Bargaining games : a comparison of Nash's solution with the Coco-value

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June 19, 2016

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1 Introduction

Quite often in life two (or more parties) have to conduct negotiation over an issue, with the payoff to each party depending on the outcome of the negotiation. Negotiations between employers and employees on working conditions, nations negotiating treaties, or executives negotiating corporate mergers and acquisition are examples of such bargaining processes. In each of these case, there is a range of outcomes available, only if both parties can come to an agreement, but sometimes negotiations do not lead to an agreement : employees can leave their place of work, countries can impose high tariffs, hurting mutual trade, or negotiations on mergers can fail and the acquisition do not take place.

One of the early problems of game theory was finding a solution to these situations called bargaining games, where two players had the ability to cooperate for their mutual benefit. All the difficulty of this problem lies in the fact that each player will want to maximize his utility, thus both parties can fail to reach an agreement, and therefore end up with a final outcome which would hurt the interests of the two players.

To solve this problem, we have to find an agreement that would be accepted by both parties. One way of finding such solution is to follow the axiomatic approach, where the desired properties are satisfied. Then show that there exists a unique solution satisfying these properties. This was the approach advocated by John Nash in his paper *The Bargaining problem.* In this paper, we will consider unstructured bargaining games, where we do not study the process of the bargaining, we only propose a solution to both players. First we will assume that the two players cannot exchange utility, i.e. they only have their goods to trade, and they cannot exchange money, and derive the Nash solution from a set of four axioms. Second we will study an example where we first assume the utility of each player is not transferable, then we will allow it and see how it impacts the Nash solution. At last, we will compare the Nash solution in the case of transferable utility with another solution, the Coco-value.

2 Bargaining games and Nash solution

2.1 The model

We present the model of Nash Bargaining. This part is inspired from Chapter 10 of the book *Game theory, 2013* by Maschler, Solan and Zamir.
At first we introduce a few notations. Let x and $y \in \mathbb{R}^n$. Denote $x \geq y$ if $x_i \geq y_i$ for all $i = 1, ..., n$. Denote $x > y$ if $x \geq y$ and $x \neq y$. Denote $x \gg y$ if $x_i > y_i$ for all $i = 1, ..., n$.

Definition 2.1. *A bargaining game is an ordered pair (S,d) in which :*

- $S \subseteq \mathbb{R}^2$ *is a non empty, compact and convex set called the set of alternatives.*
- $d = (d_1, d_2) \in S$ *is called the disagreement point.*
- *There exists* $x \in S$ *satisfying* $x \gg d$.

Denote the set of all bargaining games $\mathcal{F}$.

We interpret a bargaining game as a situation in which two players need to agree on an alternative $x = (x_1, x_2) \in S$. If they come to such an agreement, Player 1's payoff is then x_1 and Player's 2 payoff is x_2. If the players cannot come to an agreement, they receive $d = (d_1, d_2)$ as a payoff. The assumptions appearing in the definition of a bargaining game are justified as follows :

- The set S of alternatives is bounded, i.e. the maximal and minimal outcomes of each player are bounded.
- The set S of alternatives is closed, therefore the boundary of every sequence of possible outcomes is still in S. Without this assumption it may be the case that there are no optimal solutions. For instance, if the set of alternatives is $S = \{(a, a), 0 \leq a < 1\}$ c, the players do not have a most-preferred alternative : for every agreement $(x, x) \in S$, $(y, y) = (\frac{x+1}{2}, \frac{x+1}{2}) \in S$ and $(y, y) \gg (x, x)$, which means that to every alternative in S, there always exists an alternative that is stricly preferered by both players.
- The set S of alternatives is convex, i.e. a weigthed average of possible alternatives is also an alternative. For example both players can agree on x with probability $\frac{1}{3}$ and y with probability $\frac{2}{3}$ therefore $\frac{1}{3}$ x + $\frac{2}{3}$ y $\in S$.
- We assume that there exists x such that $x \gg$ d, otherwise both players can profit from not reaching an agreement and this case would be degenerate.

Definition 2.2. *A value function* φ *is a function associating to every bargaining game* $(S, d) \in \mathcal{F}$ *with an alternative* $\varphi(S, d) \in S$

The interpretation we give to a value function φ is that if two players are playing a bargaining game (S, d), the point $\varphi(S, d)$ is the alternative that an arbitrator would propose that the players accepted as an agreement. A value function φ can be seen as an arbitrator who would propose to every bargaining game (S, d) an agreement $\varphi(S, d)$.

2.2 The axiomatic approach : properties of the Nash solution

We will now present 4 properties that one may require from value functions of bargaining games. These properties were first proposed by John Nash in 1950, and they are the mathematical expression of principles that could guide an arbitrator who is called upon a bargain.

2.2.1 Symmetry

The first reasonable axiom an arbitrator may require is fairness : if the game is symmetric, i.e. both players have the same bargaining power, then the solution proposed to each player should not favor a player.

Definition 2.3. *A bargaining game* $(S, d) \in \mathcal{F}$ *is symmetric if the following two properties are satisfied:*

1. $d_1 = d_2$ *(the disagreement point is symmetric).*
2. *If* $x = (x_1, x_2) \in S$ *then* $(x_2, x_1) \in S$

Geometrically, symmetry implies that S is symmetric with respect to the main diagonal in $\mathbb{R}^2$, where the disagreement point is located. The symmetry property forbids the arbitrator from giving preference to one party over the other when both parties have the same bargaining power.

Definition 2.4. *A value function* φ *is symmetric if for every symmetric bargaining game* $(S, d) \in \mathcal{F}$ *the vector* $\varphi(S, d) = (\varphi_1(S, d), \varphi_2(S, d))$ *satisfies* $\varphi_1(S, d) = \varphi_2(S, d)$.

2.2.2 Efficiency

One could argue that a solution to a bargaining game should come up with an agreement which is desirable for both players. We therefore do not want to propose an alternative that can be improved upon, i.e. that is strictly preferred by one player and does not harm the interests of the other player. If such an alternative exists, the arbitrator will prefer it to the proposed alternative.

Definition 2.5. *An alternative* $x \in S$ *is an efficient point of* S *if there does not exist* $y \in S, y > x$. *Denote by PO(S) , where PO stands for Pareto Optimatility, the set of efficient points of S.*

Definition 2.6. *A value function* φ *is efficient if for all* $(S, d) \in \mathcal{F}$, $\varphi(S, d) \in PO(S)$.

Remark 2.1. *PO(S) is on the boundary of S, and therefore* $\varphi(S, d)$ *is on the boundary of S if* φ *is efficient.*

2.2.3 Covariance under positive affine transformations

When the axes of a bargaining game represent monetary payoffs, it is reasonable to require that the value function should be *independent of units of measurement.* In other words, if we measure the payoff to one player in cents instead of dollars, we get a different bargaining game (in which the coordinate corresponding to each point is 100 times larger). In this case we want the coordinate corresponding to the solution to change by the same ratio.

Another desirable property to adopt is *covariance under translations*, i.e. if we add a constant to each player's payoff, the solution will change by the same constant: the amount of money that each player has at the start of the bargaining process should not change the profit that each player gets by bargaining.

Definition 2.7. *A value function* φ *is covariant under changes of units of measurement if for each bargaining game* $(S, d) \in \mathcal{F}$, *and every vector* $a \in \mathbb{R}^2$ *such that* $a \gg 0$,

$$\varphi(aS, ad) = a\varphi(S, d) = (a_1\varphi_1(S, d), a_2\varphi_2(S, d))$$

where $ax = (a_1x_1, a_2x_2)$ for all $x \in \mathbb{R}^2$ and $aS = \{ax, x \in S\}$.

Definition 2.8. *A value function* φ *is covariant under translations if for each bargaining game* (S, d) $\in \mathcal{F}$, *and* $b \in \mathbb{R}^2$,

$$\varphi(S + b, d + b) = \varphi(S, d) + b = (\varphi_1(S, d) + b_1, \varphi_2(S, d) + b_2)$$

By combining those two properties we get the property of covariance under positive affine transformations.

Definition 2.9. *A value function* φ *is covariant under positive affine transformations if for every bargaining game* $(S, d) \in \mathcal{F}$, *for every vector* $a \in \mathbb{R}^2$, $a \gg 0$ *and* $b \in \mathbb{R}^2$,

$$\varphi(aS + b, ad + b) = a\varphi(S, d) + b$$

2.2.4 Independance of irrelevant alternatives

Suppose S⊆T and $\varphi(T,d) \in S$. In the bargaining game (T,d), the arbitrator checked all the alternatives in T knowing the preferences of the players, and decided that the best alternative is $\varphi(T,d)$, which happens to be located in S. What happens now if we restrict the game to (S,d) ? One could assume that the arbitrator will still propose $\varphi(T,d)$ because if there was a more desirable agreement in S, it would still be the solution proposed by the arbitrator in T, and therefore that alternative should have been chosen instead of $\varphi(T,d)$.

Definition 2.10. *A value function φ satisfies the property of independence of irrelevant alternatives if for every bargaining game $(T,d) \in \mathcal{F}$, every subset $S \subseteq T$,*

$$\varphi(T,d) \in S \Rightarrow \varphi(S,d) = \varphi(T,d)$$

2.3 Existence and uniqueness of the Nash solution

We will now prove that there exists a unique value function that satisfies all the properties of symmetry, efficency, covariance under positive affine transformations and independence of irrelevant alternatives.

Definition 2.11. *An alternative $x \in S$ is individually rational if $x \geq d$.*

Denote $D = \{x \in S, x \geq d\}$, the set of individually rational alternatives. Since $d \in S, D \neq \varnothing$.

Theorem 2.1. *There exists a unique value function $\mathcal{N}$ satisfying symmetry, efficency, covariance under positive affine transformations and independence of irrelevant alternatives. Moreover the vector $\mathcal{N}(S,d)$ is the individually rational alternative that maximizes the area of the rectangle whose bottom left vertex is d and whose top-right vertex is $\mathcal{N}(S,d)$.*

The point $\mathcal{N}(S,d)$ is called the *Nash agreement point* and $\mathcal{N}$ the Nash solution.
We will call $f(x)$ the Nash product of $x \in S$, $f(x) = (x_1 - d_1)(x_2 - d_2)$ which is the area of the square whose botton left vertex is d and top right vertex is x.
We will prove that for every bargaining game, there exists an unique alternative maximizing f.

Lemma 2.1. *For every bargaining game $(S,d) \in \mathcal{F}$ there exists a unique point in the set*

$$argmax_{x \in S, x \geq d}(x_1 - d_1)(x_2 - d_2)$$

Proof : If we translate all the points in the plane by adding $-d$ to each point, we get the bargaining game $(S - d, (0,0))$. Since the area of the rectangle is unchanged by translation, the points at which the Nash product is maximized for the bargaining game (S,d) are translated to the points at which the Nash product is maximized for the bargaining game $(S - d, (0,0))$. We can assume without loss of generality that $d = (0,0)$, and then

$$f(x) = x_1 x_2$$

Note that $D = \{x \in S, x \geq d\}$, the set of individually rational points in S, is the intersection of the compact and convex set S, with the closed and convex set $\{x \in \mathbb{R}^2, x \geq d\}$, so it is again a compact and convex set. It is also non-empty because $d \in D$.

Since f is continuous and D is compact, there exists at least one point y in D that maximizes the Nash product. Suppose by contradiction that there exist two distinct points y and z that maximizes f. In particular,

$$y_1 y_2 = z_1 z_2$$

For every c >0, the function $g : x \mapsto \frac{c}{x}$ is strictly convex. For $c = y_1 y_2$ both (y_1, y_2) and (z_1, z_2) are on the graph of the function g because $y_1 y_2 = z_1 z_2$. Therefore $w = \frac{1}{2}$ y + $\frac{1}{2}z$ is above the graph and therefore $f(w) > f(y) = f(z)$ and $z \in D$ by convexity of the set D. Impossible.

□

So the Nash product admits a unique maximizer on the set D of the individually rational alternatives so this value function is well-defined. Given that by assumption there exists $x \in S$ such that $x \gg d$, $x \in D$, therefore $\mathcal{N}(S,d) \gg d$.
We will now show that Nash's value function satisfies the four properties we stated earlier.

Lemma 2.2. *The value function $\mathcal{N}$ satisfies the properties of symmetry, efficency, covariance under positive affine transformations and independence of irrelevant alternatives.*

Proof : We will check each of the proprieties separately.

Symmetry : Let (S, d) be a symmetric bargaining game, and let

$$y^* = \mathcal{N}(S, d) = argmax_{x \in S, x \geq d}(x_1 - d_1)(x_2 - d_2)$$

Denote by z^* the point $z^* = (y_2^*, y_1^*)$, i.e. the point obtained by permutting the coordinates of y^*. Since S is symmetric, and $y^* \in S$, we deduce that $z^* \in S$. Since $d_1 = d_2$, the area defined by the rectangle whose bottom left vertex is d and top right is y^*, is the same as the area of the rectangle defined by d and z :

$$f(y^*) = (y_1^* - d_1)(y_2^* - d_2) = (z_2^* - d_1)(z_1^* - d_2) = (z_1^* - d_1)(z_2^* - d_2) = f(z)$$

But by the previous lemma, f admits an unique maximizer, therefore $y^* = z^*$, i.e $y_1^* = y_2^*$.

Efficiency : Suppose by contradiction that $y^* = \mathcal{N}(S, d)$ is not efficient, i.e there exists $z \in S$, such that $z > y^*$. Then the area of the rectangle defined by d and z is strictly greater than $f(y^*)$, which is absurd given that $y^* = argmax_{x \in S, x \geq d}\ (x_1 - d_1)(x_2 - d_2)$.

Covariance under positive affine transformations : A translation does not change the area of a rectangle, and the multiplication by $a = (a_1, a_2)$ (in the sense we defined earlier) multiplies the area by $a_1 a_2$. It follows that if prior to the application of the affine transformation the Nash product maximizes at y, then after the application of the transformation $x \to ax + b$, the Nash product maximizes at $ay + b$.

Independence of irrelevant alternatives : This comes from a general fact : let S $\subseteq$ T, let g : $T \to \mathbb{R}$ be a function, and let $w \in argmax_{x \in T}\ g(x)$. If $w \in S$, then $w \in argmax_{x \in S}\ g(x)$.
Indeed since $S \subseteq T$ and $w \in argmax_{x \in T}\ g(x)$ and $w \in S$,

$$\max_{x \in S}\ g(x) \geq g(w) = \max_{x \in T}\ g(x) \geq \max_{x \in S}\ g(x)$$

Therefore we have :

$$\max_{x \in S}\ g(x) = \max_{x \in T}\ g(x)$$

So $w \in argmax_{x \in S}\ g(x)$.

□

We now need to prove the uniqueness of the value function to finish the proof of the theorem.

Lemma 2.3. *IF the value function φ satisfies the properties of symmetry, efficency, covariance under positive affine transformations and independence of irrelevant alternatives, then it is identical to the Nash solution.*

Proof : Let φ be a value function satisfying all of the four properties. Let $(S, d) \in \mathcal{F}$, and denote $y^* = \mathcal{N}(S, d)$. We will show that $\varphi(S, d) = y^*$.

Step 1 : Applying a positive affine transformation

Since there exists $x \in S$, such that $x \gg d$, the point $y^* = \mathcal{N}(S, d) \in \{x \in S : x \geq d\}$ at which the Nash product is maximized satisfies $y^* \gg$ d. We can therefore define a positive affine transformation L over the plane shifting d to the origin, and y^* to the point (1,1). This function is given by :

$$L(x_1, x_2) = \left(\frac{x_1 - d_1}{y_1^* - d_1}, \frac{x_2 - d_2}{y_2^* - d_2} \right)$$

Since $y_1^* > d_1$ and $y_2^* > d_2$, the denominators in the definition are strictly positive. The function L is of the form $L = ax + b$, with $a = \left(\frac{1}{y_1^* - d_1}, \frac{1}{y_2^* - d_2} \right) \gg 0$, and $b = \left(\frac{-d_1}{y_1^* - d_1}, \frac{-d_2}{y_2^* - d_2} \right)$. Since the value function satisfies covariance under positive affine transformations,

$$\mathcal{N}(aS + b, ad + b) = \mathcal{N}(aS + b, (0, 0)) = ay^* + b = (1, 1).$$

Step 2 : $x_1 + x_2 \leq 2$ **for every** $x \in aS + b$

Let $x \in aS+b$. Since S is convex, the set $aS+b$ is also convex. Since $x \in aS+b$, and $(1,1) \in aS+b$, the interval connecting x and $(1,1)$ is also in $aS+b$. In other words for all $\epsilon \in [0,1]$, the point z^ϵ defined by

$$z^\epsilon = (1-\epsilon)(1,1) + \epsilon x = (1+\epsilon(x_1-1), 1+\epsilon(x_2-1))$$

belongs to aS + b. If ϵ is sufficiently close to 0 then $z^\epsilon \geq (0,0)$, and therefore $z^\epsilon \in \{w \in aS+b, w \geq (0,0)\}$. It follows that for each such ϵ,

$$f(z^\epsilon) \leq \max_{w \in aS+b, w \geq (0,0)} f(w) = f(\mathcal{N}(aS+b,(0,0))) = f(1,1) = 1$$

Hence

$$\begin{aligned} 1 \geq f(z^\epsilon) = z_1^\epsilon z_2^\epsilon &= 1 + \epsilon(x_1+x_2-2) + \epsilon^2(x_1-1)(x_2-1) \\ &= 1 + \epsilon(x_1+x_2-2+\epsilon(x_1-1)(x_2-1)) \end{aligned}$$

Therefore, for every $\epsilon > 0$ sufficiently small,

$$0 \geq \epsilon(x_1+x_2-2+\epsilon(x_1-1)(x_2-1)),$$

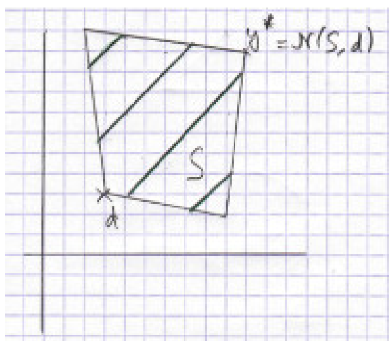
leading to the conclusion that

$$2 + \epsilon(x_1-1)(x_2-1) \geq x_1 + x_2.$$

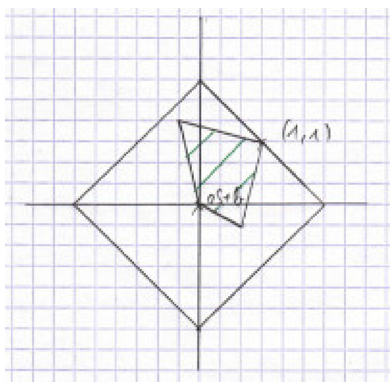
So by taking the limit as ϵ approaches 0, we have that $2 \geq x_1 + x_2$, which is what we wanted to show.

Step 3 : $\varphi(S,d) = \mathcal{N}(S,d)$

Let T be a symmetric square relative to the diagonal $x_1 = x_2$ that contains $aS+b$, with one side along the line $x_1 + x_2 = 2$. For instance we can see an example of bargaining game (S,d) before affine transformation :



Then we apply the affine transformation L which maps the disagreement point to the point (0,0) and the Nash solution to the point (1,1) :



Since $aS + b$ is compact, such a square exists. By the symmetry and efficiency of φ, one has $\varphi(T, (0, 0)) = (1, 1)$. Since the value function φ satisfies independence of irrelevant alternatives, and since $aS + b$ is a subset of T containing $(1, 1)$, it follows that $\varphi(aS + b, (0, 0)) = (1, 1)$. Since the value function φ satisfies independence of irrelevant alternatives, and since $aS + b$ is a subset of T containing $(1, 1)$, it follows that $\varphi(aS + b, (0, 0)) = (1, 1)$. Since the value function satisfies covariance under positive affine transformations, one can implement the inverse transformation L^{-1} to deduce that $\varphi(S, d) = y^*$. Since $y^* = \mathcal{N}(S, d)$, we conclude that $\varphi(S, d) = \mathcal{N}(S, d)$, as required. □

3 Example of bargaing game solved by the Nash solution

Now that we have introduced Nash's solution to bargaining problems, we will see how we can apply it and run it numerically with Python. We will then compare it to another solution of bargaining games. Note that untill now, we only talked about bargaining games without transfer of utility, which means that the agents had no mean to exchange utility through money, or any other kind of side payment. In this chapter we will later cover games with transfers of utility where the players will have the opportunity to exchange money, and see how this affects the Nash solution.

3.1 Two-players bargaining games with N objects

The following example comes from John Nash's article *The bargaining problem, 1950.* Suppose we have two rational and intelligent individuals, Bill and Jack, who are in a position to barter goods but have no money to facilitate the exchange, i.e. there cannot be any exchange of utility. Further let us assume for simplicity that the utility of either individual of a portion of the total number of goods involved is the sum of the utilities to him of the individual goods in that portion. We give below a table of goods possessed by each individual with the utility of each to each individual.

Bill's goods	Utility to Bill	Utility to Jack
book	2	4
whip	2	2
ball	2	1
bat	2	2
box	4	1
Jack's goods		
pen	10	1
toy	4	1
knife	6	2
hat	2	2

To solve this problem, we will use the Nash solution we have previously defined.
But first we need to model the problem so it fits the problem we have, i.e. from the original utilities and goods each player possesses, we have to derive the bargaining game (S, d).

In order to do so, we have to draw the convex hull of all the alternatives of the game, because as explained earlier, players can play mixed strategies, therefore any weigthed average of possible alternatives is also an alternative. So this leaves us with a convex hull derived from 2^N points, with N being the total number of objects being traded. Indeed, for any object we have two choices : to give it to Jack or to Bill, and given that there are N objects, this leaves us with 2^N alternatives.

Now to find the Nash solution, it is sufficient to look at the alternatives $z \in \partial S$ i.e. on the the boundary of S , such that $z \gg d$. Indeed we saw in the previous chapter that the Nash solution is on the boundary of the set of alternatives by efficiency of the solution. Furthermore there exists x such that $x \gg d$, we must have that $\mathcal{N}(S, d) \gg d$.

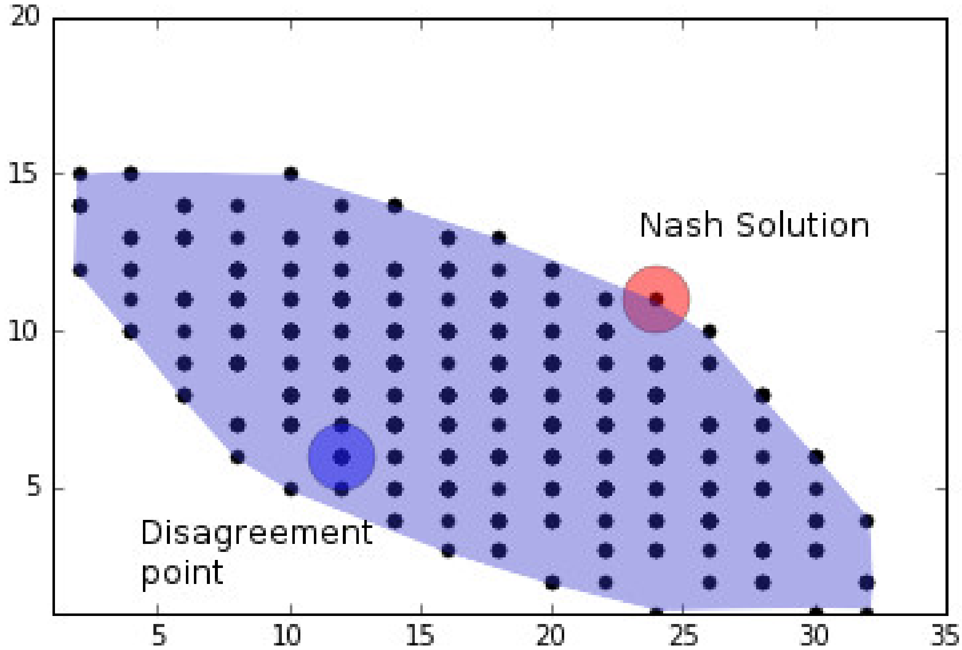
Finally, we just have to look at the points in the boundary of S and belong to $\{x \in S, x \gg d\}$, then find the point maximizing the Nash product, which we recall is $f(x) = (x_1 - d_1)(x_2 - d_2)$.

We will now give an algorithm which draws the convex hull and find the Nash solution of a given bargaining game derived from a barter game with a finite number of objects, and by the previous caculation, a finite number of alternatives.

We have implemented this algorithm on Python [1] for the Nash solution that was previously introduced. We will assume that d, the disagreement point is the alternative where each player keeps the

[1] cf *Appendix*

goods he had before the exchange. We obtain the following graph, with the Nash solution being the point highlit in red.



As it is obtained by the alorithm, the Nash solution proposes the following distribution of the goods : Bill will get the box, the pen, the toy, and the knife, whereas Jack will have the book, the whip, the ball, the bat, and the hat.

3.2 Transfers of utility with the Nash solution

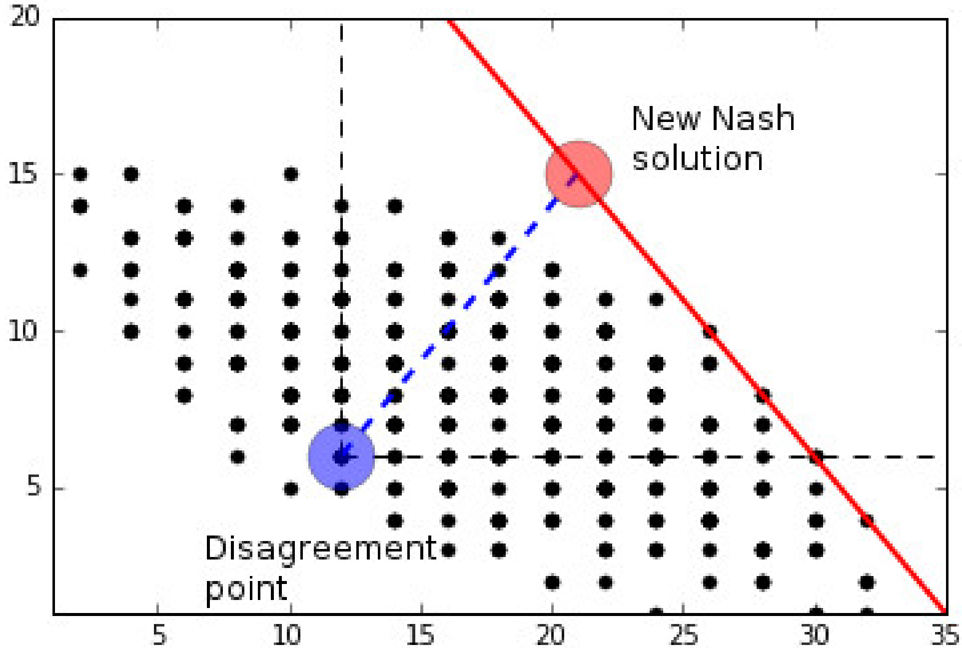
So far we have assumed that players had solely their goods as bargaining tokens, and could not use a currency for instance for their exchange. One might wonder then what might happen if we allow the players to exchange money.

If we suppose that players are able to exchange money, then new alternatives arise : for any point $x = (x_1, x_2) \in S$, if we denote $c = x_1 + x_2$, then all points belonging to the line $y = c - x$, will belong to S. In particular, this implies that the points in the boundary of S that are efficient now belong to a straight line which passes through the Nash solution, given that the Nash solution is efficient.

Now the Nash solution is even simpler than before : it is merely the intersection between the line $x + y = W$, W being the social optimum, and the line which starts from the disagreement point $d = (d_1, d_2)$ with a slope of 1. In fact, the Nash solution can be explicited quite easily given by the equations of the two lines and we find that :

$$\mathcal{N}(S,d) = \left(\frac{W - (d_2 - d_1)}{2}, \frac{W + (d_2 - d_1)}{2}\right)$$

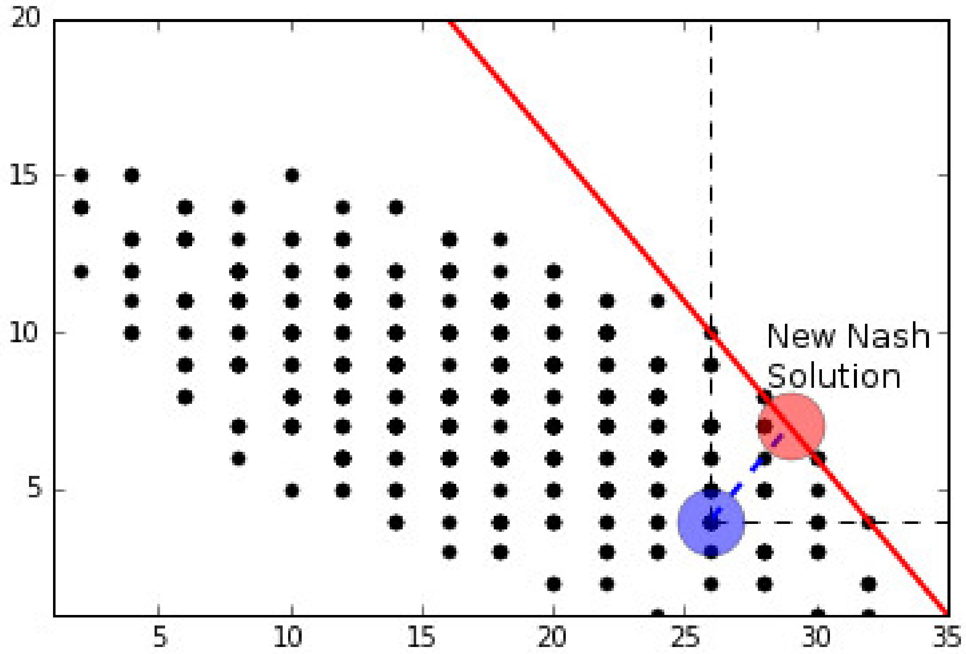
For instance, we can look at the previous example with two players trying to struck a bargain but this time with the ability to exchange utility.



As it is shown, finding the solution is quite easy, the red line is the line whose equation is $x + y = W$, and then the dotted line is simply the line starting from $d = (d_1, d_2)$ and with a slope of 1.

One intuitive but interesting remark can be seen on this graph : the disagreement point greatly affects the Nash solution. To be more precise, if for instance one player starts with a disagreement payoff far greater than the other, the Nash solution will greatly favor him. The impact of the disagreement point was already explicit without transfer utility we can directly see this on the graph. This can also be seen from the explicit formulation of the Nash solution : if for instance $d_1 \gg d_2$ (in the sense that d_1 is far greater than d_2), then if we denote $\mathcal{N}(S, d) = (\mathcal{N}_1, \mathcal{N}_2)$, then $\mathcal{N}_1 = \frac{W-(d_2-d_1)}{2} \gg \frac{W+(d_2-d_1)}{2} = \mathcal{N}_2$ (in the same sense).

If for instance we modify the previous example by assuming that before the bargaing game, player 1 owns all the objects except the last two, then we have the following Nash Solution (which clearly favors the player 1) :


New Nash
Solution

4 A comparison with the Coco-value

4.1 Introduction of the Coco-value and interpretation of both solutions

The Nash Solution to a bargaining game is not the only solution that exists for bargaining games, there exist other characterizations of a solution, which choose to replace the axiom of indepedence irrelevant alternatives to build an another solution, Kalai-Smorodinsky [1979] who proposes to replace the axiom of independence of irrelevant alternatives by the axiom of monotonicity, but we will not develop this approach here.

The other solution we wil propose is another approach that applies to a broader range of games : the Coco-value. But first, let us introduce the Coco-value for non-cooperative two-player games.

Definition 4.1. *Let* $\Gamma = (S^1, S^2, g^1, g^2)$ *be a two-player game. Denote* $\Gamma_0 = (S^1, S^2, g^1 - g^2, g^2 - g^1)$ *and* $\Gamma_c = (S^1, S^2, g^1 + g^2, g^1 + g^2)$*. We define the Coco-value of the game by :*

$$Coco(\Gamma) = \left(\frac{W(\Gamma_c) + val(\Gamma_0)}{2}, \frac{W(\Gamma_c) - val(\Gamma_0)}{2} \right)$$

with $W(\Gamma_c)$ *the social optimum of the game* $\Gamma_c = \max_{x \in S^1, y \in S^2}\ x + y$*, i.e. the best payoff that the two players can hope to achieve if they cooperate, and* $val(\Gamma_0)$ *the value of the zero-sum game* Γ_0*.*

The definition of the Coco-value shows that any two-player game can be seen as a mixture of both a zero-sum game and a pure coordination game. The Coco-value reflects what is the best alternative they can achieve by cooperating when they have the ability to exchange utility, while they still try to maximize their utility individually.

Now that we have introduced the Coco-value, we will try to apply it to bargaining games, and compare it to the Nash Solution.

From the characterization of each solution, a comparison can directly be made :

$$Coco(\Gamma) = \left(\frac{W + val(\Gamma_0)}{2}, \frac{W - val(\Gamma_0)}{2} \right)$$

$$\mathcal{N}(S, d) = \left(\frac{W + d_1 - d_2}{2}, \frac{W + d_2 - d_1}{2} \right)$$

Therefore, if we denote by $v = val(\Gamma_0)$, we have that :

$$Coco(\Gamma) = \mathcal{N}(S, d) \iff v = d_1 - d_2 \iff val(g^1 - g^2) = val(g^1) - val(g^2)$$

The condition of $val(g^1 - g^2) = val(g^1) - val(g^2)$ is often not verified and embodies the difference between the two approaches : the Nash solution gives to player 1 the difference between each player's value, while the Coco-value gives to player 1 the value of the game where the objective of the player is to maximizes his difference with the other player.

Thus the Coco-value favors the player who has the edge over the other in terms of payoffs, while the Nash solution favors the player who can assure more than the other and will be the least affected if they did not reach an agreeement.

We can see this with the following game :

(1,2)	(2,3)
(2,3)	(3,4)

Here we have that $g^1 - g^2 = -1$, $W = 9$ and $v^1 = v^2 = 3$. Therefore we have that : $Coco(\Gamma) = (4, 5)$ and $\mathcal{N}(S, d) = (4.5, 4.5)$. This game favors the player 2 in terms of difference with the other player, therefore the player 2 is favored by the Coco-value. Nonetheless, both players have the same value, therefore the Nash solution is the same for both players.

This example gives us a hunch to interpret the two value functions : the Coco-value is a solution which favors the player who earns more than the other, while the Nash solution favors the player who can assure more than the other.

4.2 Properties of the Coco-value

Now we will focus on the several properties that the two solutions share, and show that the Coco-value satisfies two of the axioms of the Nash solution : the symmetry and the efficiency.

Symmetry : If the game $\Gamma = (S^1, S^2, g^1, g^2)$ is symmetric, we have that the value of the game $\Gamma_0 = (S^1, S^2, g^1 - g^2, g^2 - g^1)$ is 0. Indeed, if we denote by A the matrix of representing the payoffs of player 1, and B the matrix of player 2, by symmetry we have that $B = A^T$, therefore we have that the matrix of the game Γ_0 is $C = A - A^T$, therefore $C = -C^T$, and if we denote by g the payoff of player 1 in the game Γ_0, we have that :

$$-val(\Gamma_0) = -\max_{X^1}\min_{X^2} g = \min_{X^1}\max_{X^2} -g = \min_{X^2}\max_{X^1} g = val(\Gamma_0)$$

with $X^i, i = 1, 2$ the probabilities on the set $S^i, i = 1, 2$. Indeed if $C = -C^T$ this means that $-g$ can be replaced by g if we swap the players. From this we conclude that $val(\Gamma_0) = 0$ and that :

$$Coco(\Gamma) = \left(\frac{W(\Gamma_c) + val(\Gamma_0)}{2}, \frac{W(\Gamma_c) - val(\Gamma_0)}{2}\right) = \left(\frac{W(\Gamma_c)}{2}, \frac{W(\Gamma_c)}{2}\right)$$

Therefore the Coco-value satisfies the axiom of symmetry.

Efficiency : If we sum the utilities of each player we have that :

$$Coco(\Gamma)_1 + Coco(\Gamma)_2 = \frac{W(\Gamma_c) + val(\Gamma_0)}{2} + \frac{W(\Gamma_c) - val(\Gamma_0)}{2} = W(\Gamma_c)$$

Therefore the Coco-value is Pareto-efficient.

Covariance under positive affine transformation : The Coco-value does not verify the property of covariance under positive affine transformation, because if we apply an affine transformation $L = ax + b$, with $a, b \in \mathbb{R}^2$ and $a \gg 0$ to the Coco-value gives uncertain results as the value used to compute the Coco-value, which was $val(g^1 - g^2)$, now becomes $val(a^1g^1 + b^1 - a^2g^2 - b^2)$ which has no reason to be linearly separable as we would need.

However, even if the Coco-value does not fully verify the propriety of covariance under positive affine transformation, it still verifies part of the propriety.

Lemma 4.1. *The Coco-value satisfies covariance under translation and same-scale transformation, i.e. if we change the utility of the two players by the same ratio $a > 0$, then the Coco-value is multiplied by a.*

Proof :

We will first prove that the Coco-value satisfies covariance under translation. Let $\Gamma = (S^1, S^2, g^1, g^2)$ a two-player game, and $b \in \mathbb{R}^2$.

Then if we note $Coco(\Gamma) = \left(\frac{W+val(\Gamma_0}{2}), \frac{W-val(\Gamma_0}{2}\right)$, then

$$Coco(\Gamma + b) = \left(\frac{W + b^1 + b^2 + val(\Gamma_0) + b^1 - b^2}{2}, \frac{W + b^1 + b^2 - val(\Gamma_0) - b^1 + b^2}{2}\right)$$

by the definition of the social optmimum which is the max both players can achieve if they cooperate, and the fact that $val(A+c) = val(A) + c$, for any $A \in \mathcal{M}_{m\times n}$ a real matrix representating a zero-sum game. This is justified by the fact that any translation does not change the set of optimal strategy of each player. Therefore we have that :

$$Coco(\Gamma + b) = \left(\frac{W + val(\Gamma_0)}{2} + b^1, \frac{W - val(\Gamma_0)}{2} + b^2\right) = \left(\frac{W + val(\Gamma_0)}{2}, \frac{W - val(\Gamma_0)}{2}\right) + b = Coco(\Gamma) + b$$

We will now prove that the Coco-value is covariant under same-scale transformation.

Let $a > 0$ and $\Gamma = (S^1, S^2, g^1, g^2)$ a two-player game.
Denote $\widetilde{\Gamma} = (S^1, S^2, a \times g^1, a \times g^2)$, the modified game Γ where each player's utility has been multiplied by the same factor a. Then we have that :

$$Coco(\widetilde{\Gamma}) = \left(\frac{a \times W + a \times val(\Gamma_0)}{2}, \frac{a \times W - a \times val(\Gamma_0)}{2}\right) = a \times \left(\frac{W + val(\Gamma_0)}{2}, \frac{W - val(\Gamma_0)}{2}\right) = a \times Coco(\Gamma)$$

□

This means that even though the Coco-value does not verify covariance under positive affine transformations, it is almost covariant under change of units of measurement, because as we remind, a solution φ verifies this property if for every $a \in \mathbb{R}^2, a \gg 0$ and $(S, d) \in \mathcal{F}$:

$$\varphi(aS, ad) = a\varphi(S, d) = (a_1\varphi_1(S, d), a_2\varphi_2(S, d))$$

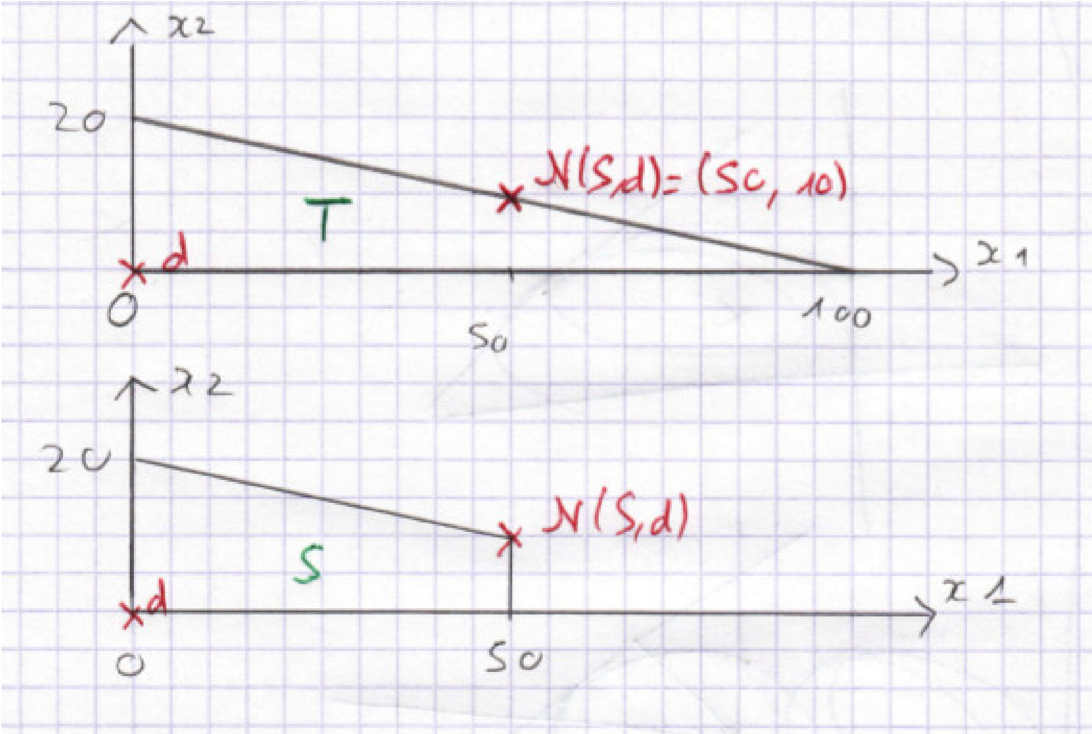
which means a_1 and a_2 need not be the same, while for the Coco-value we need to multiply the utilities of both players by the same factor.

This restriction to same-scale transformation is not so bothersome, as one may require that if we change the scale of the utilities, for instance switching from dollars to euros, both players should change simultaneously, given that if they do not both trade in the same currency, we will have a deal with euros on one side and dollars on the other, which would be irrelevant.

4.3 Critique of the Nash solution and monotonicity of the Coco-value

The most controversial property of the Nash solution is the independence of irrelevant alternatives. Luce and Raifa [1957] gave an example which questions the relevance of this axiom.

Consider the following games :



The Nash solution for both games is $(50, 10)$. Suppose the players start negotiating in the set T, and see $(50, 10)$ as a desirable outcome. We then suppose that for a reason, players end up bargaining in the set of alternatives S. Player 2 can now claim that $(50, 10)$ is unreasonable : in the bargaining game $(T, (0, 0))$, both players compromised to arrive at the outcome $(50, 10)$, but in the bargaining game $(S, (0, 0))$, $(50, 10)$ gives to player 1 his best possible outcome while player 2 does not get his highest possible payoff. It is therefore reasonable for player 2 to demand more than 10, by claiming that player 1 should also compromise and receive less than he would from his best alternative.

To answer this critique, Kalai and Smoridinsky [1975] proposed an another axiomatic approach where the axiom of independence from irrelevant alternatives is replaced by the propriety of monotonicity. We will now present this property, and show it is verified by the Coco-value.

Definition 4.2. *A value function φ satisfies monotonicity if for every $(S, d) \in \mathcal{F}$ and $(T, d) \in \mathcal{F}$, such that $S \subseteq T$:*

$$\varphi(S, d) \leq \varphi(T, d)$$

This means that adding more alternatives to the game does not make a player's situation worse. This property may be desirable because in some cases, as the one shown earlier, the axiom of independence of irrelevant alternatives leads to unreasonable outcomes.

Now let us prove that the Coco-value verifies this property :

Proof :

To prove this, we will simply prove that if we add more strategies to a player, the Coco-value of the game with his added strategies should be larger than the previous one. Let $\Gamma = (S^1, S^2, g^1, g^2)$, and T^1 a set such that $S^1 \subseteq T^1$. Denote by $\widetilde{\Gamma} = (T^1, S^2, g^1, g^2)$, and for a given Γ, $Coco(\Gamma) = (Coco^1(\Gamma), Coco^2(\Gamma))$.

Given that $S^1 \subseteq T^1$, we have that $val(\Gamma_0) \leq val(\widetilde{\Gamma_0})$, therefore :

$$Coco^1(\Gamma) = \frac{W + val(\Gamma_0)}{2} \leq \frac{\widetilde{W} + val(\widetilde{\Gamma_0})}{2} = Coco^1(\widetilde{\Gamma})$$

5 Conclusion

In this paper, we concentrated on bargaining games, where two individuals have the opportunity to collaborate where the action of each player will affect the final outcome for both players . We followed the axiomatic approach to tackle this problem, and found that if a value function satisfied the properties of symmetry, efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives, then it is identical to the Nash solution.

We then studied a simple example of bargaing games given by John Nash in his 1950 article *The bargaining problem*, and saw how we could solve numerically a bargaining game with a finite number of objects by using the Nash solution.

Finally, we compared the Nash solution with the Coco-value, and showed that these solutions differ by their approach : the Nash solution favored the player who can assure the most, while the Coco-value favored the player who could ensure the most in the worst case compared to the other player. We then compared several properties shared by the solutions : symmetry and effiency. In the end we saw that the axiom of independence from irrelevant alternatives is open to critics. Kalai and Smoridinsky proposed an another solution in 1975, where they would replace the axiom of independence with monotonicity.

6 Appendix

References

[1] John F. Nash. *The bargaining problem, 1950.* Econometrica 18, 155-62.

[2] Maschler, Solan and Zamir. *Game theory, 2013.* Cambridge University Press.

[3] Kalai. *Other solutions to Nash bargaining problem, 1975.* Econometrica, 43, 513-18.

[4] Raiffa and Schlaifer. *Games and decision, 2013.* New York : Wiley.

Memoire_Code

June 15, 2016

In [1]:
```
%pylab inline
from scipy.spatial import ConvexHull
import sys
print(sys.version)
```

```
Populating the interactive namespace from numpy and matplotlib
2.7.11 |Anaconda 4.0.0 (64-bit)| (default, Dec  6 2015, 18:08:32)
[GCC 4.4.7 20120313 (Red Hat 4.4.7-1)]
```

In [2]:
```
#Vecteur des préférences
Y1 = array([2,2,2,2,4,10,4,6,2])
Y2 = array([4,2,1,2,1,1,1,2,2])
```

In [94]:
```
#Répartition intiale des biens
a0 = array([0,0,0,0,0,1,1,1,1])
n = len(a0) #Nombre d'objets
```

In [88]:
```
#Répartition secondaire des biens
a0 = array([0,0,0,0,0,0,0,1,1])
n = len(a0) #Nombre d'objets
```

In [89]:
```
def toBinary(N): #Convertit en binaire
    return ''.join(str(1 & int(N) >> i) for i in range(n)[::-1])
```

In [90]:
```
def isEqual(X,y): #Renvoie le vecteur des booleens
    Y = y * ones(len(X))
    Z = [int(x) for x in (X==Y)]
    return Z
```

In [91]:
```
def payOff(Y1,Y2,a):
    X = zeros(2)
    X[1] = dot(a,Y2)
    X[0] = dot(Y1,isEqual(a,0))
    return X
```

In [95]:
```
cloud = zeros((2**n,n))
points = zeros((size(cloud),2))

for i in range(2**n): #Ensemble des répartitions possibles
    cloud[i,:] = array(list(toBinary(i)))

points = [] #Ensemble des points
for i in range(2**n):
    coord = payOff(Y1,Y2,cloud[i,:])
    points.append([coord[0], coord[1]])
```

```
d0 = payOff(Y1,Y2,a0) #Point de desaccord
X = [x[0] for x in points]
Y = [x[1] for x in points]
```

```
---------------------------------------------------------------------------

ValueError                                Traceback (most recent call last)

<ipython-input-95-4f4aacb2e27e> in <module>()
  7 points = [] #Ensemble des points
  8 for i in range(2**n):
----> 9     coord = payOff(Y1,Y2,cloud[i,:])
 10     points.append([coord[0], coord[1]])
 11


<ipython-input-91-89fc4de80f2e> in payOff(Y1, Y2, a)
  2     X = zeros(2)
  3     X[1] = dot(a,Y2)
----> 4     X[0] = dot(Y1,isEqual(a,0))
  5     return X


ValueError: setting an array element with a sequence.
```

In [8]:

```
#Récupération des points de l'enveloppe convexe
hull = ConvexHull(points)

hull_indices = unique(hull.simplices.flat)
hull_points = [points[i] for i in hull_indices]

X_hull = [x[0] for x in hull_points]
Y_hull = [x[1] for x in hull_points]
```

In [14]:

```
#Calcul de la solution de Nash

 w = 0
 index_w = 0
 maxi = 0
 index_max = 0
 for i in hull_indices:
     product = (X[i] - d0[0]) * (Y[i] - d0[1])
     new_w = X[i] + Y[i]
     if new_w > w:
         w = new_w
         index_w = i
     if product > maxi:
         index_max = i
         maxi = product
 print("Utilite du joueur 1 "+ str(X[index_max]))
 print("Utilite du joueur 2 "+ str(Y[index_max]))
 for i in range(n):
```

```
            print("Le bien "+ str(i+1) + " revient au joueur " + str(int(cloud[index_max,i]+1)))
        print(w)
```

```
Utilite du joueur 1 24.0
Utilite du joueur 2 11.0
Le bien 1 revient au joueur 2
Le bien 2 revient au joueur 2
Le bien 3 revient au joueur 2
Le bien 4 revient au joueur 2
Le bien 5 revient au joueur 1
Le bien 6 revient au joueur 1
Le bien 7 revient au joueur 1
Le bien 8 revient au joueur 1
Le bien 9 revient au joueur 2
36.0
```

In [87]:
```
#Graphique

Xw = X[index_w]
Yw = Y[index_w]

dx = d0[0]
dy = d0[1]

inter = (w - (dy - dx) )/2

plot(X,Y,'ko', markersize = 5)
plot(X[index_max],Y[index_max], 'ro', markersize=25,alpha=.25)
plot(Xw,Yw, 'bo', markersize=25,alpha=.25)
plt.plot([w-10,10], [10,w-5], 'r-', lw=2)
plt.plot([dx,inter], [dy,w-inter])
```

Out[87]: [<matplotlib.lines.Line2D at 0x7fa9b4506a10>]