The asset pricing in finite and discrete time-horizon

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Final year project
under the direction of Xiaolu TAN

June 28, 2017

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1 Introduction

1.1 Definition and concept

Financial markets allow the transfer of liquidities between investors and agents with financing needs. They can provide information on economic agents but this information isn't always accurate or timely (for instance company's stock prices and their financial health are not always available to all agents of the market at the same time). This brings us to define the notion of market efficiency. A market is efficient if at each time, prices and return rates of financial instruments reflect completely all the information available at that time. Securities always trade at their fair value and only unpredictable events can cause price variations.
In 1979, Harrison and Kreps show that the absence of arbitrage, under some regularity conditions, is equivalent to the existence of a risk neutral probability measure. Under this probability measure, the expected rate of return of a risky asset is equal to the risk-free rate and thus, economic agents can not reasonably expect to make more money by investing in the risky assets rather than in the risk-free one.
We study here the no-arbitrage conditions and the conditions for the completeness of a market. The Dalang-Morton-Willinger theorem (1990) is a generalisation of the Harrison-Pliska theorem (1981) as it also works for non finite probability spaces.

1.2 Notations and initial framework

We consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ representing the possible outcomes affecting prices and their probability to happen.

Definition 1. In order to represent the information accumulated at a time, we introduce the notion of filtration. We call filtration on $(\Omega, \mathcal{F})$ an increasing sequence $\mathcal{F}_n$ of sub $\sigma - algebras\ \mathcal{F}$.

Definition 2. We will denote S the price process. A stochastic process is a collection of random variables (S_n) defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and assuming values in $\mathbb{R}^m$.

Definition 3. S is an $\mathcal{F}$-adapted process if for $\forall n \in \mathbb{N}$, S_n is $\mathcal{F}_n$-measurable . Recall that the price process take into account all the past informations, therefore S must be $\mathcal{F}$-adapted.

Definition 4. $(\theta)_{t=0}^{T}$ is a predictable process if $\forall n \in \mathbb{N}$, θ_{n+1} is $\mathcal{F}_n$-measurable. Here, $(\theta)_{t=0}^{T}$ denotes the strategy of the agent, it is predictable as it only depends on the previous portfolio changes.

Definition 5. (M_n) is a martingale with respect to the filtration $(\mathcal{F}_n)$ if:

- (M_n) is adapted $(\mathcal{F}_n)$
- (M_n) is integrable for all n
- $E[M_{n+1} \mid \mathcal{F}_n] = M_n$

2 The First fundamental theorem of asset pricing

2.1 The no-arbitrage condition

An arbitrage is a set of financial operations with zero initial capital, which ill generate non-negative cash flows in the future, with a positive probability that at least one of the positive cashflows will be positive.

Let $L^0(\Omega, \mathcal{F}_\mathcal{T}, \mathbf{P})$ the set of random variables measurable with respect to the σ-algebra $\mathcal{F}_\mathcal{T}$.
If the agent chooses the strategy $(\theta)_{t=0}^{T}$, then his total gain or loss would be given by $H = \sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) >$ We can now consider the linear subspace of $L^0(\Omega, \mathcal{F}_\mathcal{T}, \mathbf{P})$ representing all the possible gains, with θ running through the set of predictable m-dimensionnal processes :

$$K = \{H \mid H = \sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) >\}$$

Let us introduce the convex cone :

$$C = K - L^0_+ = \{k \in K \mid \exists l \in L^0 \; such \; that \; k \geqslant l\}$$

C denotes the set of all super-replicable contingent claims with no initial investment.

2.2 The Dalang-Morton-Willinger Theorem

Definition 6. Given a measurable space $(\Omega, \mathcal{F})$, if a σ-finite measure ν is abolutely continous with respect to a σ-finite measure μ, then we denote by $\frac{d\nu}{d\mu}$ the Radon-Nikodym's derivative.

Theorem 1. *The market satisfies the no-arbitrage condition if it satisfies any of the following equivalent statements :*

1. $C \cap L^0_+ = \{0\}$
2. $C \cap L^0_+ = \{0\}$ *and* $C = \overline{C}$
3. *There is a probability measure Q with bounded and positive Radon–Nikodym's derivative dQ/dP, such that the coordinates of the m-dimensional process S are martingales under Q.*

Remark. The first statement is the basic interpretation of no arbitrage ($H \geqslant 0 \implies H = 0$)

2.3 A look at the case $T = 1$

The convergence in probability implies the existence of a sub-sequence almost surely convergent and the almost surely convergence implies the convergence in probability when the measure is finite. Thus, a set $Z \subset L^0$ is closed if and only if the limit of every almost surely convergent sequence from Z is in Z. With this in mind, we show the following lemma which allows us to avoid measurable selection arguments in further proofs.

Lemma 1. Let (η_n) be an $\mathbb{R}^m$-valued sequence of measurable functions. If for every outcome ω the sequence $(\eta_n(\omega))$ is bounded, then there is a strictly increasing, measurable sequence of integers (σ_k) such that for every outcome the sequence (η_{σ_k}) is convergent. If $sup_n||\eta_n|| = \infty$ then there is a strictly increasing, measurable sequence of integers, (σ_k) such that for every outcome ω, $lim_{k\to\infty}||\eta_{\sigma_k}(\omega)|| = \infty$

Remark. Fixing ω, $(\eta_n(\omega))$ is just a sequence of vectors of $\mathbb{R}^m$ so that any norm can be used here.

Proof. The existence of a strictly increasing sequence such that (η_{σ_k}) is convergent stems from Bolzano-Weierstrass theorem. The essential aspect of this lemma lies in the measurability of the subsequence. Looking first at the case m=1, for all ω, $\eta_\infty(\omega) = \liminf_n \eta_n(\omega)$ exists by definition and is finite. η_∞ is measurable as a limit of measurable functions. Now, let us construct a measurable sequence (σ_k) such that $\eta_{\sigma_k} \to \eta_\infty$. Letting $\sigma_0 = 0$, we define the random variable :

$$\sigma_k = \inf\{n \in \mathbb{N} \mid n > \sigma_{k-1} \text{ } and \text{ } ||\eta_n - \eta_\infty|| \leqslant \frac{1}{k}\}$$

For $m \geqslant 1$, by induction, we construct a strictly increasing function σ_1 making the first coordinate converge. We then compose this by σ_2 so that the first two coordinates of $\eta_{\sigma_2\circ\sigma_1}$ converge. We repeat the process until all m coordinates converge.
Similary, we prove the second part by constructing :

$$\sigma_k = \inf\{n \in \mathbb{N} \mid n > \sigma_{k-1} \text{ } and \text{ } ||\eta_n|| \geqslant k\}$$

□

This lemma allows us to keep finite-dimensionnal sub-spaces closedness when changing stability under scalar multiplication by stability under measurable function multiplication.

Lemma 2. (Stricker) Let $f_1, ..., f_m$ $\mathcal{A}$-measurable functions with $\mathcal{F} \subset \mathcal{A}$.

$$L = \{f \mid f = \sum_{i=1}^{m} f_i\theta_i, \ \theta_i \in L^0(\Omega, \mathcal{F}, \mathbf{P})\}$$

L is a closed sub-space of $L^0(\Omega, \mathcal{A}, \mathbf{P})$

Remark. If at least one of the f_i is a non-zero constant function, then $L^0(\Omega, \mathcal{F}, \mathbf{P}) \subset L$

Proof. The stability under addition and scalar multiplication is clear. Letting $(l_n) \in L^{\mathbb{N}}$ such that :

$$l_n = \sum_{i=1}^{m} f_i\theta_{i,n} = F \cdot \Theta_n \xrightarrow{\mathbf{P}} l_\infty$$

where $F = (f_1, ..., f_m)$ and $\Theta_n = (\theta_{1,n}, ..., \theta_{m,n})$. Without loss of generality, we assume $l_n \xrightarrow{a.s.} l_\infty$ and we want to show $l_\infty \in L$. The idea is to find a strictly increasing measurable sequence (σ_k) such that $\Theta_{\sigma_k} \to \Theta_\infty$ and then $F \cdot \Theta_{\sigma_k} \to F \cdot \Theta_\infty = l_\infty$ resulting in $l_\infty \in L$. In order to use Lemma 1, we need to show that for all ω, $\Theta_n(\omega)$ is bounded. Assume that this is not the case and let :

$$\Omega_0 = \{\omega \in \Omega \mid \sup_n ||\Theta_n(\omega)|| = \infty\}$$

For $\omega \in \Omega_0$, let us divide $l_n(\omega)$ by $||\Theta_n(\omega)||$:

$$\frac{l_n(\omega)}{||\Theta_n(\omega)||} = F \cdot \Theta_n(\omega)$$

Since $\frac{\Theta_n(\omega)}{||\Theta_n(\omega)||}$ is bounded, there is a strictly increasing measurable sequence σ_k such that $\frac{\Theta_{\sigma_k}(\omega)}{||\Theta_{\sigma_k}(\omega)||}$ is convergent. Let us define :

$$\Omega_1 = \{\omega \in \Omega_0 \mid \sup_n ||\Theta_{\sigma_k}(\omega)|| = \infty\}$$

From this step,for all $\omega \in \Omega_1$, $||\Theta_{\sigma_k}|| \to \infty$ and since l_n is convergent,

$$\frac{l_{\sigma n}(\omega)}{||\Theta_{\sigma_k}||} \to 0$$

Thus, we have :

$$F \cdot \Theta_\infty(\omega) = 0,\ \omega \in \Omega_1$$

Θ_∞ is the limit of vectors of norm equal to one, therefore $\Theta_\infty(\omega)$ is not zero for all $\omega \in \Omega_1$, so that at least one coordinate is not zero. Denote it $\theta_{j,\infty}$. For $\omega \in \Omega_1$,

$$\sum_{i=1}^{m} f_i(\omega)\theta_{i,\infty}(\omega) = 0$$

Since $\theta_{j,\infty}$ is not zero, we obtain an expression with m-1 measurable weights :

$$f_j(\omega) = \frac{\sum_{i \neq j} f_i(\omega)\theta_{i,\infty}(\omega)}{\theta_{j,\infty}(\omega)}$$

Substituting $\theta_{j,\infty}$ back to the definition of l_{σ_n},

$$\begin{aligned} l_{\sigma_n}(\omega) &= F \cdot \Theta_{\sigma_n}(\omega) \\ &= \sum_{i \neq j} f_i(\omega)\theta_{i,\sigma_n}(\omega) + f_j(\omega)\theta_{j,\sigma_n}(\omega) \\ &= \sum_{i \neq j} f_i(\omega)\ (\theta_{i,\sigma_n}(\omega) + \theta_{j,\sigma_n}(\omega)\frac{\theta_{i,\infty}(\omega)}{\theta_{j,\infty}(\omega)}) \end{aligned}$$

Hence, we can change one coordinate of $\Theta_{\sigma_n}(\omega)$ to zero and assume that for every $\omega \in \Omega_1$, only m-1 coordinates in $\Theta_{\sigma_n}(\omega)$ are not zero. If Θ_{σ_n} is not bounded, we repeat the argument. After at most m steps, we denote Ω_m the last set and we can assume that there exists $i \in [\![1, m]\!]$ such that $l_n = f_i\theta_{i,n}$. If $\theta_{i,n}$ is not bounded for all ω, then $f_i(\omega) = 0$. Replacing $\theta_{i,n}$ by $\theta_{i,n}\mathbb{1}_{\Omega_m^C}$, we obtain that Θ_n remains measurable and bounded. As there are at most m steps, we can assume that Θ_n is bounded, so $l_\infty \in L$.
$\square$

2.4 Intermediary lemmas

We now generalize the lemma 2 to multiple periods (*e.g.* $T \geqslant 1$).

Lemma 3. (Kabanov-Stricker) Let $(X_{t=1}^{T})$ be an m-dimensionnal adapted process and :

$$L = \{f \mid f = \sum_{t=1}^{T} \sum_{i=1}^{m} X_{i,t}\theta_{i,t}\} = \{f \mid f = \sum_{t=1}^{T} X_t \theta_t\}$$

where $(\theta)_{t=1}^{T}$ runs through the set of m-dimensionnal predictable processes. L is a closed subspace of L^0

Proof. We prove this lemma by induction with respect to T. We have proved the case $T = 1$ with the previous lemma. Let T $\geqslant$ 1 and assume the lemma holds for $T - 1$. Let

$$f_n = \sum_{t=1}^{T} X(t) \cdot \theta(t)$$

$$b_n = X(1) \cdot \theta_n(1)$$

$$c_n = f_n - b_n$$

If $\theta_n(1)$ is bounded, then taking a subsequence, we could assume $\theta_n(1)$ is convergent. One can select the indexes of the subsequence in an $\mathcal{F}_0$-measurable way so that $(\theta_{\sigma_k}(t))_{t=2}^{T}$ remain predictable. The convergence of (b_n) implies the convergence of (c_n) and we would have :

$$c_\infty = \sum_{t=2}^{T} X(t) \cdot \theta_\infty(t)$$

Now, if $\theta_n(1)$ is not bounded,then as in the previous lemma, for $\omega \in \Omega_m$, we divide by $||\theta_n(1)||$:

$$\frac{f_n(\omega)}{||\theta_n(1)||} = X(1) \cdot \frac{\theta_n(1)}{||\theta_n(1)||} + \sum_{t=1}^{T} X(t) \cdot \frac{\theta(t)}{||\theta_n(1)||}$$

Using Lemma 1, one can assume that $\frac{b_n}{||\theta_n(1)||}$ is convergent so that the other part is also convergent. With θ_∞ on the unit sphere, we get :

$$X(1) \cdot \theta_\infty(1) + \sum_{t=1}^{T} X(t) \cdot \theta_\infty(t) = 0$$

Let t_0 the first time period where $\theta(t_0) \neq 0$. Similarly to the previous lemma, we find an expression of coordinates of $X(t)$ with $\mathcal{F}_{t-1}$-measurable weigths. The procedure stops after at most m steps, which results in the first case where $\theta_n(1)$ is bounded.

□

We will now enunciate two lemmas without proving them, the idea of the proof of Lemma 4 being similar to the previous lemma and the proof of the Kreps-Yan's lemma relying on arguments of duality of L^1.

Lemma 4. If $L \cap L_0^+(\Omega, \mathcal{A}, \mathbf{P}) = 0$ then $C = L - L_0^+(\Omega, \mathcal{A}, \mathbf{P})$ is a closed cone in $L^0(\Omega, \mathcal{A}, \mathbf{P})$

Lemma 5. (Kreps-Yan) Let C a closed convex cone in $L^0(\Omega, \mathcal{A}, \mathbf{P})$ and assume that $L_+^1 \subseteq C$ and $C \cap L_+^1 = 0$, then there is a probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{A})$ equivalent to $\mathbf{P}$ such that $\frac{d\mathbf{Q}}{d\mathbf{P}} \in L^\infty(\Omega, \mathcal{A}, \mathbf{P})$ and $\forall c \in C$,

$$\mathrm{E_Q}(c) = \int_\Omega c d\mathrm{Q} = \mathrm{E_P}(c \cdot \frac{d\mathbf{Q}}{d\mathbf{P}}) \leqslant 0$$

2.5 Proof of the theorem

Let us now begin the proof of the first fundamental theorem.
1) $\implies$ 2) If $L \cap L_0^+(\Omega, \mathcal{A}, \mathbf{P}) = 0$ then by Lemma 4, C is closed in L^0
2) $\implies$ 3) A sequence is convergent in probability if and only if every of its subsequences has a subsequence converging almost surely to the same variable. Thus, the convergence in L^0 remains under equivalent change of measure. For instance, given a random variable S, we can define a new probability measure equivalent to P such that η remains integrable under this measure.

$$\mathrm{P_e}(A) = c \int_A exp(-||s||)d\mathrm{P}, \forall A \in \mathcal{A}$$

Without loss of generality, we assume that for all $t \in [\![1, T]\!]$, S_t is integrable. Since the convergence in L^1 implies the convergence in probability, the cone $C_0 = \bar{C} \cap L^1$ is closed in L^1. Using the hypothesis $C_0 \cap L_+^1 = 0$, by the Kreps-Yan lemma, there is an equivalent probability measure Q with dQ/dP $\in L^\infty$ such that :

$$\forall k \in C_0,\ \mathrm{E_Q}(k) \leqslant 0$$

Then choosing $k = \pm(S(t) - S(t-1), \theta(t))$ with θ measurable, we get that :

$$\mathrm{E_Q}(S(t) - S(t-1), \theta(t)) = 0$$

In particular, taking $\theta(t) = \mathbb{1}_F$ with $F \in \mathcal{F}_{t-1}$, we obtain the definition of a martingale under $\mathbf{Q}$ that is :

$$\mathrm{E_Q}(S(t) - S(t-1) \mid \mathcal{F}_{t-1}) = 0$$

3) $\implies$ 4) Assume there exists an equivalent measure $\mathbf{Q}$ such that S is a martingale under $\mathbf{Q}$. Let $h \in C \cap L_+^0$, there is a predictable process $(\theta)_{t=1}^T$ such that

$$0 \leqslant h \leqslant \sum_{t=1}^T < S(t) - S(t-1), \theta(t) >$$

We have to show that h is almost surely zero under $\mathbf{P}$ which is equivalent almost surely zero under $\mathbf{Q}$. Hence it is sufficient to prove that

$$0 \leqslant \mathrm{E_Q}(h) \leqslant \mathrm{E_Q}(\sum_{t=1}^T < S(t) - S(t-1), \theta(t) >)$$

Let $\epsilon > 0$. $\mathbb{1}_{||\theta(1)|| \leqslant n_1)}$ is $\mathcal{F}_0$-measurable and $\theta(1)\mathbb{1}_{||\theta(1)|| \leqslant n_1)}$ is bounded by construction so it is predictable. Given S is a $\mathbf{Q}$-martingale,

$$\begin{aligned}
\mathrm{E}_{\mathrm{Q}}(< S(1) - S(0), \theta(1)\mathbb{1}_{||\theta(1)|| \leqslant n_1} >) &= \mathrm{E}_{\mathrm{Q}}(\mathrm{E}_{\mathrm{Q}}(< S(1) - S(0), \theta(1)\mathbb{1}_{||\theta(1)|| \leqslant n_1} > | \mathcal{F}_0)) \\
&= \mathrm{E}_{\mathrm{Q}}(< \mathrm{E}_{\mathrm{Q}}(S(1) - S(0) \mid \mathcal{F}_0), \theta(1)\mathbb{1}_{||\theta(1)|| \leqslant n_1} >) \\
&= \mathrm{E}_{\mathrm{Q}}(< 0, \theta(1)\mathbb{1}_{||\theta(1)|| \leqslant n_1} >) \\
&= 0
\end{aligned}$$

Now we repeat the process by multiplying by $\mathbb{1}_{||\theta(2)|| \leqslant n_2}$,

$$\begin{aligned}
\mathrm{E}_{\mathrm{Q}}(h \cdot \mathbb{1}_{||\theta(1)|| \leqslant n_1}\mathbb{1}_{||\theta(2)|| \leqslant n_2}) \leqslant\ & \mathrm{E}_{\mathrm{Q}}(< S(1) - S(0), \theta(1)\mathbb{1}_{||\theta(1)|| \leqslant n_1}\mathbb{1}_{||\theta(2)|| \leqslant n_2} >) \\
&+ \mathrm{E}_{\mathrm{Q}}(\sum_{t=2}^{T} < S(t) - S(t-1), \theta(t)\mathbb{1}_{||\theta(1)|| \leqslant n_1}\mathbb{1}_{||\theta(2)|| \leqslant n_2} >)
\end{aligned}$$

By dominated convergence theorem for some n_2,

$$\mathrm{E}_{\mathrm{Q}}(< S(1) - S(0), \theta(1)\mathbb{1}_{||\theta(1)|| \leqslant n_1}\mathbb{1}_{||\theta(2)|| \leqslant n_2} >) \leqslant \frac{\epsilon}{T}$$

We repeat the procedure by multiplying each time by $\mathbb{1}_{||\theta(1)|| \leqslant n_t}$ and obtain :

$$\mathrm{E}_{\mathrm{Q}}(h \cdot \prod_{t=1}^{T} \mathbb{1}_{||\theta(t)|| \leqslant n_t}) \leqslant \epsilon$$

Finally, by tending n_t to ∞, we conclude by monotone convergence theorem that

$$\mathrm{E}_{\mathrm{Q}}(h) \leqslant \epsilon$$

3 The second fundamental theorem of asset pricing

3.1 The completeness of the market

Definition 7. The market is complete if every contingent claim is replicable, *e.g.* for every $\mathcal{F}_T$-measurable claim H_T, there is a predictable m-dimensionnal process $(\theta)_{t=1}^{T}$ and a real number λ such that

$$H_T = \lambda + \sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) >$$

Theorem 2. *Assume the no-arbitrage condition holds on a market defined by the m-dimensionnal price process $(S)_{t=1}^{T}$. This market is complete if and only if there exists a unique risk-neutral measure on $(\Omega, \mathcal{F}_T)$*

Proof. 1) $\implies$ 2) Assume by contradiction that the market is complete and there exist two different risk-neutral measures $\mathbf{Q}$ and $\mathbf{R}$. There exists $F \in \mathcal{F}_T$ with $\mathrm{Q}(F) \neq \mathrm{R}(F)$. By definition of completeness, there exist $\lambda \in \mathbb{R}$ and a predictable process $(\theta)_{t=1}^{T}$ such that

$$\mathbb{1}_F = \lambda + \sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) >$$

The idea is to calculate the expected value of each side with respect to the measures Q and $\mathcal{R}$. It is sufficient to show that

$$\mathrm{E}_{\mathrm{P}}(\sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) >) = 0$$

Indeed, by equivalence of the measures, we would have

$$\lambda = \mathrm{P}(F) = \mathrm{Q} = \mathrm{R}$$

which is absurd. Note that we cannot use the additivity of the integral if $\theta(\omega)$ is not bounded for all ω. Using the same argument as in the proof of the first fundamental theorem by multiplying by $\mathbb{1}_{||\theta(t)|| \leqslant n_t}$ and using monotone convergence theorem, we conclude that the expected value is indeed zero.

2) $\implies$ 1) We are going to show the negation. Assume the market is not complete. By no-arbitrage condition, there exists $\mathbf{Q}$ such that the coordinates of $(S)_{t=1}^{T}$ are martingales under $\mathbf{Q}$. Let us define

$$L = \{H \mid H = \lambda + \sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) >\}$$

with $\lambda \in \mathbb{R}$ and θ a predictable process. As the market is not complete, there exists a claim $H_T \notin L$. By a change of measure, we can assume without loss of generality that coordinates of S and H_T are integrable. By the first fundamental theorem, we assume that the Radon-Nikodym's derivative $\mathrm{d}\mathbf{Q}/\mathrm{d}\mathbf{P}$ is bounded hence coordinates of S and H_T are integrable under the risk-neutral measure $\mathbf{Q}$. We want to exhibit an integrable function that is not in L. We have to show that L is closed in $L^1(\Omega, \mathcal{F}_T, \mathbf{Q})$ As we work on probability measures, $\mathbb{1} \in L^1$ and $L_s = L \cap L^1$ is the direct sum of the hyperplane K and of a one-dimensionnal subspace. If the vector $\mathbb{1}$ is in K then L is closed. If $\mathbb{1} \notin L$, then let

$$l_n = \lambda_n \cdot \mathbb{1} + r_n \to l_\infty \quad with\, \lambda_n \in \mathbb{R}\, and\, r_n \in K$$

Let d the distance between K and the real line. Since K is closed and $\mathbb{1} \notin K$, $d > 0$. The sequence (l_n) is convergent, so there exists $c \in \mathbb{R}$

$$\forall n, ||l_n||_1 \leqslant c$$

By definition, if $r_n \in K$, then $-\frac{r_n}{\lambda_n} \in K$. Hence,

$$\begin{aligned} c &\geqslant ||\lambda_n \cdot \mathbb{1} + r_n|| \\ &= \mid \lambda_n \mid ||\mathbb{1} - (-\frac{r_n}{\lambda_n})|| \\ &\geqslant \mid \lambda_n \mid \cdot d \end{aligned}$$

with d positive, so that $\mid \lambda_n \mid \leqslant \frac{c}{d}$, $e.g.$ λ_n is bounded. Denoting (λ_n) the convergent subsequence, we have that $\lambda_n \mathbb{1}$ is also convergent. Thus, $r_n = l_n - \lambda_n \mathbb{1}$ converges. Since $r_n \in K$ closed subspace, the limite of (r_n) is in K. Finally, we proved that the limit of a subsequence of l_n is in L, so $l_\infty \in L$ and L is closed.

Because H_T is integrable and does not belong to the subspace L, there exists a function in L^1 that is not in L. Using the geometrical version of the Hahn-Banach theorem, there exists $z \in L^\infty$ which separates the subspace L and the variable H_T. For all $l \in L$,

$$< z, l >= \int z \cdot l \, d\mathrm{Q} = \mathrm{E}_\mathrm{Q}(z \cdot l) = 0$$

Taking the predictable strategy $\theta = 0$ and $\lambda = 1$, 1 is in L.

$$< z, 1 >= \int z \, d\mathrm{Q} = 0$$

Let

$$g = 1 + \frac{z}{2||z||_\infty} \quad and \quad \mathrm{R}(A) = \int_A g \, d\mathrm{Q}$$

Note that $\mid \frac{z}{2||z||_\infty} \mid \leqslant 0.5$, hence the derivative g=dR/dQ is positive and bounded.

$$R(\Omega) = \mathrm{E}_\mathrm{Q}(1) + \frac{\mathrm{E}_\mathrm{Q}(z)}{2||z||_\infty} = 1$$

thus, **R** and **Q** are equivalent probability measures. Taking a predictable bounded strategy θ and $\lambda = 0$, as S is a martingale under **Q**, we have :

$$\begin{aligned}
\mathrm{E}_\mathrm{R}(\sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) >) &= \mathrm{E}_\mathrm{Q}(\sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) > (1 + \frac{z}{2||z||_\infty})) \\
&= \mathrm{E}_\mathrm{Q}(\sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) >) \\
&= \sum_{t=1}^{T} \mathrm{E}_\mathrm{Q}(< S(t) - S(t-1), \theta(t) >) \\
&= \sum_{t=1}^{T} \mathrm{E}_\mathrm{Q}(\mathrm{E}_\mathrm{Q}(< S(t) - S(t-1), \theta(t) >| \mathcal{F}_{t-1})) \\
&= \sum_{t=1}^{T} \mathrm{E}_\mathrm{Q}(< \mathrm{E}_\mathrm{Q}(S(t) - S(t-1) \mid \mathcal{F}_{t-1}), \theta(t) >) \\
&= \sum_{t=1}^{T} \mathrm{E}_\mathrm{Q}(0, \theta(t)) \\
&= 0
\end{aligned}$$

In particular, taking the strategy that is zero except in T where it is $\mathbb{1}_F$ with $F \in \mathcal{F}_{t-1}$,

$$\mathrm{E}_\mathrm{Q}((S(t) - S(t-1))\mathbb{1}_F) = 0$$

$$e.g. \quad \int_F S(t) \, d\mathrm{R} = \int_F S(t-1) \, d\mathrm{R}$$

then by definition of a martingale,

$$\mathrm{E}_{\mathrm{R}}(S(t)|\mathcal{F}_{t-1}) = S(t-1)$$

Finally, we found a measure $\mathbf{R}$ such that S is an $\mathbf{R}$-martingale, so the risk-neutral measure is not unique.

□

3.2 Pricing European contingent claims on finite and discrete time-horizon

In this paper, we focus on valuing European options and ignore the American options which can be exercised at any time up to the maturity date.

Let H_T be a contingent claim H_T of terminal date T. H_T is $\mathcal{F}_T$-measurable. Our goal is to find its price at time $t = 0$. Assuming there are no arbitrage opportunities and that the market is complete, we can replicate H_T with $\lambda \in \mathbb{R}$ and θ predictable strategy

$$H_T = \lambda + \sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) >$$

By contradiction, assume that the risk-free amount of asset λ is not unique, *e.g.* there exist λ_1, λ_2 with $\lambda_1 > \lambda_2$ and θ_1, θ_2 such that for $i = 1, 2$

$$H_T = \lambda_i + \sum_{t=1}^{T} < S(t) - S(t-1), \theta_i(t) >$$

Then

$$\lambda_2 - \lambda_1 + \sum_{t=1}^{T} < S(t) - S(t-1), \theta_2(t) - \theta_1(t) >= 0$$

Thus, $\theta_2 - \theta_1$ is an arbitrage strategy. Denoting λ as a function of H_T, the no-arbitrage price is $\pi(H_T) = \lambda$. By the first fundamental theorem, there exists a martingale measure $\mathbf{Q}$ with bounded derivative $\mathrm{d}\mathbf{P}/\mathrm{d}\mathbf{Q}$. Again, with a change of measure, we can assume H_T is integrable under $\mathbf{P}$ and thus $\mathbf{Q}$. Therefore, we have

$$\mathrm{E}_{\mathrm{Q}}(H_T) = \lambda + \mathrm{E}_{\mathrm{Q}}(\sum_{t=1}^{T} < S(t) - S(t-1), \theta(t) >) = \lambda = \pi(H_T)$$

Remark. We used the completeness assumption for the existence of a replicating strategy. Assuming there exists a replicating strategy for H_T, by uniqueness of λ, this price does not depend of the martingale measure used.

4 Conclusion

A key assumption in our model is that agents can only make decisions for finite and discrete time periods, which means they can't modify their portfolios between two time periods. Furthermore, this model can not be used to price other derivatives such as American options, as it is much harder to determine the optimal early-exercise strategy.

In 1973, Black and Scholes delivered a breakthrough with their continuous model used to price European options. Their idea relied on a trading strategy consisting of replicating every option through trades between the risky and the risk-free assets, combined with the understanding that two investment strategies bearing the same risk should have the same price (in a no-arbitrage context). One could study the intertemporal asset pricing (Merton, 1973), building on this more realistic framework to price American options and take into account dividend distribution.

5 References

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