the literature on the subject, for example [7,15]. We first give the definition of a semi algebraic set.

Definition 1.2.1 *A set $F \subset \mathbb{R}^n$ is a semi algebraic set (resp. a basic semi algebraic set) if it can be written as a finite union and intersection (resp. as a finite intersection) of sets of the form $\{x \in \mathbb{R}^n, P_k(x) \leq 0\}$ and $\{x \in \mathbb{R}^n, P_k(x) < 0\}$, where the P_k are multivariate polynomials.*

A fundamental result [93], named Tarski-Seidenberg theorem, is the following:

Theorem 1.2.2 *Let $F \subset \mathbb{R}^n$ be a semi algebraic set, and $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection on the first $n-1$ coordinates. Then $\pi(F)$ is a semi algebraic set.*

An easy corollary, that we will also call Tarski-Seidenberg theorem for convenience, is

Corollary 1.2.3 *Let $F \subset \mathbb{R}^n$ be a semi algebraic set, and $f : \mathbb{R}^n \to \mathbb{R}^m$ a polynomial mapping. Then $f(F)$ is semi algebraic.*

A function from a real interval to $\mathbb{R}$ is said to be semi algebraic if its graph is. Germs of semi algebraic functions are characterized by the following proposition:

Proposition 1.2.4 *If $f :]a,b] \to \mathbb{R}$ is semi algebraic, then there exists $c \in]a,b]$ such that f admits a Puiseux expansion on $]a,c]$: there exists $(M,p) \in \mathbb{Z} \times \mathbb{N}^*$ and reals $a_k, k \geq M$ such that*

$$f(x) = \sum_{k=M}^{+\infty} a_k x^{\frac{k}{p}} \ \forall x \in]a,c].$$

It turns out that semi algebraic sets are very useful tools for studying finite games, and we give here two well known examples. The first one is due to Bewley and Kohlberg [9]:

Proposition 1.2.5 *If Γ is a finite stochastic game, v_λ converges as λ goes to 0.*

Sketch of the proof. Writing optimality conditions in (1.1.6), one sees that the set $\{(\lambda, (x_\lambda(\omega))_{\omega\in\Omega}, (y_\lambda(\omega))_{\omega\in\Omega}, (v_\lambda(\omega))_{\omega\in\Omega}), \ \lambda \in]0,1], \ x_\lambda(\omega) \in X_\lambda(\omega), \ y_\lambda(\omega) \in Y_\lambda(\omega) \ \forall \omega \in \Omega\}$ is defined by polynomial inequalities and hence is semi algebraic. By Tarski-Seidenberg theorem, for any fixed ω, the function $\lambda \to v_\lambda(\omega)$ is semi algebraic, and Proposition 1.2.4 implies the existence of a Puiseux expansion

$$v_\lambda(\omega) = \sum_{k=M}^{+\infty} a_k \lambda^{\frac{k}{p}} \ \forall x \in]0, \lambda_0].$$

Since v_λ is bounded M is nonnegative and $v_\lambda(\omega)$ tends to a_0 as λ goes to 0. ■

Another application is the following, concerning now one-shot N-person games:

Proposition 1.2.6 *Let Γ be an N-player finite game in which player i action set is $\mathcal{A}^i$ and payoff is g^i. Then all these sets are semi algebraic:*

- *The set* NE(Γ) *of Nash equilibria of* Γ.
- *The set* NEP(Γ) *of Nash equilibrium payoffs of* Γ.

Proof. $\sigma \in$ NE(Γ) if and only if it satisfies the following polynomial inequalities:

- $\sigma^i(a^i) \geq 0$, for all i and $a^i \in \mathcal{A}^i$
- $\sum_{a^i\in\mathcal{A}^i} \sigma^i(a^i) - 1 \leq 0$ and $-\sum_{a^i\in\mathcal{A}^i} \sigma^i(a^i) + 1 \leq 0$ for all i.
- $\sigma^i(a^i)[g^i(b^i, \sigma^{-i}) - g^i(a^i, \sigma^{-i})] \leq 0$ for all i and a^i, b^i in $\mathcal{A}^i$.

Hence NE(Γ) is a (basic) semi algebraic set. By Tarski-Seidenberg theorem, NEP(Γ) is also a semi algebraic set.

■

1.2.2 o-minimal structures

O-minimal structures will play a role in Section 2.1.1, we recall here their definition as well as basic results. Some references on the subject are [22, 28, 29].

For a given p in $\mathbb{N}$, the collection of subsets of $\mathbb{R}^p$ is denoted by $\mathcal{P}(\mathbb{R}^p)$.

Definition 1.2.7 (o-minimal structure, [22, Definition 1.5]) *An* o-minimal *structure on* $(\mathbb{R}, +, .)$ *is a sequence of Boolean algebras* $\mathcal{O} = (\mathcal{O}_p)_{p\in\mathbb{N}}$ *with* $\mathcal{O}_p \subset \mathcal{P}(\mathbb{R}^n)$*, such that for each* $p \in \mathbb{N}$*:*

(i) if A *belongs to* $\mathcal{O}_p$*, then* $A \times \mathbb{R}$ *and* $\mathbb{R} \times A$ *belong to* $\mathcal{O}_{p+1}$ *;*

(ii) if $\Pi : \mathbb{R}^{p+1} \to \mathbb{R}^n$ *is the canonical projection onto* $\mathbb{R}^n$ *then for any* A *in* $\mathcal{O}_{p+1}$*, the set* $\Pi(A)$ *belongs to* $\mathcal{O}_p$ *;*

(iii) $\mathcal{O}_p$ *contains the family of real algebraic subsets of* $\mathbb{R}^n$*, that is, every set of the form*

$$\{x \in \mathbb{R}^n : g(x) = 0\},$$

where $g : \mathbb{R}^n \to \mathbb{R}$ *is a real polynomial function ;*

(iv) the elements of $\mathcal{O}_1$ *are exactly the finite unions of intervals.*

A subset of $\mathbb{R}^p$ which belongs to an o-minimal structure $\mathcal{O}$, is said to be definable in $\mathcal{O}$ or simply definable. A mapping $F : S \subset \mathbb{R}^p \to \mathbb{R}^q$ is called definable (in $\mathcal{O}$), if its graph $\{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : y \in F(x)\}$ is definable (in $\mathcal{O}$) as a subset of $\mathbb{R}^p \times \mathbb{R}^q$.

Remark 1.2.8 *By Tarski Seidenberg theorem, the class* $\mathcal{SA}$ *of semi algebraic sets is an o-minimal structure (in fact, because of axiom (iii), it is the smallest one).*

The following result is an elementary but fundamental consequence of the definition.

Proposition 1.2.9 ([29]) *Let* $A \subset \mathbb{R}^p$ *and* $g : A \to \mathbb{R}^q$ *be definable objects.*
(i) Let $B \subset A$ *a definable set. Then* $g(B)$ *is definable.*
(ii) Let $C \subset \mathbb{R}^q$ *be a definable set. Then* $g^{-1}(C)$ *is definable.*

Because of the above definition and proposition, definable sets behave qualitatively as semi-algebraic sets. The reader is referred to [22, 29] for a comprehensive account on the topic.

We now give an example of use of the axioms characterizing an o-minimal structure, that will be useful later on when considering games played on definable sets.

Example 1.2.10 *Let us consider nonempty subsets* A, B *of* $\mathbb{R}^p, \mathbb{R}^q$ *respectively, and* $g : A \times B \to \mathbb{R}$ *a definable function. Note that the projection axiom applied on the graph of* g *ensures the definability of both* A *and* B*. Set* $h(x) = \inf_{y\in B} g(x, y)$ *for all* x *in* A *and let us establish the definability of* h*; note that the domain of* h*, i.e.* $\operatorname{dom} h = \{x \in A : h(x) > -\infty\}$ *may be smaller than* A *and possibly empty. The graph of* h *is given by*

$$\textit{graph}\, h := \{(x, r) \in A \times \mathbb{R} : (\forall y \in B, g(x, y) \geq r) \ \textit{ and } \ (\forall \epsilon > 0, \exists y \in B, g(x, y) < r + \epsilon)\}.$$

Consider the sets

$$T = \{(x,r) \in A \times \mathbb{R} : \forall \epsilon > 0, \exists y \in B, g(x,y) < r + \epsilon\},$$

$$S_0 = \{(x,y,r,\epsilon) \in A \times B \times \mathbb{R} \times (0,+\infty) : g(x,y) - r - \epsilon < 0\}.$$

S_0 is definable by Proposition 1.2.9(ii). Projecting S_0 via $\Pi(x,y,r,\epsilon) = (x,r,\epsilon)$, one obtains the definable set $S_1 = \{(x,r,\epsilon) \in A \times \mathbb{R} \times (0,+\infty) : \exists y \in B, g(x,y) - r - \epsilon < 0\}$. Introducing $\Pi'(x,r,\epsilon) = (x,r)$, we see that T can be expressed as

$$(A \times \mathbb{R}) \setminus \Pi'(E)$$

with $E := (A \times \mathbb{R} \times (0,+\infty)) \setminus S_1$. Since the complement operations preserve definability, T is definable. Using this type of idea and Definition 1.2.7, we can prove similarly that

$$T' = \{(x,r) \in A \times \mathbb{R} : \forall y \in B, g(x,y) \geq r\}$$

is definable. Hence $\operatorname{graph} h = T \cap T'$ is definable and thus h is definable.

The most common method to establish the definability of a set is thus to interpret it as the result of a finite sequence of basic operations on definable sets (projection, complement, intersection, union). This idea is conveniently captured by the notion of a first order definable formula. *First order definable formulas* are built inductively according to the following rules:

- If A is a definable set, $x \in A$ is a first order definable formula
- If $P(x_1,\ldots,x_p)$ and $Q(x_1,\ldots,x_q)$ are first order definable formulas then (not P), (P and Q), and (P or Q) are first order definable formulas.
- Let A be a definable subset of $\mathbb{R}^p$ and $P(x_1,\ldots,x_p,y_1,\ldots,y_q)$ a first order definable formula then both
$$(\exists x \in A, P(x,y))$$
$$(\forall x \in A, P(x,y))$$
are first order definable formulas.

Note that Proposition 1.2.9 ensures that "$g(x_1,\ldots,x_p) = 0$" or '$g(x_1,\ldots,x_p) < 0$" are first order definable formulas whenever $g : \mathbb{R}^p \to \mathbb{R}$ is definable (e.g. polynomial).
It is then easy to check, by induction, that:

Proposition 1.2.11 ([22]) *If $\Phi(x_1,\ldots,x_p)$ is a first order definable formula, then $\{(x_1,\ldots,x_p) \in \mathbb{R}^p : \Phi(x_1,\ldots,x_p)\}$ is a definable set.*

For example an easy consequence of the above proposition is

Proposition 1.2.12 *Let Ω be a definable open subset of $\mathbb{R}^n$ and $g : \Omega \to \mathbb{R}^m$ a definable differentiable mapping. Then its derivative g' is definable.*

There exists many regularity results for definable sets [29]. We will use the following fundamental lemma :

Theorem 1.2.13 (Monotonicity Lemma [29, Theorem 4.1]) *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a definable function and $k \in \mathbb{N}$. Then there exists a finite partition of I into l disjoint intervals $I_1, \ldots, I_l$ such that f restricted to each nontrivial interval I_j, $j \in \{1, \ldots, l\}$ is C^k and either strictly monotone or constant.*

We end this section by giving examples of o-minimal structures (see [29] and references therein).

Examples (a) **(globally subanalytic sets)** There exists an o-minimal structure, that contains all sets of the form $\{(x,t) \in [-1,1]^p \times \mathbb{R} : f(x) = t\}$ where $f : [-1,1]^p \to \mathbb{R}$ ($p \in \mathbb{N}$) is an analytic function that can be extended analytically on a neighborhood of the box $[-1,1]^p$. The sets belonging to this structure are called *globally subanalytic sets.*

For instance the functions

$$\sin : [-a, a] \to \mathbb{R}$$

(where a ranges over $\mathbb{R}_+$) are globally subanalytic, while $\sin : \mathbb{R} \to \mathbb{R}$ is not (else the set $\sin^{-1}(\{0\})$ would be finite by Proposition 1.2.9(ii) and Definition 1.2.7(iv)).

(b) **(log-exp structure)** There exists an o-minimal structure containing the globally subanalytic sets and the graph of $\exp : \mathbb{R} \to \mathbb{R}$.

We shall also use a more “quantitative" characteristic of o-minimal structures.

Definition 1.2.14 (Polynomially bounded structures) *An o-minimal structure is called polynomially bounded if for every definable function $\psi : (a, +\infty) \to \mathbb{R}$ there exists a positive constant C and an integer N such that $|\psi(t)| \le Ct^N$ for all t sufficiently large*

The classes of semi-algebraic sets or of globally subanalytic sets are polynomially bounded [29], while the log-exp structure is clearly not.

We have the following result, which can be viewed as an analogous to the Puiseux development of Proposition 1.2.4 in the more general case of o-minimal structures:

Corollary 1.2.15 ([29]) *If $\epsilon > 0$ and $\phi : (0, \epsilon) \to \mathbb{R}$ is definable in a polynomially bounded o-minimal structure there exist $c \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that*

$$\phi(t) = ct^\alpha + o(t^\alpha), \quad t \in (0, \epsilon).$$

Chapter 2

Zero-sum repeated games

2.1 Sufficient conditions for convergence of values

In this section we give proofs of convergence of v_n and v_λ in different settings. In the first subsection we introduce games definable in an o-minimal structure and give conditions for such games to admit an asymptotic value. In the second subsection we give new proofs of convergence, using comparison theorems, for absorbing games and recursive games in the finite/compact framework. In the third subsection we consider a more abstract setting and general nonexpansive operators defined on some space $\mathcal{F}$, and investigate the asymptotic properties of Ψ depending on the geometry of $\mathcal{F}$.

2.1.1 Algebraic approach

This section follows [BGV15]. As we said in the introduction the algebraic approach is very adapted to finite games ; here we try to use the same ideas for stochastic games in the finite/compact setting, in which in addition both g and ρ are definable in some fixed o-minimal structure (for example the semi algebraic one), see Section 1.2.2 for definitions. We say that such a stochastic game is definable in the o-minimal structure. One example would be when I and J are intervals and g and ρ are rational functions.

To prove existence of an asymptotic value in this context, it is natural to consider the two following questions:

(a) Under which conditions the Shapley operator of a definable game is definable (in the *same* o-minimal structure) ?

(b) If a Shapley operator of a game is definable, what are the consequences in terms of games values?

Let us first answer to b) with the following proposition. Part (i) generalizes a similar result in [64] in the specific case of the semi algebraic structure.

Proposition 2.1.1 *If Ψ is definable, then the following assertions hold true.*

(i) The limits of v_λ and v_n exist and coincide, i.e.

$$\lim_{n\to+\infty} v_n = \lim_{\lambda\to 0} v_\lambda := v_\infty.$$

(ii) If Ψ is definable in a polynomially bounded o-minimal structure, there exists $\theta \in (0,1]$ such that

$$\|v_n - v_\infty\| = O(\frac{1}{n^\theta}).$$

Sketch of the proof. The proof is similar in spirit to Proposition 1.2.5 in the finite case. Since Ψ is definable one can write the function $\lambda \to v_\lambda$ using a first order formula, and by Proposition 1.2.11 this function is thus definable. The monotonicity lemma (Proposition 1.2.13) then implies that for every ω, $v_\lambda(\omega)$ is piecewise monotone, and thus converges since it is bounded. It also implies that v_λ is of bounded variation, which implies convergence of v_n to the same limit [64], and the estimate (ii) in the polynomially bounded case. ■

Remark 2.1.2 *The fact that the curve v_λ is of bounded variation implies [60] that the stronger notion of uniform value holds as soon as the stage payoffs are observed.*

Answering part a) is unfortunately more difficult ; in particular there exists games definable in the semi algebraic structure such that neither Ψ nor $\lambda \to v_\lambda$ are semi algebraic.

Example 2.1.3 *Consider the following stochastic game with two states $\{\omega_1, \omega_2\}$ and action sets $[0,1]$ for each player. The first state is absorbing with payoff 0, while for the second state, the payoff is*

$$g(x, y, \omega_2) = \frac{1+x}{2(1+xy)^2}$$

and the transition probability is given by

$$1 - \rho(\omega_1|x, y, \omega_2) = \rho(\omega_2|x, y, \omega_2) = \frac{(1+x)y}{2(1+xy)^2},$$

for all (x, y) in $[0,1]^2$.

Then on $0 \le f_1 < f_2 \le 1$ one has $\Psi_1(f_1, f_2) = \Big(f_1, f_1 + \frac{f_2 - f_1}{2\ln(1+f_2-f_1)}\Big)$, and $v_\lambda = \frac{\lambda(e^{\frac{1-\lambda}{2}}-1)}{1-\lambda}$ in the nonabsorbing state.

Observe however that v_λ still converges in this example, and in fact both Ψ and v_λ are definable in the larger o-minimal structure of globally subanalytic sets.

To answer positively question a) one thus needs more structure on the game.

Definition 2.1.4 *A function $h : I \times J \times \Omega$ to $\mathbb{R}$ is separably definable if*

$$h(i, j, \omega) = \sum_{k=1}^{K} a_k(i, \omega) b_k(j, \omega).$$

where K is a positive integer, and $a_k(\cdot, \omega)$ and $b_k(\cdot, \omega)$ are continuous definable functions. A stochastic definable game (Ω, X, Y, g, ρ) is separably definable*, if both g and $\rho(\omega'|\cdot)$, for every ω', are separably definable.*

The most natural example of separably definable games are games with semi-algebraic action spaces and polynomial reward and transition functions. One then establishes

Proposition 2.1.5 *Separably definable games have a definable Shapley operator.*

Sketch of the proof. Let Γ be such a game and Ψ its Shapley operator. Fix $\omega \in \Omega$, we prove that $f \to \Psi(f)(\omega)$ is definable. Since g and $\rho(\omega'|\cdot)$ are separably definable, one can write

$$\Psi(f)(\omega) = \mathrm{Val}_{X\times Y}\left\{\sum_{k=1}^{K} a_k^0(x)b_k^0(y) + \sum_{\omega'\in\Omega} f(\omega')\sum_{k=1}^{K} a_k^{\omega'}(x)b_k^{\omega'}(y)\right\}$$

where each a_k^0, b_k^0, $a_k^{\omega'}$, $b_k^{\omega'}$ is continuous, linear and definable. This can then be written as

$$\Psi(f)(\omega) = \sup_{S\in\mathcal{L}}\inf_{T\in\mathcal{M}} SM(f)T^t.$$

where $\mathcal{L} \subset \mathbb{R}^{K(1+|\Omega|)}$ is the convex hull of the set $\left\{ (a_k^w(i))_{1\le k\le K, w\in\Omega\cup\{0\}},\ i \in I \right\}$, $\mathcal{M} \subset \mathbb{R}^{K(1+|\Omega|)}$ is the convex hull of the set $\left\{ (b_k^w(j))_{1\le k\le K, w\in\Omega\cup\{0\}},\ i \in I \right\}$, and $M(f)$ is a diagonal matrix whose diagonal entries consist either of 1 or of some $f(\omega')$. The function $f \to M(f)$ is clearly definable. By Caratheodory's theorem, $\mathcal{L}$ and $\mathcal{M}$ can be expressed using a first order formula, and are thus definable by Proposition 1.2.11. One concludes using Example 1.2.10 and its dual. ∎

Corollary 2.1.6 *Let Γ be a game in the finite/compact setting, such that $\rho(\omega'|\cdot)$ is separably definable for every ω, and g is continuous. Then v_λ and v_n converge to a common limit.*

Sketch of the proof. When the payoff g is also separably definable it is an immediate consequence of propositions 2.1.1 and 2.1.5. In general, one uses Stone-Weierstrass theorem to approximate the continuous g by polynomials (that are separably definable), and use the fact that the Shapley operator of a game depends continuously of its payoff function. ∎

In particular,

Corollary 2.1.7 *Let Γ be a game in the finite/compact setting with a definable transition probability, and either switching control or finitely many actions on one side. Then v_λ and v_n converge to a common limit.*

Remark 2.1.8 *In both cases the result no longer hold without the definable hypothesis, see [SV15, 97] and Section 2.2.*

Leaving the game theoretic framework for an instant, let us finally give an application of proposition 2.1.1 (in the case of the $log - exp$ structure) to nonlinear Perron-Frobenius theory:

Corollary 2.1.9 (Growth minimization) *Assume that T is a self-map of $(\mathbb{R}_+^*)^d$, every coordinate of which can be written as*

$$[T(f)]_i = \inf_{p \in \mathcal{M}_i} \langle p, f \rangle \qquad 1 \leq i \leq d, \tag{2.1.1}$$

where $\mathcal{M}_i$ is a subset of $\mathbb{R}_+^d$. Assume in addition that each set $\mathcal{M}_i$ is definable in the log-exp structure. Then, the growth rate $\chi(T) = \exp(\lim_{n\to\infty} n^{-1} \log T^n(e))$ does exist and is independent of the choice of $e \in \operatorname{int} C$.

2.1.2 Analytic approach

This section is based on [SV13] in which we give new proofs of convergence of the discounted values via a comparison principle in three frameworks : finite/compact absorbing games, finite/compact recursive games, and games with incomplete information and standard signaling. We explain here in detail the idea of the proof in case of absorbing and recursive games games, which are very different from the original ones in [45] and [31] respectively. The proof for games with incomplete information, which is in the same spirit that the two we will present, use tools quite similar than those used by Laraki in [49] so we omit it here.

a) Absorbing games

Without loss of generality an absorbing game has three states: the nonabsorbing one ω as well as two absorbing states 1^* and -1^* with absorbing payoff 1 and -1 respectively. Since the only interesting starting state is ω we drop all references to it in the formulas when there is no ambiguity. Denote $p(i,j) = \rho(\omega|i,j,\omega)$ and $p^*(i,j) = 1 - p(i,j)$ the nonabsorbing and absorbing probabilities respectively. Also define, for $p^*(i,j) < 0$, the absorbing payoff given (i,j):

$$\overline{g}^*(i,j) = \frac{\rho(1^*|i,j,\omega) - \rho(1^*|i,j,\omega)}{p^*(i,j)}.$$

$\overline{g}^*(i,j)$ is not defined when $p^*(i,j) = 0$, but in that case we use the convention that $p^*(i,j)\overline{g}^*(i,j) = 0$. Then the Shapley operator can be simply rewritten as a function from $\mathbb{R}$ to itself, where the only variable is the value in starting state ω, as

$$\Psi(f) = \operatorname{Val}_{(x,y)\in X\times Y} \left\{ g(x,y) + p(x,y)f + p^*(x,y)\overline{g}^*(x,y) \right\}. \tag{2.1.2}$$

where g, p, and $p^*\overline{g}^*$ are bilinearly extended on $X \times Y$.

Lemma 2.1.10
i) Let $f \in \mathbb{R}$ such that $f \geq \Phi(0,f)$ and $y \in Y_0(f)$. Then, for any $x \in X$,

$$p^*(x,y) > 0 \implies f \geq \overline{g}^*(x,y).$$

ii) Let $f \in \mathbb{R}$ such that $f \leq \Phi(0,f)$ and $x \in X_0(f)$. Then, for any $y \in Y$,

$$p^*(x,y) > 0 \implies f \leq \overline{g}^*(x,y).$$

Proof. Given $x \in X$ and $y \in Y_0(f)$,

$$f \geq \Phi(0, f) \geq p(x, y)f + p^*(x, y)\overline{g}^*(x, y)$$

and $p(x, y) = 1 - p^*(x, y)$, hence the result. ■

Given $\lambda \in]0, 1[$, $x \in X$ and $y \in Y$, let $r_\lambda(x, y)$ be the induced payoff in the discounted game by the corresponding stationary strategies.

Lemma 2.1.11

$$r_\lambda(x, y) \leq \begin{cases} g(x, y), & \textit{if } p^*(x, y) = 0, \\ \max(g(x, y), \overline{g}^*(x, y)), & \textit{if } p^*(x, y) > 0. \end{cases}$$

Proof.

$$r_\lambda(x, y) = \lambda g(x, y) + (1 - \lambda)\left[p(x, y) r_\lambda(x, y) + p^*(x, y)\overline{g}^*(x, y)\right];$$

hence

$$r_\lambda(x, y) = \frac{\lambda g(x, y) + (1 - \lambda)p^*(x, y)\overline{g}^*(x, y)}{\lambda + (1 - \lambda)p^*(x, y)}.$$

■

Corollary 2.1.12 *v_λ converges as λ goes to 0.*

Proof. Suppose, on the contrary, that there are two sequences $v_{\lambda_n} \to v$ and $v_{\lambda'_n} \to v'$ with $v > v'$. Up to an extraction, one can assume that $x_{\lambda_n} \in X_{\lambda_n}(v_{\lambda_n})$ converges to x and, similarly, $y_{\lambda'_n} \in Y_{\lambda'_n}(v_{\lambda'_n})$ converges to y. By continuity $v' = \Phi(0, v')$ and $y \in Y_0(v')$. For n large enough that $v' < v_{\lambda_n}$, if $p^*(x_{\lambda_n}, y) > 0$ then the first assertion in Lemma 2.1.10 gives

$$\overline{g}^*(x_{\lambda_n}, y) \leq v' < v_{\lambda_n} \leq r_{\lambda_n}(x_{\lambda_n}, y)$$

hence $v_{\lambda_n} \leq g(x_{\lambda_n})$ by Lemma 2.1.11 . If $p^*(x_{\lambda_n}, y) = 0$, Lemma 2.1.11 also gives immediately $v_{\lambda_n} \leq g(x_{\lambda_n})$. Going to the limit we get $v \leq g(x, y)$. A dual reasoning yields $v' \geq g(x, y)$, a contradiction. ■

We now identify the limit v of the absorbing game, with a different formula than the one proved in the finite case in [19, 50]

Definition 2.1.13 *Define the function $W : X \times Y \to \mathbb{R}$ by*

$$W(x, y) := \operatorname{med}\left(g(x, y), \sup_{x'; p^*(x', y) > 0} \overline{g}^*(x', y), \inf_{y'; p^*(x, y') > 0} \overline{g}^*(x, y')\right),$$

where $\operatorname{med}(\cdot, \cdot, \cdot)$ *denotes the median of three numbers, with the usual convention that a supremum (resp., an infimum) over an empty set equals $-\infty$ (resp., $+\infty$).*

Corollary 2.1.14 *The limit v is the value of the zero-sum game, denoted by Υ, with action spaces X and Y and payoff W.*

Proof. It is enough to show that $v \leq w := \sup_x \inf_y W(x,y)$ as a dual argument yields the conclusion. Assume, by contradiction, that $w < v$.

Let $\varepsilon > 0$ with $w + 2\varepsilon < v$. Consider $x \in X_0(v)$ an accumulation point of $x_\lambda \in X_\lambda(v_\lambda)$ and let y be an ε-best response to x in the game Υ. Lemma 2.1.10 *ii)* implies that

$$\inf_{y'; p^*(x,y')>0} \overline{g}^*(x,y') \geq v > w + \varepsilon \geq W(x,y),$$

so that

$$W(x,y) = \max\left(g(x,y), \sup_{x'; p^*(x',y)>0} \overline{g}^*(x',y) \right).$$

Thus, $\sup_{x'; p^*(x',y)>0} \overline{g}^*(x',y) \leq w+\varepsilon < v-\varepsilon$ and, similarly, $g(x,y) < v-\varepsilon$. The corresponding inequalities hold with x_λ, for λ small enough:

$$p^*(x_\lambda,y)[\overline{g}^*(x_\lambda,y) - (v-\varepsilon)] \leq 0, \qquad g(x_\lambda,y) \leq v-\varepsilon,$$

leading by Lemma 2.1.11 to $v_\lambda \leq v - \varepsilon$, a contradiction. ■

b) A general result, and application to recursive games

We now use the same techniques than in Part a) but for general finite/compact games. The proposition below will only ensure convergence of v_λ for certain classes of these games (for example recursive ones), but it is interesting to state it in general to understand what argument is missing in other classes.

Proposition 2.1.15 *Let Γ be a finite/compact game and v and v' be two accumulation points of v_λ: $v_{\lambda_n} \to v$ and $v_{\lambda'_n} \to v'$. Assume that* $\max_\Omega v(\omega) - v'(\omega) > 0$. *For every ω let $\mathbf{x}_{\lambda_n}(\omega) \in X_{\lambda_n}(\omega)$ and $\mathbf{y}_{\lambda'_n}(\omega) \in Y_{\lambda'_n}(\omega)$ be optimal strategies of Player 1 and 2 in state ω in the λ_n and λ'_n discounted game respectively, and let $\mathbf{x}(\omega)$ and $\mathbf{y}(\omega)$ be accumulation points in $X_0(\omega)$ and $Y_0(\omega)$ respectively.*
Define $\Omega_1 := \mathrm{Argmax}_\Omega(v - v')$ and $\Omega_2 := \mathrm{Argmax}_{\Omega_1} v = \mathrm{Argmax}_{\Omega_1} v'$. Then there exists ω_0 in Ω_2 such that $v(\omega_0) \leq g(\mathbf{x}(\omega_0), \mathbf{y}(\omega_0), \omega_0)$ on Ω_2.

Proof. Up to extraction, there exists $\omega_0 \in \Omega_2$, which realizes the maximum of v_{λ_n} on Ω_2 for every n. Write x_{λ_n} for $\mathbf{x}_{\lambda_n}(\omega_0)$, x for $\mathbf{x}(\omega_0)$ and y for $\mathbf{y}(\omega_0)$; we assume by contradiction that

$$v(\omega_0) > g(x,y,\omega_0).$$

By optimality of x_{λ_n} we get:

$$\begin{aligned} v_{\lambda_n}(\omega_0) \ &\leq \lambda_n g(x_{\lambda_n}, y, \omega_0) + (1-\lambda_n) \ \Bigg[\sum_{\omega' \in \Omega_2} \rho(\omega' | x_{\lambda_n}, y, \omega_0) v_{\lambda_n}(\omega') \\ &\qquad + \sum_{\omega' \in \Omega \setminus \Omega_2} \rho(\omega' | x_{\lambda_n}, y, \omega_0) v_{\lambda_n}(\omega') \Bigg], \end{aligned} \tag{2.1.3}$$

so, by definition of ω_0,

$$(1 - (1-\lambda_n)\rho(\Omega_2 | x_{\lambda_n}, y, \omega_0))\, v_{\lambda_n}(\omega_0) \leq \lambda_n g(x_{\lambda_n}, y, \omega_0) + (1-\lambda_n) \sum_{\omega' \in \Omega \setminus \Omega_2} \rho(\omega' | x_{\lambda_n}, y, \omega_0) v_{\lambda_n}(\omega').$$

For simplicity, denote $\rho_n := \rho(\Omega_2|x_{\lambda_n}, y, \omega_0)$. If $\rho_n = 1$ for infinitely many n, going to the limit immediately yields $v(\omega_0) \leq g(\mathbf{x}(\omega_0), \mathbf{y}(\omega_0), \omega_0)$ and the requested contradiction, hence we assume that it is not the case. Up to an extraction, μ_n defined by $\mu_n(w') = \dfrac{\rho(\omega'|x_{\lambda_n}, y, \omega_0)}{1-\rho_n}$ is thus a probability measure on $\Omega\backslash\Omega_2$ and converge to some μ. Denote $\alpha_n = \frac{\lambda_n}{1-(1-\lambda_n)\rho_n} \in [0,1]$. We now get an analogue of Lemma 2.1.11:

$$\begin{aligned}
v_{\lambda_n}(\omega_0) &\leq \alpha_n g(x_{\lambda_n}, y, \omega_0) + \frac{1-\lambda_n}{1-(1-\lambda_n)\rho_n} \sum_{\omega' \in \Omega\backslash\Omega_2} \rho(\omega'|x_{\lambda_n}, y, \omega_0) v_{\lambda_n}(\omega') \\
&= \alpha_n g(x_{\lambda_n}, y, \omega_0) + (1-\alpha_n) \sum_{\omega' \in \Omega\backslash\Omega_2} \frac{\rho(\omega'|x_{\lambda_n}, y, \omega_0)}{1-\rho_n} v_{\lambda_n}(\omega') \\
&\leq \max\left(g(x_{\lambda_n}, y, \omega_0), \sum_{\omega' \in \Omega\backslash\Omega_2} \mu_n(\omega') v_{\lambda_n}(\omega') \right).
\end{aligned}$$

Going to the limit and using that $v(\omega_0) > g(x, y, \omega_0)$ yields

$$v(\omega_0) \leq \sum_{\omega' \in \Omega\backslash\Omega_2} \mu(\omega') v(\omega') \tag{2.1.4}$$

On the other hand, since $\omega_0 \in \Omega_2$,

$$\begin{aligned}
v'(\omega_0) &= \Phi(0, v')(\omega_0) \\
&\geq \left[\sum_{\omega' \in \Omega_2} \rho(\omega'|x_{\lambda_n}, y, \omega_0) v'(\omega') + \sum_{\omega' \in \Omega\backslash\Omega_2} \rho(\omega'|x_{\lambda_n}, y, \omega_0) v'(\omega') \right],
\end{aligned}$$

so using the fact that v' is constant on Ω_2, and going to the limit,we get an analogue to Lemma 2.1.10:

$$v'(\omega_0) \geq \sum_{\omega' \in \Omega\backslash\Omega_2} \mu(\omega') v'(\omega'). \tag{2.1.5}$$

Substracting (2.1.5) from (2.1.4) yields

$$(v - v')(\omega_0) \leq \sum_{\omega' \in \Omega\backslash\Omega_2} \mu(\omega')(v - v')(\omega'),$$

and since $\omega_0 \in \Omega_1 = \text{Argmax}_\Omega(v - v')$, this implies that the support of μ is included in Ω_1 and that (2.1.4) is an equality. This, in turn, forces the support of μ to be included in $\Omega_2 = \text{Argmax}_{\Omega_1} v$, a contradiction to the construction of μ. ■

Corollary 2.1.16 *If Γ is a finite/compact recursive game then v_λ converges as λ goes to 0.*

Proof. Assume by contradiction that there are two accumulation points v and v' with $\max_\Omega\{v - v'\} > 0$, denote $\Omega_1 = \text{Argmax}_\Omega(v - v')$ and $\Omega_2 := \text{Argmax}_{\Omega_1} v$. Since the game

is recursive, Proposition 2.1.15 implies that $v(\omega_0) \leq 0$ for some $\omega_0 \in \Omega_2$, which implies that $v(\cdot) \leq 0$ on Ω_1. A dual argument yields that $v'(\cdot) \geq 0$ on Ω_1, a contradiction. ■
The characterization of the limit due to Everett [31] can also be obtained using Proposition 2.1.15, we refer the interested reader to [SV13].

Unfortunately Proposition 2.1.15 is not enough to prove convergence in general, as it (and its dual) only gives the existence of two (possibly distinct) states ω_0 and ω_0' such that

$$g(\mathbf{x}(\omega_0'), \mathbf{y}(\omega_0'), \omega_0') \leq v'(\cdot) \leq v(\cdot) \leq g(\mathbf{x}(\omega_0), \mathbf{y}(\omega_0), \omega_0)$$

on Ω_1, which is not a contradiction. Observe however that if Ω_1 is a singleton (in particular, if the game is absorbing) we arrive to a contradiction. In fact, one can prove the following stronger result, that will be useful for guessing in which games the discounted values v_λ may not converge (see Section 2.2):

Corollary 2.1.17 *Let Γ be a finite/compact game and v, v', Ω_1 and Ω_2 be as in Proposition 2.1.15. Then Ω_2 contains at least two elements.*

Proof. Assume by contradiction that $\Omega_2 = \{\omega_0\}$. By Proposition 2.1.15, $v(\omega_0) \leq g(\mathbf{x}(\omega_0), \mathbf{y}(\omega_0), \omega_0)$. If we considered the dual of Proposition 2.1.15, the sets considered would be $\Omega_1' := \operatorname{Argmin}_\Omega(v' - v) = \Omega_1$ but $\Omega_2' := \operatorname{Argmin}_{\Omega_1} v' = \operatorname{Argmin}_{\Omega_1} v \neq \Omega_2$ so this would not work.

Notice however that in the proof of Proposition 2.1.15, the only properties of Ω_2 that we used are that v and v' are constant on it, and that if

$$v(\omega_0) = \sum \mu(\omega')v(\omega')$$

where μ has a support in Ω_1, then μ has a support in Ω_2. Both properties still hold if one replaces Ω_2 by $\operatorname{Argmin}_{\Omega_1} v$. Using now the dual of this modified Proposition 2.1.15 one gets $v'(\omega_0) \geq g(\mathbf{x}(\omega_0), \mathbf{y}(\omega_0), \omega_0)$, a contradiction. ■

2.1.3 Geometric approach

This section follows [GV12]. We prove asymptotic results on v_n using only the geometric structure of the set $\mathcal{F}$ on which Ψ is defined.

Definition 2.1.18 *We say that $\delta : E \times E \to \mathbb{R}$ is a* hemi-metric *on a set E if the two following conditions are satisfied for all $(e_1, e_2, e_3) \in E^3$:*

a) $\delta(e_1, e_3) \leq \delta(e_1, e_2) + \delta(e_2, e_3)$

b) $\delta(e_1, e_2) = \delta(e_2, e_1) = 0$ *if and only if* $e_1 = e_2$.

We then say that (E, δ) is a hemi-metric space.

Notice that a hemi-metric is generally not a metric, since it is neither symmetric nor non-negative. To any hemi-metric, one can canonically associate a metric by the following easy lemma.

Lemma 2.1.19 *For any hemi-metric δ, the function $d(e_1, e_2) = \max(\delta(e_1, e_2), \delta(e_2, e_1))$ is a metric on X.*

In the sequel, E is equipped with the topology induced by the metric d. We shall say that (E, δ) is *complete* when the associated metric space (E, d) is complete. The usual definition of a geodesic is extended to hemi-metric spaces:

Definition 2.1.20 *A geodesic joining a point $e \in E$ to a point $e' \in E$ is a map $\gamma : [0, 1] \to E$ such that $\gamma(0) = e$, $\gamma(1) = e'$, and such that for all $0 \le s \le t \le 1$,*

$$\delta(\gamma(s), \gamma(t)) = (t - s)\delta(e, e').$$

We will consider hemi metric space with the following good geometry:

Definition 2.1.21 *We say that (E, δ) is* metrically star-shaped *with centre e° if there exists a family of geodesics $\{\gamma_e\}_{e\in E}$, such that γ_e joins the centre e° to the point e, and such that the following inequality is satisfied for every $(e, e') \in X^2$ and $s \in [0, 1]$:*

$$\delta\left(\gamma_e(s), \gamma'_e(s)\right) \le s\delta(e, e'). \tag{2.1.6}$$

If the hemi-metric δ is not a metric, we also require that for any e, the quantity $\delta(e, \gamma_e(s))$ tends to 0 as s goes to 1.

The condition (2.1.6) is a form of *metric convexity* [70]. In particular any Busemann space [70] is metrically star-shaped, but our definition is less demanding since we only require the inequality (2.1.6) to be satisfied for one specific choice of geodesics.

Example 2.1.22 *Let E be the set $\mathcal{F}$ of continous functions from some compact set Ω to $\mathbb{R}$, and define the top function on $\mathcal{F}$:*

$$\mathbf{t}(f) = \max_{\omega\in\Omega} f(\omega).$$

Then it is easy to verify that $\delta(f, f') := \mathbf{t}(f' - f)$ is a hemi-metric, that its associated metric is the uniform norm, and that all straight lines are geodesics. Hence $(\mathcal{F}, \delta)$ is metrically star shaped.

A metrically star-shaped space (E, δ) with centre x° being given, we now consider $\Psi : E \to E$ non-expansive with respect to the hemi-metric δ, meaning that for all $(e, e') \in E^2$,

$$\delta(\Psi(e), \Psi(e')) \le \delta(e, e').$$

Definition 2.1.23 *To each non-expansive mapping Ψ, we associate the two following quantities:*

$$\overline{\chi}(\Psi) = \inf_{e\in E} \delta\left(e, \Psi(e)\right) \quad , \tag{2.1.7}$$

$$\chi(\Psi) = \lim_{k\to+\infty} \frac{\delta\left(e, \Psi^k(e)\right)}{k} = \inf_{k\ge 1} \frac{\delta\left(e, \Psi^k(e)\right)}{k} \quad . \tag{2.1.8}$$

Thus, the number $\chi(\Psi)$ measures the *linear escape rate* of the orbits of Ψ. It is well defined and the limit is independent of the choice of $e \in X$ by classical subadditivity and non expansiveness arguments.

Lemma 2.1.24 *The following inequality is satisfied for any non-expansive mapping Ψ:*

$$\rho(\Psi) \leq \overline{\rho}(\Psi).$$

Proof. Observe that for any $x \in X$ and $k \geq 1$,

$$\delta\left(x, \Psi^k(x)\right) \leq \sum_{l=0}^{k-1} \delta\left(\Psi^l(x), \Psi^{l+1}(x)\right) \leq k\delta(x, \Psi(x))$$

and take the infimum on both k and x. ■

Recall now the definition of the horofunction boundary: this was defined by Gromov [36] for a metric space but the same construction can be performed with a hemi-metric. Let us fix an arbitrary point $\bar{e} \in E$ (the *basepoint*). We define a map m from E to the set of functions from E to $\mathbb{R}$ by associating to any $e \in E$ the following function $m(e)$:

$$m(e) : e' \to [m(e)](e') := \delta(\bar{e}, e) - \delta(e', e).$$

Denote by $\mathscr{M}$ the closure of $m(E) := \{m(e) \mid e \in E\}$ for the topology of pointwise convergence, its elements are called *Martin* functions. The elements of the *boundary* $\mathscr{H} := \mathscr{M} \setminus m(E)$ are called *horofunctions*. The horofunctions represent the direction at infinity in the space (E, δ).

Example 2.1.25 *If (E, δ) is an Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, then it is easy to verify that $\mathscr{H}$ is the set of functions of the form $e \to \langle u, e \rangle$ for u of norm 1.*

Example 2.1.26 *In the case of the top hemi-metric and finite Ω, $\mathscr{M}$ is the set of functions of the form*

$$m(f) = \inf_{\omega \in A} (\alpha_\omega + f(\omega))$$

where A is any nonempty set in Ω and the α_ω are such that $\inf_{\omega \in A} \alpha_\omega = 0$; and $\mathscr{H}$ is the set of such functions with $A \neq \Omega$.

Proof. Take 0 as a basepoint and let $m(f_n)$ be a sequence of functions

$$[m(f-n)](f) := \mathbf{t}(f_n) - \mathbf{t}(f_n - f).$$

Up to extraction assume that $\mathbf{t}(f_n) = f_n(\omega_0)$ for some $\omega_0 \in \Omega$; since $m(f' + c) = m(f')$ for every constant c one may also assume that $f_n(\omega_0) = 0$. Up to extraction again, for every ω in Ω, $f_n(\omega)$ converges to some $-\alpha_\omega$ with $\alpha_\omega \in [0, +\infty]$ and $\alpha_{\omega_0} = 0$. This proves the result with $A = \{\omega, \alpha_\omega < +\infty\}$. ■

We can now prove our main result

Theorem 2.1.27 *Let (E, δ) be a complete metrically star-shaped hemi-metric space, and let $\Psi : E \to E$ be non-expansive. Then there exists a Martin function $h \in \mathscr{M}$ such that for all $e \in E$,*

$$h(\Psi(e)) \geq h(e) + \overline{\chi}(\Psi) \ .$$

This implies in particular that $\overline{\chi}(\Psi) = \chi(\Psi)$

Proof. Let e° and $\{\gamma_e\}_{e\in X}$ as in Definition 2.1.21; for any $\lambda \in [0, 1[$ denote by $r^\lambda : X \to X$ the function $y \to \gamma_y(\lambda)$ and recall that r^λ is λ-contracting by definition. By completeness, we can thus define for any λ the point $e_\lambda \in X$ as the only solution of the fixed point equation

$$\Psi(r^\lambda(e_\lambda)) = e_\lambda.$$

e_λ is the natural generalization of the λ-discounted value of a game in this more general setting. Then for any $e \in E$,

$$\begin{aligned}
\delta(e, e_\lambda) - \delta(\Psi(e), e_\lambda) &= \delta(e, e_\lambda) - \delta\left(\Psi(e), \Psi(r^\lambda(e_\lambda))\right) \\
&\geq \delta(e, e_\lambda) - \delta\left(e, r^\lambda(e_\lambda)\right) \quad \text{(by non-expansiveness)} \\
&\geq \delta(e, e_\lambda) - \delta\left(e, r^\lambda(e)\right) - \delta\left(r^\lambda(e), r^\lambda(e_\lambda)\right) \\
&\geq (1-\lambda)\delta(e, e_\lambda) - \delta\left(e, r^\lambda(e)\right) \quad \text{(since } r^\lambda \text{ is an } \lambda\text{-contraction)} \\
&\geq (1-\lambda)\delta(e^\circ, e_\lambda) - (1-\lambda)\delta(e^\circ, e) - \delta\left(e, r^\lambda(e)\right) \\
&= \delta(r^\lambda(e_\lambda), e_\lambda) - (1-\lambda)\delta(e^\circ, e) - \delta\left(e, r^\lambda(e)\right) \\
&\geq \delta\left(\Psi(r^\lambda(e_\lambda), \Psi(e_\lambda))\right) - (1-\lambda)\delta(e^\circ, e) - \delta\left(e, r^\lambda(e)\right) \\
&\qquad \text{(by non-expansiveness)} \\
&= \delta\left(e_\lambda, \Psi(e_\lambda))\right) - (1-\lambda)\delta(e^\circ, e) - \delta\left(e, r^\lambda(e)\right) \\
&\geq \overline{\rho}(\Psi) - (1-\lambda)\delta(e^\circ, e) - \delta\left(e, r^\lambda(e)\right).
\end{aligned}$$

Since the space $\mathscr{M}$ is compact, the family of functions $(m(e_\lambda))_{0<\lambda<1}$ admits a limit point $h \in \mathscr{M}$ as λ tends to 1. Passing to the limit in the previous inequality, and using the additional assumption in Definition 2.1.21, we deduce that $-h(e) + h(\Psi(e)) \geq \overline{\chi}(\Psi)$. Adding those inegalities for e, $\Psi(e)$, $\cdots$ we get

$$k\overline{\chi}(\Psi) \ \leq \ h(\Psi^k(e)) - h(e) \leq \delta(e, \Psi^k(e))$$

which yields $\overline{\chi}(\Psi) \leq \chi(\Psi)$. Then, the result follows by Lemma 2.1.24. ■

Hence asymptotically $\Psi^k(e)$ goes in the "direction" h. In particular considering a Shapley operator Ψ and the top hemi metric one gets

Corollary 2.1.28 *For any compact/compact game, $\mathbf{t}(v_n)$ converges as n tends to infinity to $\overline{\chi}(\Psi) = \inf_{f\in\mathcal{F}} \mathbf{t}(\Psi(f) - f)$. Moreover, there exists an asymptotically best starting state ω for Player 1: $v_n(\omega) \geq \overline{\chi}(\Psi)$ for all n. In particular $v_n(\omega)$ converges to $\overline{\chi}(\Psi)$*

Sketch of the proof. Assume that Ω is finite for simplicity. Then the characterization of the horofunctions in example 2.1.26, as well as Theorem 2.1.27, immediately gives the existence of ω such that $\Psi^n(0)(\omega) \geq \Psi^{n-1}(0)(\omega) + \overline{\chi}(\Psi)$ hence $\mathbf{t}(v_n) \geq v_n(\omega) \geq \overline{\chi}(\Psi)$ for all n. We can conclude since by (2.1.8) $\mathbf{t}(v_n)$ converges to $\chi(\Psi) = \overline{\chi}(\Psi)$ ■

Remark 2.1.29 *Of course a dual argument shows the existence of a best starting state for Player 2.*

Corollary 2.1.28 in itself is not a new result, in fact it is a particular case of the following result [46, 64] which, in turns, can be viewed as a particular case of Theorem 2.1.27 when $(E, \|\cdot\|)$ is a Banach set and δ is a suitable hemi metric whose associated metric is $\|\cdot\|$:

Proposition 2.1.30 *Let $(X, \|\cdot\|)$ be a Banach space, let $T : X \to X$ be non-expansive and assume that $\overline{\rho}(T) > 0$. Then for any $x \in X$, there exists a continuous linear form ϕ of norm one, such that*

$$\phi(T^k(x)) \geq \phi(x) + k\overline{\rho}(T) \tag{2.1.9}$$

for all $k \in \mathbb{N}$. Moreover, ϕ can be taken in the weak-star closure of the set of extreme points of the dual unit ball.

Let us conclude this section by giving, without proof, an application of Theorem 2.1.27 to a (seemingly) totally different framework. Fix an integer n, and let $U = (\mathbb{R}_+^*)^n$. One verifies that the following is an hemi-metric on U:

$$\text{RFunk}(u, u') = \log \max_{1 \leq i \leq n} \frac{u'_i}{u_i},$$

that (U, RFunk) is metrically star-shaped, and $u \to Mu$ is nonexpansive (for RFunk) for any matrix $M = \{m_{ij}\}$ with positive m_{ij}. Then Theorem 2.1.27 implies the classical Collatz-Wielandt formula characterizing the Perron root $r(M)$ of M:

$$\inf_{u \in U} \max_{1 \leq i \leq n} \frac{(Mu)_i}{u_i} = r(M) = \max_{\substack{u \in \mathbb{R}_+^n \\ u \neq 0}} \min_{\substack{1 \leq i \leq n \\ u_i \neq 0}} \frac{(Mu)_i}{u_i} \ . \tag{2.1.10}$$

The reader is referred to [GV12] for more details and applications to more general symmetric cones.

2.2 Some games for which the values do not converge

This part is based on [Vig13, SV15] and presents examples of games for which v_λ does not converge.

Let us first, using the previous section, exhibit a subclass of compact games that is likely to contain a counterexample (if such a counterexample exists). We would like for this class to be as small as possible, in order to be more likely to find a precise counterexample within. By Corollary 2.1.28 and its dual, for any game v_λ converges in at least two states, and to simplify we may as well assume these are absorbing. Since one is the best starting state for Player 1 and the other is the best starting state for Player 2, they must have different absorbing payoffs (or else v_λ would converge to a constant), say -1 and 1. Since

v_λ converges in an absorbing game, there must be at least two nonabsorbing states in a counterexample, and we consider the simplest case in which there are exactly two.

Also remark that the transitions functions, rather than the payoff functions, are most likely a source of oscillations of the values v_λ. A small variation of the payoff function g induces a small variation of all values v_λ (uniformly on λ); it is not the case for small variations of ρ. So, once again to simplify as much as possible, we assume that the payoff does not depend on the actions played by the player. If the payoff was the same in the two states the game would be recursive (up to the addition of a constant) and v_λ would converge, so the payoff in the two nonabsorbing states must be different, say -1 and 1.

It remains to understand why some transition functions would be problematic. For this let us briefly study some finite games having all the preceeding features, to understand why v_λ converge in the finite case and might not in the compact one. It turns out that such a game was already studied by Bewley and Kohlberg [11] and we consider a family of games in the same spirit[1] parametrized by some p_+^* and p_-^* in $[0,1]$.

- There are two nonabsorbing states ω_+ and ω_-, and two absorbing states 1^* and -1^*.
- Both players have two pures actions, Stay and Quit.
- The payoff in each state is independent of the actions: it is 1 in ω_+ and 1^* ; -1 in ω_- and -1^*.
- The transitions are given by the matrices in Figure 2.1

ω_-	Stay	Quit
Stay	ω_-	ω_+
Quit	ω_+	$(p_-^*)-1^* + (1-p_-^*)\omega_-$

ω_+	Stay	Quit
Stay	ω_+	ω_-
Quit	ω_-	$(p_+^*)1^* + (1-p_+^*)\omega_+$

Figure 2.1: Generalized Bewley Kohlberg game

Calculations then show that:

- $\lim v_\lambda = v$ with $v(\omega_+) = v(\omega_-) = \frac{\sqrt{p_+^*}-\sqrt{p_-^*}}{\sqrt{p_+^*}+\sqrt{p_-^*}}$.
- Optimal mixed actions in Γ_λ are given, for $k \in \{+,-\}$, by $x_\lambda(\omega_k) = y_\lambda(\omega_k) \approx \frac{\sqrt{\lambda}}{\sqrt{p_k^*}}$ as λ goes to 0 (we identify a mixed action with the probability assigned to Q).

Recall that in any one-shot zero-sum game, if an optimal action of a player is completely mixed, any optimal action of the other player is equalizing. Thus, since both x_λ and y_λ are completely mixed, they are both equalizing in Γ_λ.

Taking the mixed extension of the finite game in Figure 2.1 we get a compact game Γ^c. The (now pure) action x_λ and y_λ are optimal in Γ_λ^c. Since we want to discuss the influence of the parameters of the game on the transitions under optimal play, it is convenient to

[1] Their example is the particular case of $p_+^* = p_-^* = 1$, with slight modifications of the payoffs that do not change the asymptotics of the values

relabel the actions so that the optimal action of a player in Γ_λ depends only on λ and not on p_+^* and p_-^*. By some suitable change of variables for the actions of each player in each state we get a compact game such that the stationary strategy λ in each state is optimal (and equalizing) for each player in Γ_λ^c. We have thus constructed a compact game such that:

- There are two nonabsorbing states ω_+ and ω_-, and two absorbing states 1^* and -1^*.
- The set of actions of each player is $[0,1]$.
- In each state, for each player, the pure action λ is equalizing in Γ_λ for λ small enough.
- The transition are approximately

$$\begin{aligned}
\rho(\omega_-|i,j,\omega_+) &\approx \frac{\sqrt{i}+\sqrt{j}}{\sqrt{p_+^*}} \\
\rho(\omega_+|i,j,\omega_-) &\approx \frac{\sqrt{i}+\sqrt{j}}{\sqrt{p_-^*}} \\
\rho(1^*|i,j,\omega_+) &\approx \frac{\sqrt{i}\sqrt{j}}{p_+^*} \\
\rho(-1^*|i,j,\omega_-) &\approx \frac{\sqrt{i}\sqrt{j}}{p_-^*}
\end{aligned}$$

- The limit values satisfy

$$v(\omega_+) = v(\omega_-) = \frac{\sqrt{p_+^*}-\sqrt{p_-^*}}{\sqrt{p_+^*}+\sqrt{p_-^*}} = \frac{1-\sqrt{\frac{p_-^*}{p_+^*}}}{1+\sqrt{\frac{p_-^*}{p_+^*}}}. \tag{2.2.1}$$

While these games are compact games, there are very specific ones since they are (up to a change of variables) mixed extensions of finite games. In particular the transitions functions are linear (up to a change of variables), and this is what entails the convergence of v_λ. A natural idea is to use the additional freedom in general compact games with interval action sets to construct a similar game such that $\rho(\omega_-|i,j,\omega_+) = \frac{\sqrt{i}+\sqrt{j}}{\sqrt{p_+^*(i,j)}}$ (where p_+^* is no longer a constant but a function of i and j), and similar formulas for the other transitions. If p_-^* and p_+^* go to 0 but $\frac{p_-^*}{p_+^*}$ is slowly oscillating between two positive constants(which could not happen, by linearity, in the finite case), we expect that the value v_λ also oscillates and thus does not converge.

Let us thus consider thus the class $\mathcal{G}$ of compact stochastic games satisfying the following properties:

a) There are two nonabsorbing states ω_+ and ω_-, and two absorbing states 1^* and -1^*.

b) The action set of each player (denoted by I and J respectively) is the interval $\left[0, \frac{1}{16}\right]$.

c) The payoff depends only of the state: for all actions i and j, $g(i,j,\omega_+) = g(i,j,1^*) = 1$ and $g(i,j,\omega_-) = g(i,j,-1^*) = -1$.

d) The transition probability ρ is (jointly) continuous, and for all actions i and j, $\rho(-1^*|i,j,\omega_+) = \rho(1^*|i,j,\omega_-) = 0$.

e) In each nonabsorbing state and for each player, the pure action λ is equalizing in the discounted game Γ_λ.

Definition 2.2.1 *A pair (s,d) of continuous functions from $]0,\frac{1}{16}]$ to $\mathbb{R}$ is* feasible *if there exists a game in $\mathcal{G}$ such that*

$$\begin{aligned} v_\lambda(\omega_+) &= s(\lambda) + d(\lambda) \\ v_\lambda(\omega_-) &= s(\lambda) - d(\lambda). \end{aligned}$$

Remark that by Corollary 2.1.17, in any potential counterexample d_λ has to converge to 0. Also if $d(\lambda)$ converges very fast then the values would not change much if we replaced any transition from ω^+ to ω^- by a transition from ω^+ to ω^+, and any transition from ω^- to ω^+ by a transition from ω^- to ω^- ; but the resulting game would just be two absorbing games played in parallel, and absorbing games have an asymptotic value. Hence it makes sense to consider $d(\lambda) = \sqrt{\lambda}$. Concerning s, it has to be bounded by 1 ; and it is standard that $\lambda \frac{v_\lambda - v_\mu}{\lambda - \mu}$ is bounded for any bounded game, hence if s is differentiable $xs(x)$ has to be bounded as well.

It turns out that this necessary conditions are almost sufficient:

Proposition 2.2.2 *Let $s \in C^1(]0,\frac{1}{16}],\mathbb{R})$. Assume that s and $x \to xs'(x)$ are both bounded by $\frac{1}{16}$. Then $(s,\sqrt{\ })$ is feasible.*

Sketch of the proof. The functions $v_\lambda(\omega_+)$ and $v_\lambda(\omega_-)$ being fixed, write for each couple λ,μ the fact that the pure action λ (resp. μ) is equalizing in $\Gamma_\lambda(\omega^+)$ and $\Gamma_\lambda(\omega^-)$ (resp. in $\Gamma_\mu(\omega^+)$ and $\Gamma_\mu(\omega^-)$). This gives four equations in four unknowns $\rho(1^*|i,j,\omega_+)$, $\rho(\omega_-|i,j,\omega_+)$, $\rho(-1^*|i,j,\omega_-)$, and $\rho(\omega_+|i,j,\omega_-)$. Solving this thus gives candidates for the transition function, and one verifies that the hypotheses on s ensure that these functions are well defined, positive and continuous. ■

In particular taking $s(x) = x \sin \ln x$ one gets [Vig13]

Corollary 2.2.3 *There exists a game in $\mathcal{G}$ such that v_λ does not converge as λ goes to 0.*

Using Proposition 2.4.3 (see Section 2.4.2) one also get examples in which v_n does not converge.

Let us now present a general method to construct counterexamples [SV15]. A *configuration* C is a zero-sum repeated game with a specific starting state $\overline{\omega}$ and an exit absorbing state $\omega^* \neq \overline{\omega}$, such that the payoff is independent on the actions and is some α in $\Omega \setminus \omega^*$ and $\beta \neq \alpha$ in ω^*. If $\beta > \alpha$ we say the configuration is of type 1, and of type 2 if $\beta < \alpha$. Since there are only two payoffs, and the payoff β is only obtained in the absorbing state ω^*, the only interesting variable is the time of exit S:

$$S = \min\{n \in \mathbb{N},\ \omega_n = \omega^*\}$$

where ω_n is the state at stage n. For simplicity let us only consider discounted games. For each couple (σ, τ) of stationary strategies of the players and every discount factor λ, $d_\lambda(\sigma, \tau)$ is the expected (normalized) duration spent in $\overline{\Omega}$:

$$d_\lambda(\sigma, \tau) = E_{\sigma,\tau}[\lambda \sum_{n=1}^{S-1} (1-\lambda)^{n-1}].$$

Define the *inertia rate* of the configuration by $Q_\lambda = \sup_\tau \inf_\sigma d_\lambda(\sigma, \tau)$ if the game is of type 1, and $Q_\lambda = \inf_\tau \sup_\sigma d_\lambda(\sigma, \tau)$ if the the game is of type 2. Then clearly

Lemma 2.2.4 *For any $\alpha \neq \beta$ and discount factor λ the λ-discounted configuration has a value v_λ and*

$$v_\lambda = \alpha Q_\lambda + \beta(1 - Q_\lambda).$$

Example 2.2.5 *Assume there are only two states in $\overline{\Omega}$: the starting state $\overline{\omega}$ and an absorbing state α^*. The configuration is of type 1, and the transition from $\overline{\omega}$ to ω^* (resp. to α^*) is $a(i,j)$ (resp. $b(i,j)$), see[2] Figure 2.2*

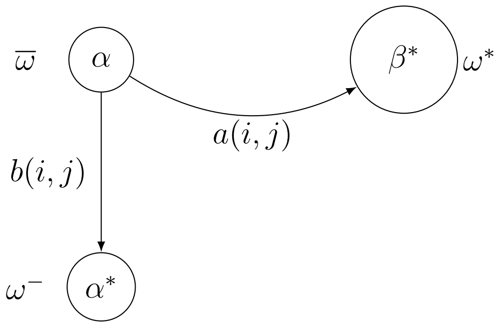

Figure 2.2: An example of configuration

Then $Q_\lambda = \min_{x\in\Delta(I)} \max_{y\in\Delta(J)} \frac{(\lambda+(1-\lambda)b(x,y))}{\lambda+(1-\lambda)(a(x,y)+b(x,y))}$.

Consider now a two person zero-sum dynamic game Γ on $\overline{\Omega}^1 \cup \overline{\Omega}^2$ generated by two configurations C^1 and C^2 of type 1 and 2 respectively, which are coupled in the following sense: the exit state from C^1, ω^{*1} is the starting state $\overline{\omega}^2$ in C^2 and vice versa. In addition the exit events are known by the players : any transition from $\overline{\Omega}^i$ to $\overline{\omega}^{-i}$ is observed by both. Finally the payoff is $\alpha^1 = -1$ on $\overline{\Omega}^1$ and $\alpha^2 = 1$ on $\overline{\Omega}^2$.

We thus obtain a reversible game (it is possible to go from $\overline{\Omega}^1$ to $\overline{\Omega}^2$ and vice versa) in which Player 1 minimize (reps. maximize) the expected time spent in $\overline{\Omega}^1$ (resp. in $\overline{\Omega}^2$).

Example 2.2.6 *Assume there are only two states in each $\overline{\Omega}^i$: the starting state $\overline{\omega}^i$ and an absorbing state. Then the game can be represented as in Figure 2.3*

[2]in the figures we usually do not write the probability of a transition from a state to itself, since it is just the complement to 1 of the other probabilities.

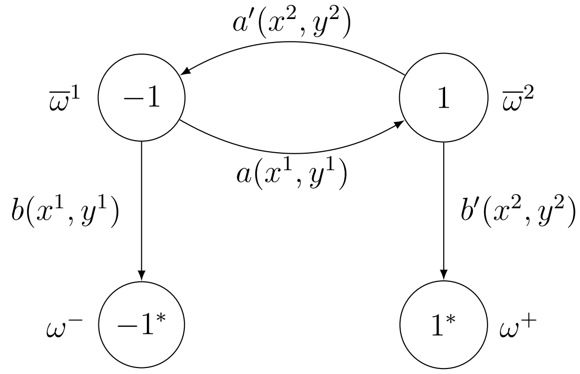

Figure 2.3: A reversible game

For example, the (mixed extension of) the original Bewley Kohlberg example given in Figure 2.1 (with $p^*_+ = p^*_- = 1$) is given by $a(x, y) = a'(x, y) = x + y - 2xy$ and $b(x, y) = b'(x, y) = xy$.

Then using Lemma 2.2.4 twice we get

Proposition 2.2.7

$$\begin{aligned} v_\lambda(\overline{\omega}^1) &= \frac{Q^2_\lambda - Q^1_\lambda - Q^1_\lambda Q^2_\lambda}{Q^1_\lambda + Q^2_\lambda - Q^1_\lambda Q^2_\lambda} \\ v_\lambda(\overline{\omega}^2) &= \frac{Q^2_\lambda - Q^1_\lambda + Q^1_\lambda Q^2_\lambda}{Q^1_\lambda + Q^2_\lambda - Q^1_\lambda Q^2_\lambda} \end{aligned}$$

Corollary 2.2.8 *Assume that Q^i_λ goes to 0 as λ goes to 0 for $i = 1, 2$. Then*

$$v_\lambda(\overline{\omega}^2) \sim v_\lambda(\overline{\omega}^1) \sim \frac{1 - \frac{Q^1_\lambda}{Q^2_\lambda}}{1 + \frac{Q^1_\lambda}{Q^2_\lambda}}. \tag{2.2.2}$$

In particular if $\frac{Q^1_\lambda}{Q^2_\lambda}$ has more than one accumulation point, then v_λ does not converge.

One can apply this to different frameworks :

Example 2.2.9 *For every s, the game defined in Proposition 2.2.2 can be written as in Figure 2.3. Then one computes that $Q^1_\lambda \sim \sqrt{\lambda}(1 + s(\lambda))$ and $Q^2_\lambda \sim \sqrt{\lambda}(1 - s(\lambda))$. Hence $\frac{Q^1_\lambda}{Q^2_\lambda}$ has more than one accumulation point for $s(x) = x \sin \ln x$ and we recover the fact that v_λ does not converge in this example.*

Example 2.2.10 *If one take the configuration coming from the left half of the Bewley-Kohlberg example, one computes that $Q^1_\lambda = \sqrt{\lambda}$.*

Consider now a configuration of type 2 as in (the dual of) Figure 2.2, controlled by Player 2 with $J = [0, 1]$, $a(j) = \sqrt{j}(2 + \sin(\ln(-\ln j)))$ and $b(j) = j$. Then one computes that

$$Q^2_\lambda \sim \frac{2\sqrt{\lambda}}{2 + \sin(\ln(-\ln \lambda))}.$$

Hence combining these configurations one gets a finite/compact game with finitely many actions for player 1 and in which v_λ does not converge.

Example 2.2.11 *Consider a configuration of type 1 in Figure 2.2 controlled by Player 1. If $I = [0,1]$, $a(i) = \sqrt{i}$ and $b(i) = i$, one compute easily that $Q^1_\lambda \sim 2\sqrt{\lambda}$.*

Combining this with the second configuration of the previous example, one gets a finite/compact game with perfect information in which v_λ does not converge. A similar game was already constructed in [97].

Some other examples that were constructed by Ziliotto in [97] can be rewritten as merging of two configurations, see [SV15] for details.

Finally let us stress the fact that the divergence of v_λ in all known examples seems to be explained by the meeting of two phenomenon:

- Micro "oscillations" of the inertia rate, or of the transition probabilities under optimal play.
- Reversibility of the game, which induces a division in the computation of the values (see formulas (2.2.1) and (2.2.2)).

When either the first phenomenon doesn't exist (e.g. in finite games) or the second one (e.g. in absorbing games) such examples do not appear.

2.3 Limit games and comparison with continuous time processes.

This part is based on [Vig10,SVV10,SV16,SV17]. In the first section we study games with vanishing duration as defined in Section 1.1.3. We study their asymptotic properties in two directions: when the time horizon or discount factor is fixed while the stage duration goes to 0 ; and when the stage duration is fixed while the time horizon goes to infinity or discount factor goes to 0.

In the second section we study asymptotic properties of the game under optimal play. For every discount factor λ one normalizes the game so that it is played on the time interval $[0,1]$. We then study the behavior, as λ goes to 0 and for a fixed t, of the accumulated payoff during time $[0,t]$, or of the occupation measure at time t.

2.3.1 Games with varying stage duration

This section follows [Vig10,SV16]. We are interested in the links between values of games with varying stage duration and iterations of the fractional Shapley operator (see Section 1.1.3). Our approach is in terms of operators and make few assumptions on the game: compact/compact framework when we study exact games, and finite/compact framework when we study the discretization of a continuous process.

a) **Exact games**

Let us first consider the values (in finite horizon or discounted) of exact games, see Section 1.1.3). We denote V_n^h the unormalized value of an n-stage game with stage duration h, that is $V_n^h = \Psi_h^n(0)$ with $\Psi_h = (1-h)Id + h\Psi$. One can write it as

$$V_n^h - V_{n-1}^h = -h(Id - \Psi)(V_{n-1}^h)$$

which is a discretization of the evolution equation in continuous time

$$\dot{f}_t = -(Id - \Psi)f_t, \quad f_0 = z. \tag{2.3.1}$$

Denote $f(z)$ the solution of this continuous equation. Then the comparison between the iterates of Ψ and the solution $f_t(z)$ of the differential equation (2.3.1) is given by the generalized Chernoff's formula [17, 62], see, e.g., Brézis [16], p.16:

Proposition 2.3.1

$$\|f_t(z) - \Psi^n(z))\| \leq \|z - \Psi(z)\| \sqrt{t + (n-t)^2}. \tag{2.3.2}$$

By doing a change of variables one gets

Proposition 2.3.2 *There exists a constant L such that for all n and $h \in [0,1]$*

$$\|V_n^h - f_{nh}(0)\| \leq Lh\sqrt{n}.$$

In particular,

- For a fixed t, $V_n^{\frac{t}{n}}$ tends to $f_t(0)$ as n tends to infinity. When Ψ is the Shapley operator of a game, it means that the value of a game with total length t and vanishing duration converges, to some $\widehat{V}_t$ that can be viewed as the value of the continuous time game of length t introduced in Neyman [66].
- For a fixed h, $\frac{V_n^h}{nh}$ has the same asymptotic behavior that $\frac{f_t}{t}$, independently of h. When Ψ is the Shapley operator of a game, it means that the (time normalized) asymptotic behavior of a game with fixed stage duration h is independent on h.

In fact, both of these properties are still true when h is no longer constant (that is when the game has a varying stage duration). For a sequence of step sizes $\{h_k\}$ in $[0,1]$ one defines inductively an Eulerian scheme $\{z_k\}$ by

$$z_{k+1} - z_k = h_{k+1}(\Psi - Id)(z_k)$$

or

$$z_{k+1} = \Psi_{h_{k+1}} z_k.$$

For two sequences $\{h_k\}, \{\hat{h}_\ell\}$ in $[0,1]$, with associated Eulerian schemes

$$z_{k+1} = \Psi_{h_{k+1}} z_k,$$

$$\hat{z}_{\ell+1} = \Psi_{\hat{h}_{\ell+1}} \hat{z}_\ell,$$

we obtain [Vig10] a generalization to the Banach setting of the Kobayashi formula [44], valid in Hilbert spaces.

Proposition 2.3.3

$$\|\hat{z}_\ell - z_k\| \le \|\hat{z}_0 - z\| + \|z_0 - z\| + \|z - \Psi z\| \sqrt{(\sigma_k - \hat{\sigma}_\ell)^2 + \tau_k + \hat{\tau}_\ell}, \quad \forall z \in z, \tag{2.3.3}$$

$$\|f_t(z) - z_k\| \le \|z - \Psi z\| \sqrt{(\sigma_k - t)^2 + \tau_k}, \tag{2.3.4}$$

with $z_0 = z$, $\sigma_k = \sum_{i=1}^k h_i$, $\tau_k = \sum_{i=1}^k h_i^2$, $\hat{\sigma}_\ell = \sum_{j=1}^\ell \hat{h}_j$, $\hat{\tau}_\ell = \sum_{j=1}^\ell \hat{h}_j^2$.

Combining propositions 2.3.2 and 2.3.3,

- For a fixed t, consider z_k for sequences h_i such that $\sum_{i=1}^k h_i = t$ and define h the mesh of the sequence (the maximum of the h_i). Then z_k tends to $f_t(0)$ when the mesh goes to 0. When Ψ is the Shapley operator of a game, it means that the value of a game with total length t and varying and vanishing duration converges to some $\widehat{V}_t$.

- For a fixed infinite sequence h_i with $\sum_{i=1}^{+\infty} h_i = +\infty$, $\frac{z_k}{\sum_{i=1}^k h_k}$ has the same asymptotic behavior (as k tends to infinity) that $\frac{f_t}{t}$, independently of the precise sequence of stepsizes. When Ψ is the Shapley operator of a game, it means that the (time normalized) asymptotic behavior of a game with varying stage duration is independent of the precise duration of each stage.

We consider now the same problematic but for discounted values and follow [SV16]. The λ-discounted value of the game with stage duration h is the unique fixed point of

$$v_\lambda^h = \lambda \Psi_h\left(\frac{1-\lambda h}{\lambda} v_\lambda^h\right). \tag{2.3.5}$$

It is straightforward to show that

Proposition 2.3.4

$$v_\lambda^h = v_\mu, \quad with \quad \mu = \frac{\lambda}{1 + \lambda - \lambda\, h}.$$

and we recover the convergence property in [65]:

- For a fixed λ, v_λ^h converges as h goes to 0. The limit, denoted $\widehat{v}_\lambda$, equals $v_{\frac{\lambda}{1+\lambda}}$. When Ψ is the Shapley operator of a game, it means that the value of a λ discounted game with vanishing duration converges, to some $\widehat{v}_\lambda$ that can be viewed as the value of the continuous time λ-discounted game introduced in Neyman [66].

- For a fixed h, v_λ^h has the same asymptotic behavior (as λ tends to 0) that $\widehat{v}_\lambda$, independently of h. When Ψ is the Shapley operator of a game, it means that the asymptotic behavior of a discounted game with fixed stage duration h is independent on h.

As for games with finite horizon one can also consider a sequence of stage durations $\{h_i\}$ with $h_i \le h$ and $\sum_i h_i = +\infty$ inducing a time partition H. The value of the associated λ-discounted game v_λ^H then satisfies

Proposition 2.3.5

$$\|v_\lambda^H - \widehat{v}_\lambda\| \leq 2\|\Psi(0)\|\, h\,.$$

So once again, v_λ^H converges for a fixed λ as the mesh h goes to 0. To study the asymptotic property, as λ goes to 0, of a game with varying stage duration one needs to make an hypothesis on Ψ:

Definition 2.3.6 *The operator Ψ satisfies assumption $(\mathcal{H})$ if there exists two nondecreasing functions $k :]0,1] \to \mathbb{R}^+$ and $\ell : [0,+\infty] \to \mathbb{R}^+$ with $k(\lambda) = o(\sqrt{\lambda})$ as λ goes to 0 and*

$$\|\Phi(\lambda, z) - \Phi(\mu, z)\| \leq k(|\lambda - \mu|)\ell(\|z\|) \quad (\mathcal{H}) \tag{2.3.6}$$

for all $(\lambda, \mu) \in]0,1]^2$ and $z \in \mathcal{F}$

where Φ is defined by (1.1.2). Observe that while $(\mathcal{H})$ is not verified by all nonexpansive maps, when Ψ is the Shapley operator of a game this property is automatic as long as the payoff function is bounded, which is assumed in all our game theoretic settings.

Then one obtains

Proposition 2.3.7 *Assume Ψ satisfies assumption $(\mathcal{H})$. Then for any $\{h_i\}$ with $\sum_i h_i = +\infty$ inducing a partition$(\mathcal{H})$, $\|v_\lambda^H - v_\lambda\| \leq C'\lambda$ for some constant C'*

and the asymptotic behavior (as λ goes to 0) of v_λ^H is the same as the one of v_λ.

Remark 2.3.8 *We gave results of two different types: either for a fixed stage duration h (or a sequence h_n) and a varying stage weight (horizon going to infinity or discount factor going to 0), or for a fixed stage weight and a vanishing stage duration. One may wonder what happens when both the stage duration and the stage weight go to 0. Since we proved that the asymptotic behavior of a game with vanishing duration is independent of the precise stage duration, this implies that these "double limit" yields the same asymptotic behavior as well.*

b) **Discretization**

We now use the same tools to study the discretization of a game played in continuous time, see Section 1.1.3 for the model. Recall that in contrast to the previous discussion, Ω is now assumed to be finite, and that the recursive structure of the game is given by the operators $\overline{\Psi}_h$ defined in (1.1.9). The un-normalized value $\overline{V}_n^h$ of the n-stage game with stage duration h satisfies $\overline{V}_n^h = (\overline{\Psi}_h)^n(0)$. Similarly for varying stage duration, corresponding to a partition H, one gets a recursive equation of the form $\overline{V}_H(t) = \prod_i \overline{\Psi}_{h_i}(0)$.

For small h, the operator $\overline{\Psi}_h$ is close to Ψ_h:

Lemma 2.3.9 *There exists C_0 such that*

$$\|\Psi_h(f) - \overline{\Psi}_h(f)\| \leq C_0(1 + \|f\|)h^2.$$

This implies that the results in the exact game framework concerning game with fixed length or discount factor and vanishing stage duration still hold.

Proposition 2.3.10 *There exists C depending only of the game G such that for any finite sequence $(h_i)_{i\leq n}$ in $[0,h]$ with sum t and corresponding partition H:*

$$\|\overline{V}_H(t) - \widehat{V}_t\| \leq C(\sqrt{ht} + ht + ht^2).$$

In particular for a given t, $\overline{V}_H(t)$ tends to $\widehat{V}_t$ as h goes to 0.

Remark 2.3.11 *For a given h, the right hand term is quadratic in t, hence we do not link the asymptotic behavior (as n tends to infinity) of the normalized quantity $\overline{v}_n^h = \frac{\overline{V}_n^h}{nh}$ and of $\widehat{v}_t = \frac{\widehat{V}_t}{t}$. However if n is a function of h converging slowly enough to infinity, the previous proposition can be used. For example for $n(h) = \frac{1}{h\sqrt{h}}$ (so that $t(h) = \frac{1}{\sqrt{h}}$), one has*

$$\|\overline{v}_{n(h)}^h - \widehat{v}_{t(h)}\| = O(\sqrt{h}).$$

The discounted case is similar, and we only consider fixed duration h for simplicity. The normalized value $\overline{w}_k^h$ of the discretization with mesh h of the λ-discounted continuous game satisfies the fixed point equation

$$\overline{w}_\lambda^h(\omega) = \mathrm{Val}_{X\times Y}\left[\int_0^h \lambda e^{-\lambda t} g(\omega_t, x, y) + e^{-\lambda h}\mathsf{P}^h(x,y)[\omega] \circ \overline{w}_\lambda^h\right].$$

and one obtains using the approximation property in Lemma 2.3.9

Proposition 2.3.12 *For a given λ, $\overline{w}_\lambda^h$ tends to $\widehat{w}_\lambda$ as h goes to 0.*

2.3.2 Asymptotic properties of time-normalized optimal trajectories

This section follows [SVV10, SV17].

As we will see in Section 2.4.1, a natural idea for one player games (in discrete or continuous time) is to rewrite them as optimization problems on an auxiliary set: the player optimizes his payoff on the set of all allowed trajectories. A related reformulation (see for example [12, 34]) is to consider the set of all possible occupational measures that the player can implement on the state Ω. Such an approach seems hopeless for general two player games, since no player controls these trajectories or occupational measures. Here we consider a more qualitative problem: what can be said about these occupational measures along optimal play ?

Let us consider two player stochastic games in the finite/compact setting. For any pair of stationary strategies $(\bar{x}, \bar{y}) \in X^\Omega \times Y^\Omega$, any state $\omega \in \Omega$ and any stage n, denote by $\gamma_n^{\omega,\bar{x},\bar{y}}$ the expected payoff at stage n under these stationary strategies if the starting state is ω. Also denote $q_n^{\omega,\bar{x},\bar{y}} \in \Delta(\Omega)$ the law of the state in stage n under these stationary strategies if the starting state is ω.

Definition 2.3.13 *For any $(\bar{x}, \bar{y}) \in X^\Omega \times Y^\Omega$, any discount factor λ, and any starting state ω, define*

$$Q_\lambda^{\omega,\bar{x},\bar{y}}\left(\lambda\sum_{k=1}^n (1-\lambda)^{k-1}\right) = \lambda \sum_{k=1^n} (1-\lambda)^{k-1} q_k^{\omega,\bar{x},\bar{y}}$$

and a linear interpolation between these dates

Hence for any $t \in [0,1]$, $Q_\lambda^{\omega,\bar{x},\bar{y}}(t) \in t\Delta(\Omega)$ represent the expected accumulated occupation measure[3] at (relative) time t under $\bar{x}$ and $\bar{y}$ in the λ-discounted game starting from ω. Similarly one defines $l_\lambda^{\omega,\bar{x},\bar{y}}(t)$, the expected accumulated payoff at (relative) time t under $\bar{x}$ and $\bar{y}$ in the λ-discounted game starting from ω:

Definition 2.3.14 *For any $(\bar{x},\bar{y}) \in X^\Omega \times Y^\Omega$, any discount factor λ, and any starting state ω, define the function $l_\lambda^{\omega,\bar{x},\bar{y}} : [0,1] \to \mathbb{R}$ by*

$$l_\lambda^{\omega,\bar{x},\bar{y}}\left(\lambda \sum_{k=1}^{n}(1-\lambda)^{k-1}\right) = \lambda \sum_{k=1}^{n}(1-\lambda)^{k-1}\gamma_k^{\omega,\bar{x},\bar{y}}$$

and a linear interpolation between these dates

Denote by $l_\lambda^{\bar{x},\bar{y}}$ and $Q_\lambda^{\bar{x},\bar{y}}$ the Ω-vectors of functions $l_\lambda^{\omega,\bar{x},\bar{y}}(\cdot)$ and $Q_\lambda^{\omega,\bar{x},\bar{y}}(\cdot)$ respectively. Limit behaviors for the payoff and occupation measures will be some accumulation points of the functions $l_\lambda^{\bar{x}_\lambda,\bar{y}_\lambda}$ and $Q_\lambda^{\bar{x}_\lambda,\bar{y}_\lambda}$ under optimal strategies $\bar{x}_\lambda$ and $\bar{y}_\lambda$ as λ tends to 0. More precisely, denote by X_λ^ε (resp. Y_λ^ε) the set of ε-optimal stationary strategies in Γ_λ for Player 1 (resp. for Player 2). Then we define (for simplicity we only consider the discounted framework):

Definition 2.3.15 *$l = (l^\omega)_{\omega\in\Omega}$ is a limit behavior for the accumulated payoff if :*

$$\forall\varepsilon > 0,\ \exists\lambda_0 > 0,\ \forall\lambda < \lambda_0,\ \exists\bar{x} \in X_\lambda^\varepsilon,\ \exists\bar{y} \in Y_\lambda^\varepsilon,\ \forall\omega\in\Omega,\ \forall t\in[0,1],\ \left|l^\omega(t) - l_\lambda^{\omega,\bar{x},\bar{y}}(t)\right| \le \varepsilon.$$

$Q = (Q^\omega)_{\omega\in\Omega}$ is a limit behavior for the accumulated occupation measure if :

$$\forall\varepsilon > 0,\ \exists\lambda_0 > 0,\ \forall\lambda < \lambda_0,\ \exists\bar{x} \in X_\lambda^\varepsilon,\ \exists\bar{y} \in Y_\lambda^\varepsilon,\ \forall\omega\in\Omega,\ \forall t\in[0,1],\ \left\|Q^\omega(t) - Q_\lambda^{\omega,\bar{x},\bar{y}}(t)\right\| \le \varepsilon.$$

Alternate weaker and stronger definitions : in both cases, if "$\forall\lambda < \lambda_0$" is replaced by "for some λ_n going to 0", we will speak of a weak limit behavior. If "$\exists\bar{x} \in X_\lambda^\varepsilon,\ \exists\bar{y} \in Y_\lambda^\varepsilon$" is replaced by "$\exists\varepsilon' < \varepsilon,\ \forall\bar{x} \in X_\lambda^{\varepsilon'},\ \forall\bar{y} \in Y_\lambda^{\varepsilon'}$", we will speak of a strong limit behavior.

Some immediate remarks are:
- a weak limit behaviour always exists by standard arguments of equicontinuity.
- if a limit behaviour l exists, v_λ converges uniformly to $l(1)$.
- if a strong limit behaviour exists, it is unique.
- no strong limit behaviour exists in general for the accumulated occupation measure (just consider a game where payoff is always 0).

Very natural questions include:
- Existence of a limit behaviour ? Uniqueness ? Existence of a strong limit behaviour for the accumulated payoff ?
- Qualitative properties of this limit behaviours ? In particular, does there always exist some limit l that is linear in time ?

[3] Note that we do not normalize by dividing by t, to avoid complications at $t = 0$.

Observe that the last property means that, when one embed time in $[0,1]$, the payoff is asymptotically constant under optimal play. For example consider a Markov chain with two states of payoff 0 and 1, and deterministic transition from one state to the other. While the stage payoff is either 0 or 1, one computes that $l(t) = \frac{t}{2}$ for each starting state: when one zooms in the payoff is $\frac{1}{2}$ at every time t.

In [SVV10] we prove

Proposition 2.3.16 *Assume the game is controlled by Player 1 and that v_λ converges uniformly to v when λ goes to 0. Then $l^\omega(t) = tv(\omega)$ is the strong limit behavior for the accumulated payoff.*

Sketch of the proof. Assume for simplicity that there is only one player, transitions are deterministic, Player 1 plays in pure strategies, and the payoffs are in $[0,1]$. We show that for any ϵ, if λ is small enough and $\bar{x}$ is ε-optimal in Γ_λ then

$$-3\varepsilon \leq l_\lambda^{\omega,\bar{x}}(t) - tv(\omega) \leq 3\varepsilon.$$

It is clear for $t \notin [\varepsilon, 1-\varepsilon]$. For $t \in [\varepsilon, 1-\varepsilon]$, the lower bound inequality is a direct implication of the fact that v_λ is nonincreasing on any trajectory in any one player game. For the upper bound, one uses the fact that the discounted sums $(1-\lambda)^{-N}\sum_{k=1}^{N}\lambda(1-\lambda)^{k-1}\gamma_k$ belong to the convex hull of the averages $\frac{1}{n}\sum_{k=1}^{n}\gamma_k; 1 \leq n \leq N$; the Tauberian theorem of [53] ; and the uniform convergence of v_λ. ■

A similar result is obtained in the framework of optimal control. Hence, in any regular dynamic programming problem, the payoff obtained during optimal play is essentially constant. Recall that this is not necessarily the case if one assumes only convergence of v_n and v_λ for every starting state : in [53] an example is given where for some starting state, and along any optimal trajectory,

- In the n stage game the payoff is 0 in the first half of the game, and 1 in the second half.
- In the λ-discounted game, the payoff is 0 in the first quarter of the game, 1 in the second quarter, and 0 again in the last half.

For two player games controlled by both players the result is no longer true, even if v_λ converges uniformly. See [SVV10] for an example with finite action and countable state spaces, and below for a finite/compact absorbing game ; in both examples there is no strong limit behavior for the accumulated payoff.

We can however prove some positive results in the class of absorbing games. First of all,

Proposition 2.3.17 *Let Γ be a finite/absorbing game with*[4] *a continuous payoff function. Then $l(t) = tv$ is a limit behavior when starting in the non absorbing starting, and there exists $\gamma \in [0, +\infty]$ such that $Q(t) = \frac{1-(1-t)^{1+\gamma}}{1+\gamma}$ is a limit behaviour for the occupation time in the non absorbing state when starting in this state.*

[4] usually one only needs separately continuous functions in the finite/compact setting but joint continuity seems crucial here

Example 2.3.18 *In the Big Match [38], $\gamma = 1$ hence $Q(t) = t - \frac{t^2}{2}$, meaning that at relative time t, the probability $1 - Q'(t)$ that the game has already entered an absorbing state is t.*

The proof grounds on the following Lemma, proved in [19, 50] in the finite/finite setting and extended in [SV17] to the finite/compact setting. We use the notations of Section 2.1.2

Lemma 2.3.19 *Let Γ be a finite/compact absorbing game with a continuous payoff function. Define for any $(x, x', a, y, y', b) \in \Delta(I)^2 \times \mathbb{R}^+ \times \Delta(J)^2 \times \mathbb{R}^+$,*

$$A(x, x', a, y, y', b) = \frac{g(x, y) + a\, p^*(x', y)\overline{g}^*(x', y) + b\, p^*(x, y')\overline{g}^*(x, y')}{1 + a\, p^*(x', y) + b\, p^*(x, y')}. \tag{2.3.7}$$

Then
1)

$$v = \sup_{(x,x',a)\in\Delta(I)^2\times\mathbb{R}^+} \inf_{(y,y',b)\in\Delta(J)^2\times\mathbb{R}^+} A(x, x', a, y, y', b) \tag{2.3.8}$$
$$= \inf_{(y,y',b)\in\Delta(J)^2\times\mathbb{R}^+} \sup_{(x,x',a)\in\Delta(I)^2\times\mathbb{R}^+} A(x, x', a, y, y', b) \tag{2.3.9}$$

2) Moreover, if (x, x', a) is ε-optimal in the above sup inf formula, then for any λ small enough the stationary strategy $\hat{x}_\lambda := \frac{x+\lambda a x'}{1+\lambda a}$ is 2ε-optimal in Γ_λ.

Then one consider near optimal strategies of the form given by 2) to establish Proposition 2.3.17. In the finite case one can even prove

Proposition 2.3.20 *In any finite absorbing game, $l(t) = tv$ is a strong limit behavior for the payoff .*

Sketch of the proof. Let x_λ and y_λ be familes of (to simplify) 0-optimal strategies, with limit x and y. Consider $p^*(x_\lambda, y_\lambda)$ the probability of absorption in each stage in the λ-discounted game. If $p^*(x_\lambda, y_\lambda) = o(\lambda)$ then absorption (almost) never happen in the game ; if $\lambda = o(p^*(x_\lambda, y_\lambda))$ it occurs at time $t = 0$ as λ goes to 0. In both cases the payoff is clearly constant along the play. Hence the only case to consider is when, up to extraction, $p^*(x_\lambda, y_\lambda)$ converges to some Cλ. If the absorbing payoff $g^*(x, y)$ and non absorbing payoff $g(x, y)$ are equal at the limit, then clearly the payoff is constant so assume for example that $g^*(x, y) < v < g(x, y)$. Since $p^*(x_\lambda, y_\lambda)$ tends to $C\lambda$, for every couple (i, j) of pure actions with $p^*(i, j) > 0$, at least one player plays i with probability $o(1)$. Assume for simplicity that it is always Player 1. Consider the following deviation : Player 1 no longer plays these moves i and transfer their probabilities to some arbitrary actions he plays with non vanishing probability. This deviation gives a payoff $g(x, y)$ when λ tends to 0, so by optimality of y_{λ_n} we get $g(x, y) \leq v$ and a contradiction. ■

Interestingly this is no longer true in finite/compact absorbing games.

Example 2.3.21 *We consider the following absorbing game with compact actions sets. There are three states, two absorbing 0^* and -1^* , and the non absorbing state ω, in which the payoff is 1 whatever the actions taken. The sets of action are $I = J = \{0\} \cup \{1/n,\ n \in \mathbf{N}^*\}$ with the usual distance. The probabilities of absorption are given by :*

$$\rho(0^*|x,y) = \begin{cases} 0 & \text{if } x = y \\ \sqrt{y} & \text{if } x \neq y \end{cases}$$

and

$$\rho(-1^*|x,y) = \begin{cases} y & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Then both functions $\rho(0^*|\cdots)$ and $\rho(-1^*|\cdots)$ are (jointly) continuous. It is easy to verify that $v_\lambda = \lambda$ and that for any discount factor $\lambda \in]0,1]$, the action 0 (resp. 1) is optimal for Player 1 (resp. Player 2) in the λ-discounted game. This yields $l(t) = 0$ as a limit behavior for the payoff. However, denoting $[\cdot]$ the integer part of a real, for any λ the strategy $\frac{1}{[1/\lambda]}$ is λ-optimal for Player 1 and $\sqrt{\lambda}$-optimal for Player 2 in the λ-discounted game, and this couple of near optimal strategies yield another, non linear limit behavior for the payoff: $l(t) = t - t^2$. If the player follows these near optimal trajectories the payoff is positive during the beginning of the game and negative at the end.

2.4 Comparison between v_λ, v_n, and general v_θ

In this section we study the relation between values with different families of evaluation in two settings: continuous time and one player in the first subsection, and discrete two player games in the second one.

2.4.1 Tauberian theorem in optimal control

This part presents a result from [OBV13]. We consider a deterministic dynamic programming problem in continuous time, defined by a measurable set of states Ω, a subset $\mathcal{T}$ of Borel-measurable functions from $\mathbb{R}_+$ to Ω, and a bounded Borel-measurable real-valued function g defined on Ω. Without loss of generality we assume $g : \Omega \to [0,1]$. For a given state x, define $\Gamma(x) := \{X \in \mathcal{T},\ X(0) = x\}$ the set of all feasible trajectories starting from x. We assume $\Gamma(x)$ to be non empty, for all $x \in \Omega$. Furthermore, the correspondence Γ is closed under concatenation: given a trajectory $X \in \Gamma(x)$ with $X(s) = y$, and a trajectory $Y \in \Gamma(y)$, the concatenation of X and Y at time s is

$$X \circ_s Y := \begin{cases} X(t) & \text{if } t \leq s \\ Y(t-s) & \text{if } t \geq s \end{cases} \tag{2.4.1}$$

and we assume that $X \circ_s Y \in \Gamma(x)$.

We also assume that if $X \in \Gamma(x)$, then for any $t > 0$, the trajectory $s \to X(t+s)$ is in $\Gamma(X(t))$. That is, any trajectory obtained by cutting the beginning of a feasible trajectory is also feasible.

We are interested in the asymptotic behavior of the average and the discounted values. It is useful to denote the average payoff of a play (or trajectory) $X \in \Gamma(x)$ by:

$$\gamma_t(X) := \frac{1}{t}\int_0^t g(X(s))ds \tag{2.4.2}$$

$$\nu_\lambda(X) := \lambda \int_0^{+\infty} \mathbb{E}^{-\lambda s} g(X(s))ds \ . \tag{2.4.3}$$

This is defined for $t, \lambda \in]0, +\infty[$. We define[5] the values as:

$$V_t(x) = \sup_{X\in\Gamma(x)} \gamma_t(X) \tag{2.4.4}$$

$$W_\lambda(x) = \sup_{X\in\Gamma(x)} \nu_\lambda(X) \ . \tag{2.4.5}$$

We then prove

Proposition 2.4.1

$$(A)\ W_\lambda \underset{\lambda\to 0}{\longrightarrow} V,\ \textit{uniformly on } \Omega \Longleftrightarrow (B)\ V_t \underset{t\to\infty}{\longrightarrow} V,\ \textit{uniformly on } \Omega\ . \tag{2.4.6}$$

Notice that our model is a natural adaptation to the continuous-time framework of deterministic dynamic programming problems played in discrete time and this theorem is an extension to the continuous-time framework of the main result of [53]. This result can be applied to optimal control for a model that we describe now.

Consider some controlled dynamic over $\mathbb{R}_+$

$$\begin{cases} y'(s) = f(y(s), u(s)) \\ y(0) = y_0 \end{cases} \tag{2.4.7}$$

where y is a function from $\mathbb{R}_+$ to $\mathbb{R}^n$, y_0 is a point in $\mathbb{R}^n$, u is the control function which belongs to $\mathcal{U}$, the set of Lebesgue-measurable functions from $\mathbb{R}_+$ to a metric space U and the function $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfies the usual conditions, that is: Lipschitz with respect to the state variable, continuous with respect to the control variable and bounded by a linear function of the state variable, for any control u.

Together with the dynamic, an objective function g is given, interpreted as the cost function which is to be minimized and assumed to be Borel-measurable from $\mathbb{R}^n \times U$ to $[0, 1]$. For each finite horizon $t \in]0, +\infty[$, the average value of the optimal control problem with horizon t is defined as

$$V_t(y_0) = \sup_{u\in\mathcal{U}} \frac{1}{t}\int_0^t g(y(s, u, y_0), u(s))ds \ . \tag{2.4.8}$$

One also defines, whenever the trajectories considered are infinite, for any discount factor $\lambda > 0$, the λ-discounted value of the optimal control problem, as

$$W_\lambda(y_0) = \sup_{u\in\mathcal{U}} \lambda \int_0^{+\infty} \mathbb{E}^{-\lambda s} g(y(s, u, y_0), u(s))ds \ . \tag{2.4.9}$$

[5] to be consistent with the rest of the memoir, and contrarily to what is customary in optimal control and what we used in [OBV13], we consider maximization and not minimization problems.

To apply Proposition 2.4.1 to optimal control let $\widetilde{\Omega} = \mathbb{R}^d \times U$ and for any $(y_0, u_0) \in \widetilde{\Omega}$, define $\widetilde{\Gamma}(y_0, u_0) = \{(y(\cdot), u(\cdot)) \mid u \in \mathcal{U}, u(0) = u_0 \text{ and } y \text{ is the solution of (2.4.7)}\}$. Then $\widetilde{\Omega}$, $\widetilde{\Gamma}$ and g satisfy our assumptions. Defining $\widetilde{V_t}$ and $\widetilde{W_\lambda}$ as in (2.4.4) and (2.4.5) respectively, since the solution of (2.4.7) does not depend on $u(0)$ we get that

$$\begin{aligned}\widetilde{V_t}(y_0, u_0) &= V_t(y_0)\\ \widetilde{W_\lambda}(y_0, u_0) &= W_\lambda(y_0)\ .\end{aligned}$$

Proposition 2.4.1 applied to $\widetilde{V}$ and $\widetilde{W}$ thus implies that V_t converges uniformly to a function V in Ω if and only if W_λ converges uniformly to V in Ω.

Sketch of the proof of Proposition 2.4.1. Our proof follows the one in the discrete time setting [53] and use heavily the fact that in this one player framework the limit of (the finite horizon or discounted) values is nonincreasing along any trajectory. Alternatively, one can apply directly the result of [53] to discretizations of the continuous time problem. ■

Interestingly, as in the case for discrete time, the result no longer holds when convergence is only pointwise, as we see in this adaptation in our framework of an example in [72].

Example 2.4.2 *We consider the following problem:*

- *The set of controls is* $[0, 1]$.
- *The compact state space is* $\Omega = \{(x, y) | 0 \leq y \leq \sqrt{2x} \leq 2\sqrt{2}\}$.
- *The continuous cost $g(x)$ is equal to -1 outside the segment [0.9,2.1], to 0 on [1,2], and linear on the two remainings intervals.*
- *The dynamic is The dynamic is given by* $f(x, y, u) = (y, u)$ *for* $x \in [0, 3]$*, and* $f(x, y, u) = ((4 - x)y, (4 - x)u)$ *for* $3 \leq x \leq 4$*. The inequality* $y(t)y'(t) \leq x'(t)$ *is thus satisfied on any trajectory, which implies that* Ω *is forward invariant under this dynamic.*

Then the functions V_t and W_λ converge pointwise on Ω to some $\widetilde{V}(\cdot)$ and $\widetilde{W}(\cdot)$ respectively, and $\widetilde{V}(0, 0) \neq \widetilde{W}(0, 0)$.

We point out that our result has been extended very recently for the two player case in continuous time: for differential games [42] and in a more general framework [43].

2.4.2 Comparison between some family of values in zero-sum stochastic games

We first consider the link between v_n and v_λ in two person zero sum stochastic games. In [Vig13] we prove in the compact/compact setting[6]:

[6]in [Vig13] we considered only a finite set of states, but the proof is exactly the same in the compact case

Proposition 2.4.3 *Assume that for every ω, $v_\lambda(\omega)$ is continuously differentiable, and that its derivative satisfies $\|v'_\lambda(\omega)\| = o\left(\frac{1}{\lambda}\right)$, uniformly on ω. Denote $w_n = v_\lambda$ for $\lambda = \frac{1}{n}$, then $\|w_n - v_n\|$ goes to 0 as λ goes to 0, and in particular v_λ and v_n have the same accumulation points (for the uniform norm).*

Proof. By assumptions and the mean value theorem,

$$\sup_{\mu \in [\frac{1}{n}, \frac{1}{n-1}]} \|w_n - v_\mu\|_\infty = o(\frac{1}{n}).$$

Hence v_λ and w_n have the same accumulation points.

By a argument due to Neyman(Theorem 4 in [64]),

$$\|w_n - v_n\|_\infty \leq \frac{1}{n} \sum_{i=1}^{n-1} i \|w_{i+1} - w_i\|_\infty.$$

So we only need to observe that, by the mean value theorem,

$$\begin{aligned} i\|w_{i+1} - w_i\|_\infty &= i\|v_{\frac{1}{i+1}} - v_{\frac{1}{i}}\|_\infty \\ &\leq \frac{1}{i+1} \sup_{\lambda \in [1/i, 1/i+1]} \left\| \frac{\mathrm{d}v_\lambda}{\mathrm{d}\lambda} \right\|_\infty \\ &= o(1). \end{aligned}$$

■

A Tauberian theorem was proven recently in [98] in the same compact/compact setting ; observe that it does not imply nor his implied by Proposition 2.4.3, as there are converging v_λ such that $\|v'_\lambda(\omega)\| \neq o\left(\frac{1}{\lambda}\right)$ and vice-versa. Very recently [43] presented an argument that works under an assumption which holds when v_λ converges as well as when $\|v'_\lambda(\omega)\| = o\left(\frac{1}{\lambda}\right)$.

We now turns to the link between different evaluations v_θ and the family v_λ. Following [Vig10], we prove that some evaluations v_θ are close to $\lim v_\lambda$ by looking at auxiliary equations in continuous time.

Let $\boldsymbol{\lambda} : \mathbf{R} \to]0, 1]$ be a continuous function and consider the evolution equation:

$$u(t) + u'(t) = \Phi(\boldsymbol{\lambda}(t), u(t)) \quad \text{with } u(0) = u_0 \tag{2.4.10}$$

where Φ is the operator defined by equation (1.1.2). When $\boldsymbol{\lambda}$ is a constant λ, this is a continuous analogue to (1.1.7) and it is easy to show that $u(t)$ tends to v_λ as t goes to infinity. We are interested in the asymptotic behavior of u when $\boldsymbol{\lambda}$ is a parametrization converging slowly to 0, in the spirit of [3]. Intuitively, if $\boldsymbol{\lambda}$ varies slowly enough then it is almost constant on large intervals of time so u should approximate successively the different values v_λ and thus share their asymptotic behavior.

Assume from now[7] on that Ψ satisfies hypothesis ($\mathcal{H}$) (see (2.3.6)) with $k(a) = a$ and $\ell(a) = C + a$, that is

$$\|\Phi(\lambda, f) - \Phi(\mu, f)\| \leq |\lambda - \mu|(C + \|f\|) \quad \forall f \in \mathcal{F} \quad \forall (\lambda, \mu) \in]0, 1]^2.$$

[7] We refer the interested reader to Remark 4.15 in [Vig10] for the same type of results using weaker assumptions

For Shapley operators, this is true as soon as the payoff function is bounded and thus holds in all frameworks we consider. Then, using Gronwall-type inequalities and the fact that $\Phi(\alpha, \cdot)$ is $1-\alpha$ contracting, one gets

Proposition 2.4.4 *Let $\boldsymbol{\lambda}$ be a $\mathcal{C}^1$ function from $[0,+\infty[$ to $]0,1]$ and let $L : \mathbb{R}^+ \to \mathbb{R}$ be defined by $L(t) = e^{\int_0^t \left[\frac{|\boldsymbol{\lambda}'(s)|}{\boldsymbol{\lambda}(s)} - \boldsymbol{\lambda}(s)\right] ds}$. Then the corresponding solution u of (2.4.10) satisfies:*

$$\|u(t) - v_{\boldsymbol{\lambda}(t)}\| \leq \frac{L(t)}{\boldsymbol{\lambda}(t)} \left[\|u'(0)\| + (C + C') \int_0^t \frac{|\boldsymbol{\lambda}'(s)|}{L(s)} ds \right].$$

where C is the constant in condition $(\mathcal{H})$ and $C' = \sup_{\lambda \in]0,1]} \|v_\lambda\|$.

In particular, as was intuited,

Corollary 2.4.5 *Let $\boldsymbol{\lambda}$ be a $\mathcal{C}^1$ function from $[0,+\infty[$ to $]0,1]$, such that $\frac{\boldsymbol{\lambda}'(t)}{\boldsymbol{\lambda}^2(t)}$ converges to 0 as t goes to $+\infty$, and let u be the corresponding solution of equation (2.4.10). Then $\|u(t) - v_{\boldsymbol{\lambda}(t)}\|$ goes to 0 as t goes to $+\infty$.*

One can now consider the same kind of questions in discrete time. For any sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $]0,1]$, define the discrete counterpart of equation (2.4.10) :

$$w_n = \Phi(\lambda_n, w_{n-1}) \quad \text{with } w(0) = w_0 \tag{2.4.11}$$

Remark that when Ψ is the Shapley operator of a game, w_n is the value of a game with time evaluation θ such that

$$\theta_i = \begin{cases} \lambda_{n+1-i} \prod_{j=n+2-i}^{n} (1-\lambda_j) & \text{if } 1 \leq i \leq n-1 \\ 1 - \sum_{j=1}^{n-1} \theta_j & \text{if } i = n \\ 0 & \text{if } i > n \end{cases}$$

Then one obtains the discrete version of Corollary 2.4.5 :

Proposition 2.4.6 *Let λ_n be a sequence in $]0,1]$. Assume that both λ_n and $\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}$ tend to 0 as n goes to $+\infty$. Then the solution $(w_n)_{n \in \mathbb{N}}$ of (2.4.11) satisfies*

$$\|v_{\lambda_n} - w_n\| \to 0$$

as n goes to $+\infty$.

Sketch of the proof. The sequence $\gamma_n = \frac{1}{\lambda_n}$ tends to $+\infty$ and satisfies $\gamma_n - \gamma_{n-1} \to 0$ as n goes to $+\infty$. This implies the existence of an interpolation function $\gamma : \mathbb{R} \to \mathbb{R}$ which is $\mathcal{C}^2$ and such that for all n in $\mathbb{N}$, $\gamma(n) = \gamma_n$, $\lim_{+\infty} \gamma(t) = +\infty$ and $\lim_{+\infty} \gamma'(t) = 0$. The function $\boldsymbol{\lambda} := \frac{1}{\gamma}$ thus satisfies $\boldsymbol{\lambda}(n) = \lambda_n$ and all the hypotheses of Corollary 2.4.5.

Let us denote by u the corresponding solution of equation (2.4.10). By Corollary 2.4.5 it is thus enough to show that $\|w_n - u(n)\| \to 0$ as n goes to $+\infty$ and it can be done using the fact that the $\Phi(\alpha, \cdot)$ are contracting. ■

Observe that this theorem is not a Tauberian theorem : it holds whether v_λ converges or not. And it implies in particular

Corollary 2.4.7 *v_λ converges as λ goes to 0 if and only if there exists a sequence λ_n satisfying the hypothesis of Proposition 2.4.6 such that the corresponding sequence w_n defined by (2.4.11) converges.*

2.5 Perspectives

In this section we briefly give some possible directions for future research on zero-sum repeated games.

Concerning definable games, our results in 2.1.1 do not answer the question of convergence in general. As Example 2.1.3 shows, there are semi algebraic games in which neither the Shapley operator nor the value function are semi algebraic. It is still possible however that semi algebraic games have a globally subanalytic Shapley operator, or more generally that any game definable in some o-minimal structure has a Shapley operator definable in some extension of this structure, in the spirit of [84]. Another possibility would be that the semi algebraic structure is simply not large enough, but that Shapley operators of games definable on more complex structures (as the globally subanalytic one for example) are definable in this same structure.

Outside of definable games, an interesting class of games is the following : games in the finite/compact setting such that there exists an $\varepsilon > 0$ such that for any pure strategies i and j and any $\omega \neq \omega'$, either $\rho(\omega'|i, j, \omega) = 0$ or $\rho(\omega'|i, j, \omega) \geq \varepsilon$. This class contains in particular finite games and ergodic games (games in which $\rho(\omega'|i, j, \omega) > 0$ for all i j and $\omega \neq \omega'$), both known to have an asymptotic value but for very different reasons.

Another line of research is understanding better the frontier separating games with good asymptotic behavior (i.e. convergence of values) and counterexamples. A natural idea is to try to "regularize" irregular games. We give two examples. First of all, for any game Γ say in the finite/compact framework, and any $\varepsilon > 0$, one can associate the game Γ^ε in which the transition ρ of Γ has been replaced by $(1-\varepsilon)\rho + \varepsilon u$, where u is the uniform law on $\Delta(\Omega)$. Since Γ^ε is ergodic, it admits an asymptotic value v^ε ; does v^ε converge as ε goes to 0 ? More generally, one can be said of v_λ^ε when both λ and ε tends to 0 ? A second example concerns finite stochastic games with observation of the moves and public signals on the state. When the state is observed at each stage these games have an asymptotic value ; when the state is never observed Ziliotto [97] constructed a game without asymptotic value. For any $\alpha \in]0, 1[$, consider the game where the state is publicly observed with probability α at each stage and note v^α its asymptotic value, what can be said of v^α when α goes to 0 ? Of v_λ^α when both α and λ go to 0 ?

A similar "double limit" type of problems appeared in Section 2.3.1 when we studied v_λ^h. However, as we explained in Remark 2.3.8 , the answer is simple in the context of stochastic games. The same question can be posed for stochastic games with public signals and the answer seems more difficult as our results of Section 2.3.1 do not apply. Indeed, for games with no observation of the state the asymptotic behavior of the values may depend on the stage duration, even for one player games. As an example consider a MDP with no observation of the state, with two non absorbing state $\{\omega_0, \omega_1\}$, each with payoff 0, and 2 actions. Action "Change" moves the state from ω_0 to ω_1 and vice versa ; action "Quit" absorbs with payoff -1 if the state is ω_0 and with payoff 1 if the state is ω_1. The starting state is ω_0. Then it is clear that in this game the optimal play is to play Change then Quit, for an asymptotic payoff of 1. In the same game with varying stage duration, as soon as all stage durations are less than $\frac{1}{2}$ the player will at each stage be in state ω_0 with a larger probability than in state ω_1, hence the optimal play is to always play Change for an asymptotic value of 0. This gives an other possible regularization of games with signals for which v_λ does not converge: to consider the asymptotic behavior of v_λ^h as both λ and h tends to 0.

Concerning Section 2.3.2, an interesting question concerns the behavior of near optimal strategies in games with incomplete information. In games with lack of information of one side, there are [4] ε-optimal strategies of the following form: the informed player reveals some information in the first stage, and then never reveal anything else. Hence, if one looks at the trajectory of the beliefs on the time interval $[0, 1]$, there is a discontinuity at time 0 and then nothing happens. What can be said for games with incomplete information on both sides ? Are there games in which, under optimal play, the information is gradually revealed on the time interval $[0, 1]$? What can be said in general on the information revealed say on the time interval $[0, 1/2]$, compared to the information revealed along the whole interval $[0, 1]$?

Chapter 3

Structure of the sets of equilibria and equilibrium payoffs of finite N person games

In finite bimatrix games, the structure of the set of Nash equilibria is relatively well understood: this is a finite union of convex polytopes [41]. Moreover, the possible sets of Nash equilibrium payoffs have been characterized in [52]:

Proposition 3.0.1 *A nonempty subset E of $\mathbb{R}^2$ is the set of Nash equilibrium payoffs of a bimatrix game if and only if it is of the form $E = \cup_{1 \leq i \leq m} [a_i, b_i] \times [c_i, d_i]$, where $m \in \mathbb{N}$ and a_i, b_i, c_i, $d_i \in \mathbb{R}$, with $a_i \leq b_i$, $c_i \leq d_i$.*

For finite games with three players or more, the picture is much less clear. It is easily seen that a set of Nash equilibrium or Nash equilibrium payoffs is compact ; by Nash's theorem they are nonempty, and it is well known (see Proposition 1.2.6) that they are semi algebraic. However, which semi-algebraic sets can be represented as sets of Nash equilibria or of Nash equilibrium payoffs is not immediate. Let us mention a few results. Bubelis [18] shows that for any algebraic number z, there is a 3 player game with integer pure payoffs and a unique equilibrium in which the first player gets a payoff of z. He also provides a construction relating the set of equilibrium of an N-player game to the set of equilibrium strategies of the first player in a 3 player game. Datta [24] showed that any real algebraic variety is isomorphic to the set of completely mixed Nash equilibria of a 3-player game, and also to the set of completely mixed equilibria of an N-player game in which each player has two strategies. More recently, Balkenborg and Vermeulen [6] showed that any nonempty connected compact semi-algebraic set is homeomorphic to a connected component of the set of Nash equilibria of a finite game in which each player has only two strategies, all players have the same payoffs, and pure strategy payoffs are either 0 or 1. These results show that, modulo isomorphisms or homeomorphisms, and a focus on completely mixed equilibria or connected components of equilibria, all algebraic or nonempty compact semi-algebraic sets may be encoded as sets of Nash equilibria.

In this chapter we prove results in this direction. In the first section, following [VV16], we prove that for any nonempty compact semi algebraic set E in a product of simplices, there exists a finite game with $N > n$ players, each having only two pure strategies, such that E is precisely the projection of the set of Nash equilibria of this game on its first coordinates (those corresponding to the strategies of the first n players). This result was independently obtained by Levy [54]. This implies a similar result on equilibrium payoffs, as opposed to equilibria: for any nonempty compact semi-algebraic set E in $\mathbb{R}^n$, there exists a finite game with $N > n$ players, each having only two pure strategies, such that E is precisely the set of Nash equilibrium payoffs of the first n players; that is, the projection of the set of Nash equilibrium payoffs on its first n coordinates. We also prove the same kind of results with the additional assumption that the set E is defined by polynomial with integer coefficients and the additional requirement that our games are with integer payoffs.

In the second section we prove the same kind of results but instead of projecting the set of equilibria of a game on some subset of players, we project it some subset of actions. For example, a first result states that any nonempty compact semi algebraic $E \subset \mathbb{R}^N$, $N \geq 3$, is the projection of the set of equilibria of an N player game on the set giving the probability that each player plays his first strategy. An easy consequence is that any such set is the set of equilibrium payoffs of some N player game, giving a full characterization of the sets of Nash equilibria of finite games. As in the previous section, one can additionally require that the game has integer payoffs, provided that the semi algebraic set is defined by polynomials with integer coefficients.

The proofs of the results of the first two sections are fully elementary. To be more precise, given a set and certificates of semi-algebraicity, boundedness, closedness and nonemptiness, we do not use any results from real algebraic geometry. Moreover, all proofs are fully constructive and the size of the constructed game is polynomial in the size of the semi-algebraic set. Hence this has implications on the complexity and computability of some problems involving Nash equilibria, that we will explain in the last section.

3.1 Projection on players

This section is based on [VV16]. For any k, denote Δ_k the simplex of dimension k, that is the set of probabilities on k elements.

Proposition 3.1.1 *Let $N \geq 1$ and $k_i \geq 2$ be integers, and $n' = \sum_{i=1}^n k_i$. If F be a nonempty compact semi-algebraic subset of $\prod_{i=1}^n \Delta_{k_i} \subset \mathbb{R}^{n'}$, then there exists an N-player game(with $N > n$), in which every player except the first n ones has only two actions, and such that the projection of its set of Nash equilibria on the first n' coordinates (those of the first n players) is equal to F. Moreover, if F is expressed as unions and intersections of A sets of the form $\{x,\ P(x) \leq 0\}$, where the P are polynomials of degree at most d in each variable, then $N \leq n + 1 + A + 2n'(1 + \ln_2(d))$.*

Sketch of the proof. Assume for simplicity that $k_i = 2$ for all i and that $A = 1$, that is $F = \{P(x_1, \cdots, x_n) \leq 0\}$ where we identified Δ_2 with $[0,1]$ and x_i is the probability that Player i plays his first action. Each player will be denoted by an upper case letter (X^i for the first n ones), and the probability that he plays his first action by the corresponding lower case letter. All players will have only two actions, Top and Bottom. The key element of the proof is the following. For any parameter $x \in [0,1]$, consider a game in which two specific players get the following payoffs if they play Top or Bottom

$$\text{Player } X_\alpha \quad \begin{array}{c|c} T & x \\ \hline B & x_\beta \end{array} \qquad \text{Player } X_\beta \quad \begin{array}{c|c} T & x_\alpha \\ \hline B & x \end{array} \tag{3.1.1}$$

meaning that the payoffs of players X_α and X_β are respectively $x_\alpha x + (1 - x_\alpha)x_\beta$ and $x_\beta x_\alpha + (1 - x_\beta)x$ (recall that x_α is the probability that X_α plays Top, and similarly for x_β). Then in any equilibrium, if $x_\alpha > x$ then Player X_β plays Top, hence $x_\beta = 1 > x$, hence Player 1 plays Bottom and $x_\alpha = 0$, a contradiction. A dual reasonning ensures that $x_\alpha = x$ in any equilibrium. If we do this with $x = x_\gamma$ (the probability that another player plays his first action), we see that in any equilibrium $x_\alpha = x_\gamma$. If one does the same construction with two other players, with $x = x_\alpha x_\gamma$ (which is affine in x_α and x_γ), one sees that at equilibrium some player will play Top with probability $x_\alpha x_\gamma = (x_\alpha)^2$. Continuing this way one can ensure that some players play Top with probabilities $x_1^{a_1} \cdots x_n^{a_n}$ for every a_i less than d. This allows us to construct the payoff function of the action Top of some player U in such a way that in any equilibrium U gets $P(x_1, \cdots, x_n)$ by playing Top. Let action Bottom give U a payoff of 0, then in any equilibrium in which U plays Bottom, $P(x_1, \cdots, x_n) \leq 0$ hence $(x_1, \cdots, x_n) \in F$. One now construct the payoff of the original n players (whose payoff was not yet specified) in such a way that:

- When Player U plays Bottom, they are indifferent. This will ensure that for any $e \in F$, there is an equilibrium in which U plays Bottom and the first n players play Top with probability $(e_1, \cdots, e_n)$.

- When Player U plays Top with positive probability, the first n players have to play Top with probability $(z_1, \cdots, z_n)$, where z is some fixed element of F (this is were the nonemptiness is used).

■

Remark 3.1.2 *The number N given by the proposition is near optimal as one can proves that it is impossible to do better than $\frac{n' \ln_2(d)}{\ln_2(n' \ln_2(d))}$.*

A byproduct of the proof of Proposition 3.1.1 is:

Proposition 3.1.3 *If F is a nonempty compact semi-algebraic subset of $\mathbb{R}^n$, then there exists an N-player binary game (with $N > n$) such that the projection of its set of Nash equilibrium payoffs on the first n coordinates is F.*

In fact, with a more complex construction one proves:

Proposition 3.1.4 *In both theorem above, if F is defined by inequalities involving polynomials with coefficients in $\mathbb{Z}$, then the binary game can be constructed with pure payoffs in $\mathbb{Z}$ as well.*

3.2 Projection on actions

This section is based on [Vig17]. Instead on projecting the set of equilibrium of a game on the strategies of some players, we now project it on some actions of all players. We now denote with uppercase letters the actions of the players, and with the corresponding lower case letters the probability that they are played.

We first present a result when the projection is on the first action of each player.

Proposition 3.2.1 *Let $N \geq 3$, and $F \subset [0,1[^N$ be a nonempty closed semi algebraic set. Then there exists an N-player finite game Γ, and a particular pure action profile $X^* = (X^1_*, \cdots, X^N_*)$ such that*

a) $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma)) = F$

b) $\mathrm{NEP}(\Gamma) = \{0\}$.

where NE and NEP denote the set of Nash equilibria and Nash equilibrium payoffs respectively. Before sketching the proof of this result, note that this proposition has an obvious benefit: since it does not add any player, the dimension of the set F is the same as the dimension of the game payoffs. This allows up to deduce rather easily the following result, much stronger than Proposition 3.1.3 of the previous section:

Theorem 3.2.2 *Let $N \geq 3$ be an integer. A set $F \subset \mathbb{R}^N$ is the set of equilibrium payoffs of some finite N-player game if and only if F is nonempty, compact, and semi algebraic.*

Proof of Theorem 3.2.2. Let F be a nonempty, compact, and semi algebraic subset of $\mathbb{R}^N$, and first assume that $F \subset [0,1[^N$. Let Γ be a finite game given by the conclusion of Proposition 3.2.1. Let Γ' be defined from Γ by adding 1 to the payoff of each player i iff player $i-1$ plays X^{i-1}_*. The games Γ and Γ' are strategically equivalent thus have the same set of equilibria. Because of properties a) and b), the set of equilibrium payoffs of Γ is $\{(e_N, e_1, \cdots, e_{N-1}) | (e_1, \cdots, e_N) \in F\}$. By relabeling the players one get a game Γ'', in which Player i plays the role of Player $i+1$ in Γ', whose set of equilibrium payoffs is F.

If F is not a subset of $[0,1[^N$, F being bounded one can choose $\alpha \in \mathbb{R}$ and $\beta > 0$ such that $F' := \alpha + \beta F$ is in $[0,1[^N$. By the previous argument, there is a finite game Γ'' whose set of equilibrium payoffs is F'. Then $\Gamma''' := \frac{1}{\beta}\Gamma'' - \frac{\alpha}{\beta}$ is strategically equivalent to Γ'', and thus its set of equilibrium payoffs is F.

■

There is a drawback however. In the previous section we dealed with binary players, which meant that they were either playing pure or completely mixed. Now each player will have many actions and thus many possible supports, which makes the construction substantially more difficult.

Sketch of the proof of Proposition 3.2.1. We asume for simplicity that $N = 3$. The basic idea is that if some action X^i of player i is played in some equilibrium σ while another action Y^i is not, then $g^i(X^i, \sigma^{-i}) \geq g^i(Y^i, \sigma^{-i})$, which gives an inequality satisfied by the probabilities of actions of other players in this equilibrium. For example if $g^1(X^1, \sigma^{-1}) = 0$ (meaning that Player 1 gets 0 when he plays X^1 whatever the two other players do), and $g^1(Y^1, \sigma^{-1}) = x_1^2 - x_1^3$ (meaning that when he plays Y^1 Player 1 gets 0, except if Player 2 plays X_1^2 and Player 3 doesn't play X_1^3 in which case he gets 1, or if Player 3 plays X_1^3 and Player 2 does not play X_1^2 in which case he gets -1), then if at an equilibrium X^1 is played and not Y^1 then $x_1^2 \leq x_1^3$.

Asume for a moment that we can guarrantee that all equilibria have some fixed support, and that each action in this support gives a payoff of zero whatever the actions of the other players. Then clearly part b) of the proposition would be satisfied. Also by the argument of the previous paragraph we would have a family of inequalities (and thus also equalities) involving the probabilities of all actions at equilibrium, for example $x_1^2 = x_1^3$. Since there are three players it is possible to have an unplayed action of player 1 with payoff $x_1^2 x_1^3$ equals to $(x_1^2)^2$ at equilibrium, and hence to enforce at equilibrium equalities like $x_2^2 = (x_1^2)^2$. Hence we could ensure that some probabilities are polynomials in other ones, and, again using inequalities, that these polynomials take nonpositive values and hence that the desired tuple of probabilities is in a prescribed semi algebraic set F. This is in the same spirit that what was done by Datta [24] when studying the links between algebraic sets and completely mixed (that is, with fixed full support) equilibria.

The problem is that one cannot hope that this works so easily. It won't work for empty semi algebraic sets (since any finite game has a Nash equilibrium, a key difference with completely mixed ones), and empty semi algebraic sets look like[1] nonempty ones ! So there will be other equilibria to deal with. Since the number of actions used in the previous paragraph to construct various inequalities may be large (depending of the degrees of the polynomials in the definition of F), there could be also a large number of other equilibria, and ensuring that each of them projects in F may be difficult.

To avoid this we construct the game such that, in addition to the previous "adapted" equilibria with fixed support constructed in a first step, there is only one other "inadapted" equilibrium. Basically, we do this by giving large payoffs, outside of adapted equilibria, only to two specific actions of each player: the one on which we project[2] and another one. We also define the payoffs of these two specific actions for each player so that each player wants to play the first one only if the next player plays the second one with large probability. By a circular argument since $N = 3$ is odd there will then be only one inadapted equilibrium. By constructing the payoffs of the two specific actions according to the coordinates of some given element $\widehat{z}$ of the nonempty F, one ensures that in this inadapted equilibrium the desired tuple of probabilities equals $\widehat{z} \in F$. Some other small tricks are needed to ensure part b) of the proposition. ∎

Remark 3.2.3 *By doing the construction carefully, one ensures that the game has a number of actions per player independent on N and polynomial in the number and maximum degree of polynomials involved in the definition of F.*

One can generalize Proposition 3.2.1 in two directions. The first one is to consider $\mathbb{Z}$-semi algebraic sets, that is semi algebraic sets for which all polynomials involved in the definition are with integer coefficients. Then

[1] meaning that it is computationaly hard [7] to decide if a semi algebraic set is empty or not just looking at the polynomials involved in its definition

[2] in the first step this action had a payoff of 0, so a difficulty is to not disrupt the construction of the first step by modifying this payoff

Proposition 3.2.4 *Let $N \geq 3$, and $F \subset [0,1[^N$ be a nonempty closed $\mathbb{Z}$-semi algebraic set. Then there exists an N-player finite game Γ with pure payoffs in $\mathbb{Z}$, and a particular pure action profile $X^* = (X^1_*, \cdots, X^N_*)$ such that*

a) $\mathrm{Proj}_{X_*}(\mathrm{NE}(\Gamma)) = F$

b) $\mathrm{NEP}(\Gamma) = \{0\}$.

In particular, reasoning as for Theorem 3.2.2 this proves

Theorem 3.2.5 *Let $N \geq 3$ be an integer. A set $F \subset \mathbb{R}^N$ is the set of equilibrium payoffs of some finite N-player game with integer pure payoffs if and only if F is nonempty, compact, and $\mathbb{Z}$-semi algebraic.*

Remark that in the particular case where F is a singleton, this implies that for $N \geq 3$ every N-uple $(z_1, \cdots, z_n)$ of algebraic numbers in $[0,1[$ (resp. of algebraic numbers) there exists a game with integer pure payoffs and with a unique equilibrium in which each player i plays his first strategy with probability z_i (resp. in which each player i gets a payoff of z_i). This generalizes a result of Bubelis [18] that shows that for every algebraic $z_1 \in [0,1[$ there is a 3 player game with integer pure payoffs and with a unique equilibrium in which player 1 gets a payoff of z_1.

The second direction in which we generalize is projecting on several actions of each player instead on just the first one.

Definition 3.2.6 *Let $N \in \mathbb{N}^*$ and $(T_1, \cdots, T_N) \in (\mathbb{N}^*)^N$. Let $F \subset \mathbb{R}^{T_1+\cdots+T_N}$ and denote the coordinates of any element $z \in F$ as $z_{i,t}$ for $i = 1$ to N and $t = 1$ to T_i. F is $(T_1, \cdots, T_N)$-admissible if the following properties are satisfied for all $z \in F$:*

- $z_{i,t} \geq 0$ *for all i and t*
- *for all i,* $\sum_{t=1}^{T_i} z_{i,t} \leq 1$.

F is strongly $(T_1, \cdots, T_N)$-admissible if the second property is replaced for all $z \in F$ by

- *for all i,* $\sum_{t=1}^{T_i} z_{i,t} < 1$.

We now generalize Proposition 3.2.1 to projection of the set of Nash equilibria on the T_i first actions of each player i. Clearly such a projection is always nonempty, closed, semialgebraic and $(T_1, \cdots, T_N)$-admissible. We now prove a reciprocal:

Proposition 3.2.7 *Let $N \geq 3$ and $(T_1, \cdots, T_N) \in (\mathbb{N}^*)^N$. Let $F \subset \mathbb{R}^{T_1+\cdots+T_N}$ be a nonempty closed semi algebraic set and assume it is strongly $(T_1, \cdots, T_N)$-admissible. Then there exists an N-player game Γ, and T_i special actions $X^i_{*,1}, \cdots, X^i_{*,T_i}$ for each player i, such that*

a) $\mathrm{Proj}_{\{X^i_{*,t}\}}(\mathrm{NE}(\Gamma)) = F$

b) $\mathrm{NEP}(\Gamma) = \{0\}$.

3.3 Applications

We now apply the construction of the previous section. Firstly, it implies that certain problems on equilibrium are computationally hard, since they are at least as hard as some problems on semi-algebraic sets. Recall that for 2 player games many problems involving equilibrium sets are already known to be NP-Hard [35]. We prove that for three players the problems are at least as hard as deciding whether or not a compact $\mathbb{Z}$-semi algebraic set is empty ; the complexity of this problem being known to lie somewhere between NP and $PSPACE$ [82].

Proposition 3.3.1 *The problem of deciding whether or not a compact $\mathbb{Z}$-semi algebraic set is empty can be reduced in polynomial time to any of these problems:*

a) Given a 3-player game with integer pure payoffs, to determine if there is more than one equilibrium.

b) Given a 3-player game with integer pure payoffs,, to determine if there is an infinite number of equilibria.

c) Given a 3-player game with integer pure payoffs, to determine if the number of mixed equilibria is not odd (meaning either even of infinite).

d) For some fixed nonempty $\mathbb{Z}$-semi algebraic set E strictly included in $\mathbb{R}^3$, and given a 3-player game with integer pure payoffs, to determine whether there is one equilibrium payoff in E.

e) In particular, given a 3-player game with integer pure payoffs, to determine whether there is one equilibrium with positive (or negative, or 0) payoff for one (or several) players.

Sketch of proof. We prove d) : let E be such a set and assume we have an algorithm to determine whether a game has one equilibrium payoff in E. Let $e \in E$ and $e' \notin E$. For a given compact and closed semi-algebraic set $F \in \mathbb{R}^N$, consider the set $(\{e\} \times F) \cup (\{e'\} \times \{0\}^N) \subset \mathbb{R}^{N+3}$. It is compact nonempty and semi algebraic, up to some rescalling one can write it as a strongly admissible (T_1, T_2, T_3) subset for some $T_1+T_2+T_3 = N+3$. By Proposition 3.2.7 one can construct in polynomial time a game Γ such that this set is the projection on some actions of the 3 players of the set of equilibria of Γ. In particular the projection of the set of equilibria of Γ on the first action of all players is, up to some rescaling, $\{e, e'\}$ if F is nonempty and $\{e'\}$ if F is empty. Reasoning as in the proof of Theorem 3.2.1 one gets a game Γ' such that it has an equilibrium payoff in E if and only if F is nonempty. Hence one can decide whether or not F is empty by running the algorithm to decide if there is an equilibrium payoff of Γ' in E. ■

Also for 3 players the results of NP-hardness holds even if the pure payoff are restricted to be in some fixed set, for example

Proposition 3.3.2 $3 - SAT$ *can be reduced in polynomial time to any of these problems (that are thus NP-Hard).*

– *Given a 3-player game with pure payoffs in $\{-1, 0, 1\}$, to determine if there is more than one equilibrium.*

– *Given a 3-player game with pure payoffs in $\{-1, 0, 1\}$, to determine if there is an infinite number of equilibria.*

– *Given a 3-player game with pure payoffs in $\{-1, 0, 1\}$, to determine if the number of mixed equilibria is not odd (meaning either even of infinite).*

– *Given a 3-player game with pure payoffs in $\{-1, 0, 1\}$, to determine if there is an equilibrium in which the first player plays is first action with positive probability.*

– *Given a 3-player game with pure payoffs in $\{-1, 0, 1\}$, to determine if there is an equilibrium in which the first player plays is first action with probability $\frac{1}{2}$.*

Another kind of application concerns the computability of some problems involving Nash equilibrium. Recall that is was proved by Matiyasevich [58], based on previous works by Davis, Putnam and Robinson [25, 26, 77] (see also [27] for a very interesting survey) that Hilbert tenth problem (on $\mathbb{N}^*$) is undecidable, that is

Theorem 3.3.3 *There is no algorithm that decide whether a given polynomial in k variables with integer coefficients has a zero in $(\mathbb{N}^*)^n$. The results holds even if one only considers polynomials with a fixed number n of variables and with a fixed degree, provided they both are larger that some explicit constants.*

Denote by $\frac{1}{\mathbb{N}^* \setminus \{1\}}$ the set $\left\{\frac{1}{n}, n \in \mathbb{N}^* \setminus \{1\}\right\}$.

Proposition 3.3.4 *There exists no algorithm which solve the following decision problems, given a finite game with integer payoffs:*

a) Is there an equilibrium in which all players play there first action with a probability in $\frac{1}{\mathbb{N}^* \setminus \{1\}}$?

b) Is there an equilibrium in which the payoff of all players is in $\frac{1}{\mathbb{N}^* \setminus \{1\}}$?

This is true even for a fixed number of players and actions, provided they are larger than some computable constants.

Proof. Let $P \in \mathbb{Z}[z_1, \cdots, z_N]$ be a polynomial in N variables with integer coefficients and degree at most d in each variable. The function

$$(z_1, \cdots, z_N) \longrightarrow z_1^d \cdots z_N^d P\left(\frac{1 - z_1}{z_1}, \cdots, \frac{1 - z_N}{z_N}\right)$$

can be continuously extended to a polynomial Q in $\mathbb{Z}[z_1, \cdots, z_N]$ with degree at most d in each variable. Since $z \to \frac{1-z}{z}$ is a bijection mapping $\frac{1}{\mathbb{N}^* \setminus \{1\}}$ onto $\mathbb{N}^*$, Q has a zero in $\left(\frac{1}{\mathbb{N}^* \setminus \{1\}}\right)^N$ if and only if P has a zero in $(\mathbb{N}^*)^N$.

Let $\widehat{z} = \left(\frac{2}{5}, \cdots, \frac{2}{5}\right)$ and

$$F = (\{z,\ P(z) = 0\} \cup \{\widehat{z}\}) \cap \left[0, \frac{1}{2}\right]^N.$$

$F \subset [0,1[^N$ is nonempty, closed and $\mathbb{Z}$-semi algebraic, and $\widehat{z} \in \mathbb{Q}^N \bigcap F$. Also, $F \cap \left(\frac{1}{\mathbb{N}^* \setminus \{1\}}\right)^N = \emptyset$ if and only if P has no zero in $(\mathbb{N}^*)^N$. By Proposition 3.2.4 and Theorem 3.2.5, one can explicitely construct two games Γ_1 and Γ_2 with integer coefficients such that $\text{Proj}_{X_*}(\text{NE}(\Gamma_1)) = F$, and $\text{NEP}(\Gamma_2) = F$. Hence if an algorithm existed to answer a) or b), then one would be able to solve Hilbert's tenth problem, which is impossible by Theorem 3.3.3. ■

It is easy to see that Theorem 3.3.3 still holds if one replace $\mathbb{N}^*$ by $\mathbb{N}$ or $\mathbb{Z}$. If one replace it by $\mathbb{Q}$ however, the so-called Hilbert tenth problem on $\mathbb{Q}$ is one of the biggest open problems in the area of undecidability in number theory [30]. By a similar reasoning than for the previous proposition, one proves that this problem is equivalent to some natural problems concerning games with pure integer payoffs:

Proposition 3.3.5 *The following problems are either all decidable or all undecidable:*

a) Hilbert's tenth problem on $\mathbb{Q}$: deciding, for every N and every polynomial with integer coefficients and N variables, whether P has a zero in $\mathbb{Q}^N$.

b) Deciding if a finite game with integer pure payoffs has an equilibrium in which for each player, his first action is played with probability in $\mathbb{Q}^*$.

c) Deciding if a finite game with integer pure payoffs has an equilibrium in which all players get a payoff in $\mathbb{Q}^*$.

3.4 Perspectives

In this section we briefly gives some possible directions for future research on this topic.

A first line of research is to better understand not only the sets of Nash equilibria but also the sets of corollated equilibria of finite games. In [52], the authors shows that for a couple $(A, B) \in (\mathbb{R}^2)^2$, there exists a 2 Player game such A is its set of equilibrium payoffs and B is its set of corollated equilibrium payoffs iff A is of the form given in Proposition 3.0.1 and B is a polytope containing the convex hull of A. What can be said for $N \geq 3$ players ? Note that a key ingredient of the proof in [52] is that given a game, one can perturb it to add some extreme points to its set of corollated equilibrium payoffs, without changing its set of equilibrium payoffs. Hence it is enough to prove the result when B is the convex hull of A. This idea cannot work directly for $N \geq 3$, are there are sets of Nash equilibrium payoffs whose convex hull is not a polytope.

Another possibility is to study sets of equilibria or equilibrium payoffs of games with dynamic structures. A first simple framework would be finite games in extensive form with perfect information: what can be said of their sets of (subgame-perfect) equilibrium payoffs ?

Another problem would be the characterization of the functions $\lambda \to v_\lambda$ (or of the functions $\lambda \to v_\lambda(\omega)$ for a fixed starting state ω) that are value functions of some 2 player zero-sum game (in either the finite or finite/compact framework). Note that even for finite MDPs a characterization was only found very recently [51]. Also remark that Proposition 2.2.2 gives a partial answer in the case of finite/compact games with four states, and that corollaries 2.1.17 and 2.1.28 give informations on the set of accumulation points of v_λ. In particular together with the examples of Section 2.2 they imply that $V \subset \mathbb{R}^4$ is the set of accumulation points of a finite/compact zero-sum stochastic games with four states iff there exists $a \leq b \leq c \leq d$ such that, up to some permutation of the coordinates, $V = \{a\} \times \{(x,x),\ x \in [b,c]\} \times \{d\}$.

Finally one can investigate the structure of parametrized games. For example, given a N-player one shot game Γ, and $a \in \mathbb{R}^N$, denote $\Gamma(a)$ the game Γ in which the payoff if everybody plays his first action has been replaced by the vector a, and f_Γ the correspondence $a \to \mathrm{NEP}(\Gamma(a))$. Which correspondences from $\mathbb{R}^N$ to itself can be represented as f_Γ for some Γ ? Some results in that direction were obtained in [54]. Understanding this would probably give us insights on the dynamics of N-players absorbing games.

Chapter 4

A minmax theorem

This very short part is based on [PV15]. We prove a minmax theorem with assumptions on the domains of a concave-convex mapping but, surprisingly, no assumption on its regularity.

Proposition 4.0.1 *Let X and Y be two nonempty convex sets and $f : X \times Y \to \mathbb{R}$ be a concave-convex mapping, i.e., $f(\cdot, y)$ is concave and $f(x, \cdot)$ is convex for every $x \in X$ and $y \in Y$. Assume that*

- *X is finite dimensional,*
- *X is bounded,*
- *$f(x, \cdot)$ is lower bounded for some x in the relative interior of X.*

Then the zero-sum game on $X \times Y$ with payoff f has a value, i.e.,

$$\sup_{x\in X} \inf_{y\in Y} f(x, y) = \inf_{y\in Y} \sup_{x\in X} f(x, y).$$

Sketch of the proof. Assume for simplicity that $X = [-1, 1]$ and $0 \leq f \leq 1$. Define, for every $\varepsilon > 0$, the set $X_\varepsilon = [-(1-\varepsilon), 1-\varepsilon]$. Since $f(\cdot, y)$ is concave, it is continuous on X_ε and thus usual minmax theorems apply [32], i.e.

$$\sup_{x\in X} \inf_{y\in Y} f(x, y) \geq \sup_{x\in X_\varepsilon} \inf_{y\in Y} f(x, y) = \inf_{y\in Y} \sup_{x\in X_\varepsilon} f(x, y).$$

The concavity of f with respect to its first variable implies, for every y and every $x > 1 - \varepsilon$,

$$\frac{f(x, y) - f(0, y)}{x} \leq \frac{f(1-\varepsilon, y) - f(0, y)}{1-\varepsilon}.$$

Since f takes value in $[0, 1]$ we get

$$f(x, y) \leq \frac{xf(1-\varepsilon, y) - (x - (1-\varepsilon))f(0, y)}{1-\varepsilon} \leq \frac{f(1-\varepsilon, y)}{1-\varepsilon} \leq f(1-\varepsilon, y) + \frac{\varepsilon}{1-\varepsilon}.$$

hence if $x > 1 - \varepsilon$ is a best reply to y then $1 - \varepsilon$ in X_ε is an $\frac{\varepsilon}{1-\varepsilon}$ best reply. Similarly if $x < -(1 - \varepsilon)$ is a best reply to y then $-(1 - \varepsilon)$ in X_ε is an $\frac{\varepsilon}{1-\varepsilon}$ best reply, which gives the result.

■

The three hypotheses are needed as we give three examples, in each of which two hypotheses are satisfied yet the minmax differ from the maxmin.

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