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# ROUTING GAMES

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# 1 INTRODUCTION

Game theory is the study of mathematical models of strategic interaction between rational decision-makers. Formally, a game is given by a set of strategies for each player, and a pay-off function, also called the utilitarian function, which represents some value the players wish to maximize, such as profit, or minimize, such as latency experienced due to congestion. If all players have chosen a strategy in their set of strategies/action, then each player receives a pay-off dependent on the choices of himself and his competitors or allies. The hypothesis of rationality means that the players don't choose there strategies arbitrarily, but choose their strategies in order to maximize their profit or minimize their pay-off.

Our goal will be to compare the solution found by selfish players to the socially optimal solution. In game theory there is a widely used best response for what course of action the players should take. It is called the Nash equilibrium, and was first suggested by the mathematician John Forbes Nash in 1950. Suppose all players have chosen their strategy. If some player could profit by unilaterally changing his strategy and thereby getting a better pay-off, then obviously the player has the incentive to just do that. We are, therefore, not in a stable situation. If, on the other hand, no player can profit by unilaterally changing his strategy, the players are said to be in a Nash equilibrium.

A central problem arising in the management of a large network is that of routing traffic to achieve the best possible network performance. In many networks, it is difficult or even impossible to impose optimal routing strategies on network traffic, leaving network users free to act according to their own interests. Hence, we will study the degradation in network performance due to selfish, uncoordinated behavior by network users in a variety of traffic models. The two major type of games we will focus on are : atomic and non atomic selfish routing games. Two basic but interesting examples we will study are Pigou's Example (1920) and the famous "paradox" discovered by Braess in 1968.

# 2 PIGOU'S EXAMPLE (1920)

The Pigou's example is a basic network composed by two parallel routes, each a single edge, that connects a source vertex s to a destination (or "sink") vertex t. Each edge has a cost that is a function of the amount of traffic, i.e the flow that uses the edge, and which corresponds to the travel time.
-The upper edge has a constant cost function $c_1(x) = 1$ (it can be 1 hour for example). Note that it is immune to congestion !
-The lower edge has a variable cost $c_2(x) = x$, which increases as the edge gets more congested.

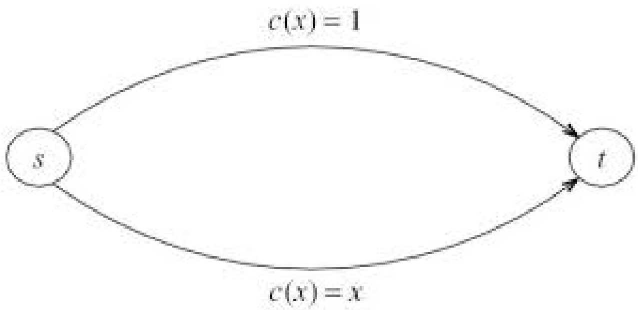


Figure 1: Pigou's example (1920)

We also assume that each selfish player wants to minimize its travel time (i.e its cost), so the lower edge, which cost is $c_2(x) = x$, is a dominant strategy, and in an equilibrium outcome, all of the players will follow this strategy, all of them will then have a cost of 1.

The cost functions c(x) are assumed to be non-decreasing, continuous and with the property that xc(x) is convex. For example, we might have a constant cost function, a linear cost function, a quadratic cost function, etc..

## 3 NON ATOMIC SELFISH ROUTING GAMES

In this section, we modelize the generalization of the basic Pigou's Example.

Model : We consider that a certain unit of flow of drivers are at the vertex S, and they are assumed to be infinitely divisible. Adding to this the fact that each player controls an infinitesimally small fraction of flow such that changing the edge (i.e deviating unilaterally) will only change his proper cost function depending on the cost of the edge he takes, hence it will not change the global cost function of the entire flow. This kind of games which such properties are called non-atomic selfish routing games.

More formally, a selfish routing game occurs in a network that is given by a directed graph G=(V,E), with vertex set V and directed edge E, together with a set $(s_1, t_1), ..., (s_k, t_k)$ of source-sink vertex pairs. Such pairs will be called commodities. We use $\mathcal{P}_i$ to denote the $(s_i, t_i)$ paths of a network. We only consider networks in which $\mathcal{P}_i \neq \emptyset$ for all i, and define the set $\mathcal{P}$ of source-sink paths :

$$\mathcal{P} = \cup_{i=1}^{k} \mathcal{P}_i$$

.

**Definition 3.1.** (Flow / feasible flows) A flow is a nonnegative vector indexed by the set $\mathcal{P}$ of source-sink paths. For a flow f and a path $P \in \mathcal{P}_i$, $f_P$ is the amount of traffic of commodity i that chooses the path P to travel from $s_i$ to $t_i$. A flow f is feasible for a vector r if it routes all of the traffic i.e for each $i \in 1, ..., k$

$$\sum_{P \in \mathcal{P}_i} f_P = r_i$$

The total amount of traffic of commodity i is indicated by $r_i$.

Each edge e of a network has a cost function $c_e : \mathbb{R}^+ \to \mathbb{R}^+$ and we always assume that cost functions are nonnegative, continuous, and non decreasing. We define a non atomic instance by a triple of the form (G,r,c).

**Definition 3.2.** (Cost of a path P) The cost of a path P with respect to a flow f is the sum of the costs of the constituent edges :

$$c_P(f) = \sum_{e \in P} c_e(f_e)$$

where $f_e = \sum_{P \in \mathcal{P} : e \in P} f_P$ denotes the amount of traffic using paths that contain the edge e.

**Definition 3.3.** (Non atomic equilibrium flow) Let f be a feasible flow for the nonatomic instance (G,r,c). The flow f is an equilibrium flow if, for every commodity $i \in 1, ..., k$ and every pair P, $\tilde{P} \in \mathcal{P}_i$ of $s_i \to t_i$ paths with $f_P > 0$, $c_P(f) \leqslant c_{\tilde{P}}(f)$.

In other words, all paths in use by an equilibrium flow f have a minimum-possible cost(given their source, sink and the congestion incurred by the flow f). In particular, all paths of a given commodity used by an equilibrium flow have equal cost.

*Remark* 3.4. The cost incurred by a player depends only on its path and the amount of flow on the edges of its path, rather than on the identities of any of the players. Game of this type are often called congestion games.

**Important hypothesis** : In this section, note that the cost functions c(x) are assumed to be **non-decreasing, differentiable**, and with the property that **xc(x) is convex**. For example, we might have a constant cost function, a linear cost function, a quadratic cost function, etc..

**Proposition 3.5.** *We define the cost of a flow f as :*

$$C(f) = \sum_{P \in \mathcal{P}} c_P(f) f_P = \sum_{P \in \mathcal{P}} \sum_{e \in P} c_e(f_e) f_P = \sum_{e \in P} \sum_{P \in \mathcal{P}} c_e(f_e) f_P = \sum_{e \in E} c_e(f_e) f_e$$

.

**Definition 3.6.** (Optimal flow)
For an instance (G,r,c), we call a feasible flow f* optimal if it minimizes the cost over all feasible flows i.e $C(f*) = \min_f C(f) = \min_f \sum_{e \in E} c_e(f_e) f_e$

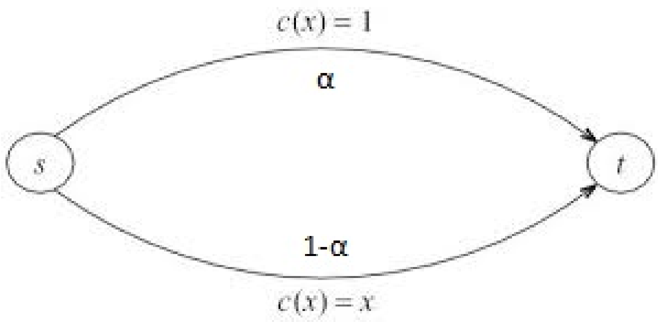


Figure 2: Pigou's example with split traffic

Let's compute the optimal flow in Pigou's example. Set $c_1(x)$ the cost of the upper edge and $c_2(x)$ the cost of the lower edge. In the case where they split between the upper and the lower edge with $f_1 = \alpha$ being the proportion of the flow that take the upper edge, $f_2 = (1-\alpha)$ is the proportion of flow that take the lower edge :

In this case, the cost function is : $C(\alpha) = c_1(f_1).f_1 + c_2(f_2).f_2 = 1.\alpha + (1-\alpha)^2 = \alpha^2 - \alpha + 1$
Optimal flow f* minimizes the cost function C(f).

$C'(\alpha) = 2\alpha - 1$
$C'(\alpha) = 0 \Longrightarrow 2\alpha = 1 \Longrightarrow \alpha = \frac{1}{2} \Longrightarrow x* = (\frac{1}{2}; \frac{1}{2})$

So splitting the traffic equally between the two edges is the optimal outcome.

## 4 BRAESS'S PARADOX (1968)

Consider the four node network shown in this figure.

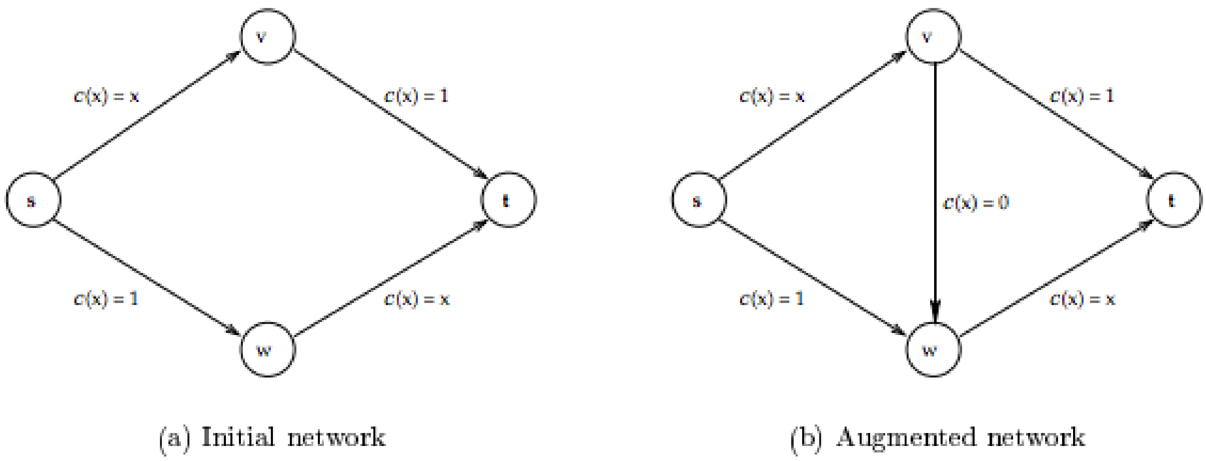


Figure 3: Braess's Paradox (1968)

There are two disjoint routes from s to t, each with combined cost 1+x, where x is the amount of traffic that uses the route. Assume that there is one unit of traffic. In the equilibrium flow, the traffic is split evenly between two routes, and all of the traffic experiences 3/2 units of cost. Now suppose than in order to decrease the cost encountered by the traffic, we build a zero-cost edge connecting the midpoints of the two existing routes. Now, what is the new equilibrium flow ? The cost of the new route s → v → w → t is never worse than that along the two original paths. As a consequence, the unique equilibrium flow routes all of the traffic on the new route. Now there is a heavy congestion on the edges (s,v) and (w,t), and all of the traffic now experiences two units of cost.

Braess's paradox thus shows that the intuitively helpful action of adding a zero-cost edge can increase the cost experienced by all of the traffic. Note that putting a cost c(x) such that $0 < c(x) < x + 1$ on the added edge will not change the paradox, since the equilibrium flow will still route all of the traffic on the new route.

*Remark* 4.1. Note that we also have the paradox in the symmetric case where we bar an edge of the network. Indeed, by preventing the flow to use an edge, the traffic will experience an inferior cost than if we don't bar the edge, which is contradictory with what we can expect. In other words, by blocking an edge, we can reduce the traffic congestion, and this was observed in April 1990 in New York. Indeed, the 42nd Street was shut down, and while disastrous traffic conditions were predicted, traffic flowed better on that day.[1]

Braess's Paradox has also remarkable analogues in several physical systems, such as mechanical systems (called the "Strings and Springs Braess Paradox") or even in the control of frequency in Electrical power grids networks.[2]

# 5 EXISTENCE, UNIQUENESS AND POTENTIAL FUNCTIONS

In this section, we will show existence and uniqueness results about equilibrium flows in nonatomic and atomic selfish routing games. We will also introduce the potential function method, which is a fundamental proof technique.

Our goal will be to show that in nonatomic selfish routing games, equilibrium flows always exist and are essentially unique, which means that all equilibrium flows of a non atomic instance have the same cost. In particular, the POS and the POA coincide in every nonatomic instance.

**Theorem 5.1.** *(Existence and uniqueness of equilibrium flows)*

*Let (G,r,c) be a nonatomic instance.*

*(a) The instance (G,r,c) admits at least one equilibrium flow.*

*(b) If $f$ and $\tilde{f}$ are equilibrium flows for (G,r,c), then $c_e(f_e) = c_e(\tilde{f}_e)$ for every edge e.*

In order to prove this theorem, we use the potential function method, which idea is to exhibit a real valued "potential function", defined on the outcomes of a game, such that the equilibria of the game are precisely the outcomes that optimize the potential function. This method has emerged as a technique in understanding the quality of equilibria, and it provides results only regarding the POS. Potential functions are also useful because they enable the application of optimization techniques to the study of equilibria.

A game is said to be a potential game if the incentive of all players to change their strategy can be expressed using a function called the potential function. The concept was introduced by Monderer and Shapley in 1996. (What make potential games attractive are their useful properties concerning the existence, uniqueness and inefficiency of their Nash equilibria).

More formally :

**Definition 5.2.** (Ordinal potential game)
A game is an ordinal potential game if there exists : $\Phi : S_1 \text{x...x} S_n \to \mathbb{R}$ such that $\forall i, s_i, s_{-i}$ and $s'_i$,

$$c_i(s_i, s_{-i}) > c_i(s'_i, s_{-i}) \iff \Phi(s_i, s_{-i}) > \Phi(s'_i, s_{-i})$$

[1] More informations here :https://www.nytimes.com/1990/12/25/health/what-if-they-closed-42d-street-and-nobody-noticed.html

[2] More informations here : https://physicsworld.com/a/beating-braess-paradox-to-prevent-instability-in-electrical-power-grids/

where $S_i$ is the set of strategies of the player i and $s_{-i}$ is the strategy ofall other players.

**Definition 5.3.** (Exact potential game)
A game is an exact potential game if

$$c_i(s_i, s_{-i}) - c_i(s_i', s_{-i}) = \Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i})$$

**Theorem 5.4.** *(Rosenthal's theorem) Every congestion game is an exact potential game.*
Proof : Consider the potential function :

$$\Phi_a(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$$

We assume that there is a flow $\tilde{f}$ where one player chooses an alternative path $\tilde{P}$ with lower cost. This means $c_{\tilde{P}}(\tilde{f}) < c_P(f) \Leftrightarrow 0 > c_{\tilde{P}}(\tilde{f}) - c_P(f)$. The edges of $\tilde{P}$ are either in P or they are not. P and $\tilde{P}$ might have some edges in common, the cost of those edges does not change as player i changes to path $\tilde{P}$ . This means that only the edges that they have not in common should be considered. Note that if player i deviates, he transfers his one unit of flow (R=1) from path P to path $\tilde{P}$. This means that edges that are now used now has an extra flow of one unit, so they now have a total flow $f_e + 1$. Hence, edges that are not anymore used have a total flow of $f_e - 1$, and we then have that $0 > c_{\tilde{P}}(\tilde{f}) - c_P(f) = \sum_{e \in \tilde{P} \setminus P} c_e(f_e + 1) - \sum_{e \in P \setminus \tilde{P}} c_e(f_e)$.

Now, let's consider $\Phi_a$. The new edges in $\tilde{P}$ (i.e $e \in \tilde{P} \setminus P$) add to $\Phi_a(f)$ one extra term $c_e(f_e + 1)$ ($+1$ because $R = 1$). The edges that are not anymore in $\tilde{P}$ , i.e $e \in P \setminus \tilde{P}$ subtracts the term $c_e(f_e)$ of $\Phi_a$.The sum for common edges in $P \cap \tilde{P}$ remains the same, i.e. : $\sum_{i=1}^{\tilde{f}_e} c_e(i) = \sum_{i=1}^{f_e} c_e(i)$. Hence, $\Phi_a(\tilde{f}) - \Phi_a(f) = \sum_{e \in \tilde{P} \setminus P} c_e(f_e + 1) - \sum_{e \in P \setminus \tilde{P}} c_e(f_e)$, which means that the difference in the cost function is exactly equal to the difference in the potential function, which is the definition of an exact potential game. Hence a congestion game is also an exact potential game.

Now, assume that for every edge e of a nonatomic instance, the function $x.c_e(x)$ is continuously differentiable and convex. Let $c_e^*(x) = (x.c_e(x))' = c_e(x) + x.c_e'(x)$ denote the marginal cost function for the edge e. For example in Pigou's example, the cost functions of the two edges are 1 and x, so the marginal cost function are $c^*(x) = 1$ and $c^*(x) = 2x$.

**Proposition 5.5.** *(Equivalence of equilibrium and optimal flows)*
*Let (G,r,c) be a nonatomic instance such that, for every edge e, the function $x.c_e(x)$ is convex and continuously differentiable. Let $c_e^*(x)$ denote the marginal cost function of the edge e. Then f* is an optimal flow for (G,r,c) if and only if it is an equilibrium flow for (G,r,c*).*

Proof : Recall that a flow f* is optimal if it is feasible and minimizes : $C(f) = \sum_{e \in E} c_e(f_e) f_e$.
We are looking for a function $h_e(x)$ for each edge e - playing the previous role of $x.c_e(x)$ such that $h_e'(x) = c_e(x)$. In order to construct a potential function for equilibrium flows, we need to "invert" the Proposition 5.5 : of what function are equilibrium flows the global minima ? Considering that $c_e(0) = 0$, we set $h_e(x) = \int_0^x c_e(y) dy$ for each edge e, which thus yields the desired potential function. Moreover, since $c_e$ is continuous and nondecreasing for each edge e, every function $h_e$ is both continuously differentiable and convex. Precisely, call :

$$\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x) dx \quad (1)$$

the potential function of a nonatomic instance (G,r,c). In this section, optimal flows are characterized the same way as in 3.6 with the following characterization : $c_P^*(f^*) \leqslant c_{\tilde{P}}^*(f^*)$.

Hence, each function $x.c_e(x)$ which is convex is now replaced by the convex function $h_e(x) = \int_0^x c_e(y) dy$, and we have therefore equilibrium flows are the global minimizers of the potential function $\Phi$.

**Proposition 5.6.** *(Potential function for equilibrium flows) Let (G,r,c) be a nonatomic instance. A flow feasible for (G,r,c) is an equilibrium flow if and only if it is a global minimum of the corresponding potential function Φ given as before.*

Proof of Theorem (5.1) : Note that the set of feasible flows of (G,r,c) can be identified with a compact subset of $|\mathcal{P}|$-dimensional Euclidean space (because it is closed and bounded in a finite dimensional space). Since edge cost function $c_e$ is continuous, $h_e(x) = \int_0^x c_e(y)dy$ is $C^1$ and continuous in particular, so $\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x)dx$ is continuous on the set of feasible flows as the sum of continuous functions on this set. By Weierstrass's Theorem (i.e Theorem of continuity in compact subsets), it achieves a minimum value on this set, and by proposition (3.5), this point corresponds to an equilibrium flow of (G,r,c). Hence, (a) is prooved.

For part (b), recall that each cost function $c_e$ is increasing, and since $h_e'(x) = c_e(x)$, hence $h_e'$ is increasing, so $h_e'' \geqslant 0$ and then we have that each $h_e$ is convex. Hence the potential function is convex as a sum of convex functions.

Now, suppose that f and $\tilde{f}$ are equilibrium flows for (G,r,c). By (3.5), they are optimal flows and by (6.5), they both minimize the potential function. Now consider all convex combinations of f and $\tilde{f}$-that is, all vectors of the form $\lambda f + (1-\lambda)\tilde{f}$ for $\lambda \in [0,1]$. All of these vectors are feasible flows as a combination of feasible flows. Since Φ is a convex function, a chord between two points on its graph lies on or above the graph of the function, but cannot pass below its graph i.e : $\Phi(\lambda f + (1-\lambda)\tilde{f}) \leqslant \lambda\Phi(f) + (1-\lambda)\Phi(\tilde{f})$ for every $\lambda \in [0,1]$. But, since both f and $\tilde{f}$ are global minima of Φ, the above inequality should be an equality for all convex combinations of f and $\tilde{f}$. Since for each $e \in E$, $h_e$ is convex, this can occur only if every $\int_0^{f_e} c_e(x)dx$ is linear between the values $f_e$ and $\tilde{f}_e$ i.e if $\int_0^{f_e+\tilde{f}_e} c_e(x)dx = \int_0^{f_e} c_e(x)dx + \int_0^{\tilde{f}_e} c_e(x)dx$ which implies that every cost function $c_e$ is constant between $f_e$ and $\tilde{f}_e$. Hence, considering $c_e = C$ constant, we have $\int_0^{f_e+\tilde{f}_e} Cdx = C(f_e(x) + \tilde{f}_e(x)) = \int_0^{f_e} Cdx + \int_0^{\tilde{f}_e} Cdx$ so $x \to \int_0^x c_e(y)dy$ is linear. Finally we have that $c_e(f_e) = c_e(\tilde{f}_e)$ which ends the proof of (b).

## 6 THE PRICE OF ANARCHY (POA) AND THE PRICE OF STABILITY (POS)

**Definition 6.1.** - Utilitarian function : sum of the players cost.
- Egalitarian function : maximum player cost.
They are called objective functions.

Given an objective function (also called an utilitarian function) and an equilibrium concept, a game may have different equilibria and objective function values, so we will study to what extent it is different. In this case, the optimization goal is to minimize the cost of a flow.

We will consider 2 important measure of the inefficiency of an equilibria that are used in routing games, and in order to introduce them, we study Pigou's example. In Pigou's example, there are many outcomes : -If all of the traffic takes the upper edge, then the cost function $c_1(x) = 1 \times 1 + x \times 0 = 1$
-If all of the traffic takes the lower edge, then $c_2(x) = 1 \times 0 + x \times 1 = x$

In the case where they split between the upper and the lower edge, we've already seen that the optimal flow f* that minimizes the cost function c(f) is f*=(1/2;1/2) i.e splitting the traffic equally between the two edges is the optimal outcome. Hence in this case, half of the traffic has a cost of 1, and the other has a cost of 1/2, the average cost of the traffic in this optimal flow is 3/4 hence the $\frac{C(f_{WorstEqu})}{C(f_{opt})} = \frac{1}{(\frac{3}{4})} = \frac{4}{3}$ where $C(f_{WorstEqu})$ is the cost of the worst Nash equilibrium flow (i.e with the highest cost) and $C(f_{opt})$ the cost of an optimal flow.

**Definition 6.2.** The Price of Anarchy (POA) : We defined the POA as: $POA = \frac{C(f_{WorstEqu})}{C(f_{opt})}$ where $C(f_{WorstEqu})$ is the cost of the worst Nash equilibrium flow (i.e with the highest cost) and $C(f_{opt})$ the cost of an optimal flow. In other words, it is the proportion between the worst possible social utility from a Nash equilibrium and the optimal social utility.

It's the most popular measure of the inefficiency of equilibria, which adopts a worst-case approach.

*Remark* 6.3. We will be looking for games for which the POA is close to 1. Hence, in those games, selfish behavior will not have consequences over the cost function, so optimality will remain.

**Definition 6.4.** (The Price of Stability (POS)) $POS = \frac{C(f_{BestEqu})}{C(f_{opt})}$ where $C(f_{BestEqu})$ is the cost of the best Nash equilibrium flow (i.e with the lowest cost) and $C(f_{opt})$ the cost of an optimal flow. In other words, it's the proportion between the best possible social utility of a Nash equilibrium and the optimal social utility.

**Proposition 6.5.** *Link between POS and the POA :*

*1) Consider a game with multiple equilibria that has at least one highly inefficient equilibrium. This game will have a large POA. The POS measures the inefficiency of games, but differentiate between games in which all equilibria are inefficient and those in which some equilibrium is inefficient.*

*2) In a game with an unique equilibrium, POA=POS, but in a general case for a game with multiple equilibria,* $1 \leqslant POS \leqslant POA$.

*Remark* 6.6. What's interesting is to bound the POA depending on the game's characteristics. Hence, for such classes of games, the equilibria are guaranteed to be optimal.

**Proposition 6.7.** *In Pigou's example, the POA is equal to the POS. In other words, the average cost incurred by the traffic is the same in all equilibria of the game. We will show later on using the potential function method that in a nonatomic instance, any equilibrium flow f has the same cost, C(f). Therefore, in a nonatomic instance, such as Pigou's example, any equilibrium flow can be used to calculate the price of anarchy.*

*Remark* 6.8. In an atomic instance, however, there might exist multiple equilibria with different total cost. Then, the price of anarchy is calculated using the worst outcome value.

Let's see another kind of Pigou's example in a non affine case.

**Example 6.9.** *Consider a network with two disjoint edges connecting a source vertex S to a destination vertex T.*
*-The upper edge has a constant cost function c(x)=1 .*
*-The lower edge has a highly non linear variable cost* $c(x) = x^p$ *for a large p.*

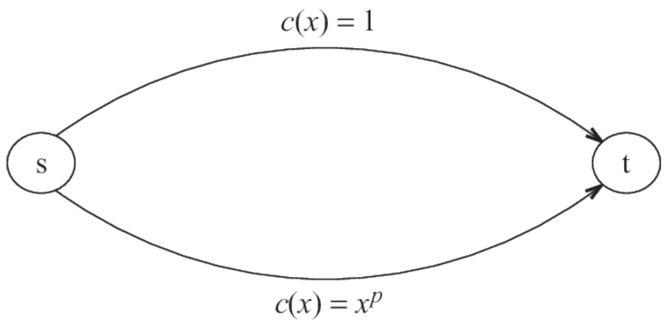


Figure 4: Non-affine Pigou's example (1920)

*As seen in the Example 1.8, the lower edge remains a dominant strategy for selfish drivers and the unique equilibrium travel time remains 1. If we again split the traffic equally between the two links, then the average cost tends to 1/2 as p* $+\infty$.
*If there was a dictator that could force a small fraction x of the traffic to travel along the lower edge, then the average cost would be* $C(x) = 1.(1-x) + x^p.x = 1 - x + x^{p+1}$, *hence* $C'(x) = -1 + (p+1)x^p = 0 \iff \frac{1}{p+1} = x^p \iff x = \sqrt[p]{\frac{1}{p+1}}$. *For p=1, we have indeed* $x = \frac{1}{2}$ *which is the previous result. If p tends to* $+\infty$, *then* $\lim_{p \to +\infty} \sqrt[p]{\frac{1}{p+1}} = \lim_{p \to +\infty} (p+1)^{-\frac{1}{p}} = \lim_{p \to +\infty} \exp\left(-\frac{1}{p}\ln(p+1)\right)$. *We have that* $\lim_{p \to +\infty} \frac{\ln(p+1)}{p} = 0$, *hence* $\lim_{p \to +\infty} \sqrt[p]{\frac{1}{p+1}} = 1$. *This implies that as p tends to* $+\infty$, *the optimal flow will have an increasing percentage of traffic over the lower edge. Thus, the cost of the network with an optimal flow is* $C(\sqrt[p]{\frac{1}{p+1}}) = 1 - \sqrt[p]{\frac{1}{p+1}} + (\sqrt[p]{\frac{1}{p+1}})^{p+1}$.

*Since the cost of the network with the equilibrium flow is 1, we have that the POA* $= \frac{1}{1-\sqrt[p]{\frac{1}{p+1}}+(\sqrt[p]{\frac{1}{p+1}})^{p+1}}$. *and* $\lim_{p\to+\infty} \frac{1}{1-\sqrt[p]{\frac{1}{p+1}}+(\sqrt[p]{\frac{1}{p+1}})^{p+1}} = \frac{1}{1-1+\frac{1}{\infty}} = \frac{1}{0} = +\infty$.

*In conclusion, the price of anarchy tends to infinity as p tends to* $+\infty$*, so the equilibrium can be arbitrarily inefficient.*

*Remark* 6.10. We can compute the POA in order types of games that are not routing games, such as the Prisoner's Dilemma, for which self-interest behavior is not Pareto optimal, so there is a conflict between the self-interest and the collective interest or "social good". Hence, in such games, the unique Nash equilibrium is inefficient, which means there is another outcome in which both players achieve a smaller cost. Hence, the outcome of rational behavior by selfish players can be inferior to a cooperative outcome.

## 7 BOUNDING THE PRICE OF ANARCHY

In the previous sections, we defined formally selfish routing networks, equilibria and the price of anarchy. Now, we introduce a simple lower bound on the price of anarchy that is based on Pigou-like networks (i.e. a network with two vertices and two edges, one commodity with r units of traffic and cost functions $c_1(x) = c(r)$ and $c_2(x) = c(x)$). Then we will establish an upper bound on the price of anarchy in general multicommodity flow networks, depending on the set of a certain type of cost functions. Common examples of sets of cost functions include linear functions, polynomials etc..

Recall that the cost functions are differentiable and non-decreasing.

**Definition 7.1.** (Pigou's bound) Let $\mathcal{C}$ be a nonempty set of cost functions that contains in particular the constant cost functions.

The Pigou bound $\alpha(\mathcal{C})$ for $\mathcal{C}$ is

$$\alpha(\mathcal{C}) = \sup_{c\in\mathcal{C}} \sup_{x,r\geq 0} \frac{r.c(r)}{x.c(x)+(r-x).c(r)} \quad (2)$$

with $\frac{0}{0} = 1$.

**Definition 7.2.** $POA(\mathcal{C}) = maxPOA$ for all networks with cost function in $(\mathcal{C})$.

$\alpha(\mathcal{C})$ is the price of anarchy for Pigou-like networks: the numerator is the cost of the equilibrium flow in a Pigou-like network, where all the traffic is routed over the lower edge. The denominator is the cost of the optimal flow: x units are routed over the lower edge and $rx$ units are routed over the upper edge. We will prove that $\alpha(\mathcal{C}) = POA(\mathcal{C})$ for all networks. The next proposition shows that this Pigou bound is a lower bound on the price of anarchy.

**Proposition 7.3.** *(Lower bound on the price of anarchy) Let C be a set of cost functions that contains all the constant cost functions. Then the POA in nonatomic instances with cost functions in C is at least* $\alpha(\mathcal{C})$ *i.e* $POA(\mathcal{C}) \geq \alpha(\mathcal{C})$.

Proof : Fix a choice of $c \in \mathcal{C}$ and $x, r \geq 0$. Recall that the cost function c is nondecreasing: If x=r, $\frac{r.c(r)}{x.c(x)+(r-x).c(r)} = \frac{r.c(r)}{r.c(r)} = 1$ and if x>r, $\frac{r.c(r)}{x.c(x)+(r-x).c(r)} = \frac{r.c(r)}{x.(c(x)-c(r))+r.c(r)}$. Since x>r and c is nondecreasing, we have that $c(x) \geq c(r)$, hence $(c(x) - c(r)) \geq 0$ and $x.(c(x) - c(r)) \geq 0$ so $\frac{r.c(r)}{x.(c(x)-c(r))+r.c(r)} \leq 1$. Thus, we have that $\alpha(\mathcal{C}) = \sup_{c\in\mathcal{C}} \sup_{x,r\geq 0} \frac{r.c(r)}{x.c(x)+(r-x).c(r)} = 1$ if $x \geq r$.

Therefore, we assume that x < r. Let G be a Pigou-like network. The lower edge has cost function $c_2(x) = c(x)$ and the upper edge has the constant cost function $c_1(x) = c(r), c \in \mathcal{C}$. Set the traffic rate to r. The equilibrium flow routes all the traffic over the lower edge, yielding cost r.c(r). Any feasible flow routes x units of traffic on the lower edge and $rx$ units on the upper edge. This yields cost x.c(x)+(r-x).c(r). By varying x, one can find the x for which x.c(x)+(r-x).c(r) is approximately minimal. Doing this for any $c \in \mathcal{C}$, one ends up with the POA being at least $\alpha(\mathcal{C})$, which means that $POA \geq \sup_{c\in\mathcal{C}} \sup_{x,r\geq 0} \frac{r.c(r)}{x.c(x)+(r-x).c(r)} = \alpha(\mathcal{C})$.

Now let's proove that the Pigou bound is also an upper bound on the price of anarchy, but in order to do so, we need an alternative characterization of the equilibrium flow.

**Proposition 7.4.** *(Variational inequality characterization) Let f be a feasible flow for the nonatomic instance (G, r, c). If flow f is an equilibrium flow, then*

$$\sum_{e\in E} c_e(f_e)f_e \leq \sum_{e\in E} c_e(f_e)f_e^*$$

*holds for every flow f* feasible for (G,r,c).*

Proof: f is an equilibrium flow if and only if $c_P(f) \leq c_{\tilde{P}}(f)$ for any P,$\tilde{P} \in \mathcal{P}_i$, for any i and for $f_P > 0$. $c_P(f) = c_{\tilde{P}}(f)$ if in addition $f_{\tilde{P}} > 0$. If any traffic deviates, creating flow f*, then that part costs an equal or higher amount since the price of the newly chosen path is equal or higher. Therefore,

$$\sum_{P\in\mathcal{P}_i} c_P(f)f_P \leq \sum_{P\in\mathcal{P}_i} c_P(f)f_P^*, \forall i$$

Summing over i, it follows :

$$\sum_{P\in\mathcal{P}} c_P(f)f_P \leq \sum_{P\in\mathcal{P}} c_P(f)f_P^*$$

$$\sum_{P\in\mathcal{P}} c_P(f)f_P \leq \sum_{P\in\mathcal{P}} c_P(f)f_P^* \iff \sum_{P\in\mathcal{P}} (\sum_{e\in P} c_e(f_e))f_P \leq \sum_{P\in\mathcal{P}} (\sum_{e\in P} c_e(f_e))f_P^*$$

By reversing the order of summation, we obtain

$$\sum_{e\in E} c_e(f_e)f_e \leq \sum_{e\in E} c_e(f_e)f_e^*$$

We can now show that the Pigou bound is also an upper bound on the price of anarchy.

**Theorem 7.5.** *(Tightness of the Pigou bound) Let $\mathcal{C}$ be a set of cost functions and $\alpha(\mathcal{C})$ the Pigou bound for $\mathcal{C}$. If (G,r,c) is a nonatomic instance with cost functions in $\mathcal{C}$, then the POA of (G,r,c) is at most $\alpha(\mathcal{C})$ i.e $POA(\mathcal{C}) \leqslant \alpha(\mathcal{C})$.*

Proof : Let f* and f be optimal and equilibrium flows, respectively. Then

$$C(f*) = \sum_{e\in E} f_e^* c_e(f_e^*) = \sum_{e\in E} (\frac{f_e^* c_e(f_e^*) + (f_e - f_e^*)c_e(f_e)}{c_e(f_e)f_e}.c_e(f_e)f_e + (f_e^* - f_e)c_e(f_e))$$

The fraction looks like the calculation of the total cost for a Pigou-like network with $x = f_e^*$ and $r = f_e$. (here we take the inverse of the fraction). The Pigou bound is the supremum of the fraction over all cost functions in $\mathcal{C}$ and all $x, r \geq 0$, therefore

$$\frac{f_e^* c_e(f_e^*) + (f_e - f_e^*)c_e(f_e)}{c_e(f_e)f_e} \geq \frac{1}{\alpha(\mathcal{C})}$$

for each edge e.

$$C(f*) \geq \frac{1}{\alpha(\mathcal{C})} \sum_{e\in E} c_e(f_e)f_e + \sum_{e\in E}(f_e^* - f_e)c_e(f_e)) \geq \frac{1}{\alpha(\mathcal{C})} \sum_{e\in E} c_e(f_e)f_e = \frac{C(f)}{\alpha(\mathcal{C})}$$

The last inequality follows from Proposition 6.5 that basically says:

$$\sum_{e\in E} c_e(f_e)f_e \leq \sum_{e\in E} c_e(f_e)f_e^*$$

i.e

$$\sum_{e\in E}(f_e^* - f_e)c_e(f_e)) \geq 0$$

Hence :

$$C(f*) \geq \frac{C(f)}{\alpha(\mathcal{C})} \implies POA = \frac{C(f)}{C(f*)} \leq \alpha(\mathcal{C})$$

Therefore, $\alpha(\mathcal{C})$ is an upper bound on the price of anarchy of (G,r,c) with $c \in \mathcal{C}$.

Hence Theorem 8.9 implies that the price of anarchy on any nonatomic instance is maximized by Pigou's bound , no matter what network is considered, thus : $\alpha(\mathcal{C}) = POA(\mathcal{C})$. The only restrictions are on the cost functions, and not on the network size, the network structure, nor the number of commodities. It is also remarkable that "the worst possible ratio" for a certain instance with cost functions in $\mathcal{C}$ can always be achieved with a network with only two parallel links.

As a last example, we will show that the price of anarchy is at most $\frac{4}{3}$ for nonatomic instances with affine cost functions.

**Theorem 7.6.** *(The price of anarchy in affine nonatomic instances) If (G,r,c) is a nonatomic instance with affine cost functions, then the price of anarchy of (G,r,c) is at most $\frac{4}{3}$.*

Proof: Theorem 8.9 says that the price of anarchy of a nonatomic instance is at most $\alpha(\mathcal{C})$. So for linear cost functions, we need to find $\alpha(\mathcal{C})$ with $\mathcal{C}$ the set of all possible affine cost functions.

$$\alpha(\mathcal{C}) = \sup_{c\in\mathcal{C}} \sup_{x,r\geq 0} \frac{r.c(r)}{x.c(x) + (r-x).c(r)}$$

with c(r)=a(r)+b, c(x)=a(x)+b, with $a, b \geq 0$ for any edge cost function. To maximize $\frac{r.c(r)}{x.c(x)+(r-x).c(r)}$ over x, we need to minimize the following expression :

$$x.c(x) + (r-x).c(r) = ax^2 + bx + ar^2 + br - arx - bx = ax^2 + ar^2 + br - arx$$

Differentiating with respect to x and equalizing to 0 yields

$$2ax - ar = 0 \iff x = \frac{1}{2}r$$

$$\alpha(\mathcal{C}) = \sup_{c\in\mathcal{C}} \frac{r.c(r)}{\frac{1}{2}r.c(\frac{1}{2}r) + (\frac{1}{2}r).c(r)} = \sup_{c\in\mathcal{C}} \frac{ar^2 + br}{\frac{1}{4}ar^2 + \frac{1}{2}br + \frac{1}{2}ar^2 + \frac{1}{2}br} = \sup_{c\in\mathcal{C}} \frac{ar + b}{\frac{3}{4}ar + b}$$

If b is increasing, the ratio is decreasing. Therefore, we choose b as low as possible, i.e. $b = 0$. Then $\sup_{c\in\mathcal{C}} \frac{ar}{\frac{3}{4}ar} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$.

# 8 ATOMIC SELFISH ROUTING

## 8.1 *The model*

As for the nonatomic one, an atomic selfish routing game is defined by a directed graph G=(V,E), k source-sink pairs $(s_i, t_i)$, and a nonnegative, continuous nondecreasing cost function $c_e : \mathbb{R}^+ \to \mathbb{R}^+$ for each edge e. We also denote an atomic instance by a triple (G,r,c).

The difference between a nonatomic and an atomic instance, is that for a nonatomic one, each commodity represents a larger population of individuals, each of whom controls a negligible amount of traffic, while in an atomic instance, each commodity represents a single player (hence atomic) who must route a significant amount of traffic on a single path.

In atomic unsplittable instances, there are k players. The hypothesis of being unsplittable means that players cannot route flow on several different paths, but must instead select a single path for routing. Different players can have identical source-sink pairs. The strategy set of player i is the set $\mathcal{P}_i$ of $s_i \to t_i$ paths, and if player i chooses the path P, then it routes its $r_i$ units of traffic on P. A flow is a nonnegative vector indexed by players and paths, with $f_P^{(i)}$ denoting the amount of traffic that player i routes on the $s_i \to t_i$ path P. A flow is feasible for an atomic instance if it corresponds to a strategy profile:for each player i, $f_P^{(i)}$ equals $r_i$ for exactly one $s_i \to t_i$ path and equals 0 for all other paths. The cost $c_P(f)$ of a path P with respect to a flow f and the cost C(f) of a flow f are defined as for nonatomic instances. An equilibrium flow of an atomic selfish routing game is a feasible flow such that no player can strictly decrease its cost by choosing a different path for its traffic.

**Definition 8.1.** (Atomic equilibrium flow) Let f be a feasible flow for the atomic instance (G,r,c). The flow f is an equilibrium flow if for every player $i \in 1, ..., k$ and every pair P, $\tilde{P} \in \mathcal{P}_i$ of $s_i \to t_i$ paths with $f_P^{(i)} > 0$, $c_P(f) \leqslant c_{\tilde{P}}(\tilde{f})$, where $\tilde{f}$ is the flow identical to f except that $\tilde{f}_P^{(i)} = 0$ and $\tilde{f}_{\tilde{P}}^{(i)} = r_i$.

This kind of equilibria are called "social Wardrop equilibria" in opposition to "selfish Wardrop equilibrium" for which no player can lower his cost by unilaterally changing routes (i.e deviating). Wardrop equilibria are used in complex networks with n players (n can be arbitrary large), and are considered as Nash equilibrium in the limite case for which each player represents an infinitesimal part of the traffic.

*Remark* 8.2. Different equilibrium flows of an atomic instance can have different costs, while all equilibrium flows of a nonatomic instance have equal cost. Secondly, the POA in atomic instances can be larger than in nonatomic ones.

The following example is an atomic instance that has affine cost functions -of the form ax+b- and its POA is 5/2.

**Example 8.3.** *(The AAE example : Awerbuch-Azar-Epstein (2005))*

*Consider the bidirected oriented triangle network shown in this figure :*

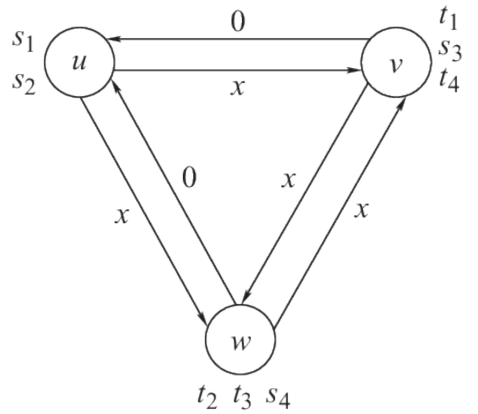


Figure 5: The AAE example (2005)

*with affine costs (either null or equal to x). Assume that there are four players each of whom needs to route one unit of traffic. The first two have source u and sinks v and w, respectively, the third has source v and sink w, and the fourth has source w and sink v. Each player has two strategies : whether he can choose a one-hop path, which means he can pass through one and only one vertex, or he can choose a two hoop path, for which he can pass through two vertices. In the optimal flow, all players route on their one-hop paths, and the total cost of this flow is 4 because each one will have a cost of one unit. This is an equilibrium flow because a flow can't have a cost strictly inferior to 4. On the other hand, if all players route on their two-hop paths, then we obtain a second equilibrium flow. Since the first two players each incur three units of cost and the last two players each incur two units of cost, this equilibrium flow has a cost of 10. The POA of this instance is therefore 10/4=2.5.*

**Proposition 8.4.** *In atomic instances with affine cost functions, different equilibrium flows can have different costs, and the POA can be as large as 5/2.*

Recall that equilibrium flows for atomic instances correspond to pure-strategy Nash equilibria, which do not always exist in arbitrary finite games. They don't always exist either in atomic selfish routing games.

*Remark* 8.5. Instances in which all players route the same amount of traffic are called unweighted. They admit at least one equilibrium flow (see next section for the proof).

## 8.2 *Existence of equilibrium flow*

We now consider equilibrium flows in atomic instances, note that an atomic instance doesn't necessarily admit an equilibrium flow, so in order to avoid this problem, we can add restrictions on the atomic instances model so that equilibrium flows are guaranteed to exist. We can also define an other equilibrium concept so that an equilibrium exists in every atomic instance.

The following theorem establishes the existence of equilibrium flows in atomic instances in which all players control the same amount of traffic.

**Theorem 8.6.** *(Existence of equilibrium flow)*

*Let (G,r,c) be an atomic instance in which every traffic amount $r_i$ is equal to a common positive value R. (This will be used in the proof to show that if every player i deviates, he will transfer his one unit of flow R from path P to path $\tilde{P}$).Then (G,r,c) admits at least one equilibrium flow.*

Proof : To prove the existence of an equilibrium flow, we prove that any global minimum of $\Phi_a$ is an equilibrium flow. We can obtain the proof of this theorem by discretizing the potential function (1) for non atomic instances and the proof of theorem (5.1)(a). Assume for simplicity that R=1. Set

$$\Phi_a(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i) \quad (3)$$

for every feasible flow f. Note that $\Phi_a$ is the same as the previous potential function $\Phi$ for nonatomic instances, except that the integral $\int_0^{f_e} c_e(x)dx$ has been replaced by the sum $\sum_{i=1}^{f_e} c_e(i)$. Since that the atomic instance (G,r,c) has a finite number of players, and each of these has a finite number of strategies, there are only a finite number of possible flows. Hence, since, we are in finite euclidean space, a global minimum of the potential function $\Phi_a$ exists, call it f. Now, we prove that f is an equilibrium flow in (G,r,c). Assume by contradiction that f is not an equilibrium flow, then the player i could strictly decrease its cost by deviating from the path P to the path $\tilde{P}$, yielding the new flow $\tilde{f}$ (strictly because according to the definition, it is an equilibrium flow if all other paths have higher or equal cost. For a flow to be a nonequilibrium flow, at least one alternative path must have strictly lower cost). In other words, we assume that there is a flow $\tilde{f}$ where one player chooses an alternative path $\tilde{P}$ with lower cost. This means $c_{\tilde{P}}(\tilde{f}) < c_P(f) \Leftrightarrow 0 > c_{\tilde{P}}(\tilde{f}) - c_P(f)$ The edges of $\tilde{P}$ are either in P or they are not. P and $\tilde{P}$ might have some edges in common, the cost of those edges does not change as player i changes to path $\tilde{P}$ . This means that only the edges that they have not in common should be considered. Note that if player i deviates, he transfers his one unit of flow (R=1) from path P to path $\tilde{P}$. This means that edges that are now used now has an extra flow of one unit, so they now have a total flow $f_e + 1$. Hence, edges that are not anymore used have a total flow of $f_e - 1$, and we then have that $0 > c_{\tilde{P}}(\tilde{f}) - c_P(f) = \sum_{e \in \tilde{P} \setminus P} c_e(f_e + 1) - \sum_{e \in P \setminus \tilde{P}} c_e(f_e)$.

Now, let's consider $\Phi_a$. The new edges in $\tilde{P}$ (i.e $e \in \tilde{P} \setminus P$) add to $\Phi_a(f)$ one extra term $c_e(f_e + 1)$ ($+1$ because $R = 1$). The edges that are not anymore in $\tilde{P}$ , i.e $e \in P \setminus \tilde{P}$ subtracts the term $c_e(f_e)$ of $\Phi_a$.The sum for common edges in $P \cap \tilde{P}$ remains the same, i.e. : $\sum_{i=1}^{\tilde{f}_e} c_e(i) = \sum_{i=1}^{f_e} c_e(i)$. Hence, $\Phi_a(\tilde{f}) - \Phi_a(f) = \sum_{e \in \tilde{P} \setminus P} c_e(f_e + 1) - \sum_{e \in P \setminus \tilde{P}} c_e(f_e)$. So, $\Phi_a(\tilde{f}) - \Phi_a(f) < 0$. Since this expression is negative, the potential function value of $\tilde{f}$ is strictly less than that of f, which contradicts our choice of f as the global minimum of the potential function $\Phi_a$. Therefore, f is an equilibrium flow of (G,r,c).

## 8.3 *Bounding the POA in atomic instances*

**Lemma 8.7.** *(Equilibrium condition)*
*Let (G,r,c) be an atomic instance in which each edge e has an affine cost function $c_e(x) = a_e x + b_e$ with $a_e, b_e \geqslant 0$. Let f and $f^*$ be equilibrium and optimal flows, respectively, for (G,r,c). Let player i use the path $P_i$ in f and the path $P_i^*$ in $f^*$. Then*

$\sum_{e \in P_i} (a_e f_e + b_e) \leqslant \sum_{e \in P_i^*} (a_e(f_e + r_i) + b_e)$ *where $r_i$ is the same amount of traffic that every player has to direct.*

Proof : Follows from the definition of an equilibrium flow in an atomic instance. $f_e^* = f_e$ if $e \in P_i \cap P_i^*$ and $f_e^* = f_e + r_i$ if $e \in P_i^* \backslash P_i$. This lemma is true not only for f being the optimal flow, but for any feasible flow.

The inequality of Lemma 8.7 holds for each player. With these inequalities, we can derive Lemma 8.8:

**Lemma 8.8.** *(Equilibrium inequality)*
*With the same assumptions and notations as in Lemma 8.7,*

$$C(f) \leqslant C(f^*) + \sum_{e \in E} a_e f_e f_e^*$$

*Proof : For each player i, multiply the inequality of Lemma 8.7 by $r_i$ :*

$$r_i \sum_{e \in P_i} (a_e f_e + b_e) \leqslant r_i \sum_{e \in P_i^*} (a_e(f_e + r_i) + b_e) \ (4)$$

*Summing the k inequality of (4) (one inequality for each player), we obtain :*

$$\sum_{i=1}^{k} r_i \sum_{e \in P_i} (a_e f_e + b_e) = C(f) \leqslant \sum_{i=1}^{k} r_i \sum_{e \in P_i^*} (a_e(f_e + r_i) + b_e)$$

*On the right term of the inequality, $r_i \leqslant f_e^*$ for any edge $e \in P_i^*$. If only player i routed his traffic over edge e, then $r_i = f_e^*$. If more players use this edge, then $r_i \leqslant f_e^*$. Therefore:*

$$\sum_{i=1}^{k} r_i \sum_{e \in P_i^*} (a_e(f_e + r_i) + b_e) \leqslant \sum_{i=1}^{k} r_i \sum_{e \in P_i^*} (a_e(f_e + f_e^*) + b_e) = \sum_{e \in E} (a_e(f_e + f_e^*) + b_e) f_e^*$$

*The equality follows by reversing the order of summation (since we sum over all edges in $P_i^*$, we multiply each sum by $f_e^*$).*

$$C(f) \leqslant \sum_{e \in E} (a_e(f_e + f_e^*) + b_e) f_e^* = \sum_{e \in E} (a_e f_e^* + b_e) f_e^* + \sum_{e \in E} a_e f_e f_e^* = C(f^*) + \sum_{e \in E} a_e f_e f_e^*$$

*The last equality finished the proof. Note that one can consider $\sum_{e \in E} a_e f_e f_e^*$ as a sort of error term: how much do the costs of any equilibrium flow and the optimal flow differ? This error term can be rewritten in terms of C(f) and $C(f^*)$ which helps us bounding the price of anarchy.*

**Theorem 8.9.** *(The Price of Anarchy in affine weighted atomic instances)*
*If (G,r,c) is an atomic instance with affine cost functions, then the price of anarchy of (G,r,c) is at most $\frac{3+\sqrt{5}}{2} \approx 2.618$.*

*Proof : Let f and f* denote equilibrium and optimal flow, respectively, for the atomic instance (G,r,c). Assume that edge e has cost function $c_e(x) = a_e x + b_e$ for $a_e, b_e \geqslant 0$. Apply the Cauchy-Schwarz inequality to the vectors $\sqrt{a_e} f_e e \in E$ and $\sqrt{a_e} f_e^* e \in E$ to obtain : $\sum_{e \in E} \sqrt{a_e} f_e . \sqrt{a_e} f_e^* = \sum_{e \in E} a_e f_e f_e^* \leqslant \sqrt{\sum_{e \in E} (\sqrt{a_e} f_e)^2} . \sqrt{\sum_{e \in E} (\sqrt{a_e} f_e^*)^2} = \sqrt{\sum_{e \in E} a_e f_e^2} . \sqrt{\sum_{e \in E} a_e (f_e^*)^2} \leqslant \sqrt{\sum_{e \in E} (a_e f_e^2 + b_e f_e)} . \sqrt{\sum_{e \in E} (a_e (f_e^*)^2 + b_e f_e^*)} = \sqrt{C(f)} . \sqrt{C(f^*)}$.The first inequality is the Cauchy-Schwartz inequality; the second inequality is true because $b_e f_e, b_e f_e^* \geqslant 0$. Combining this inequality with the inequality of Lemma 8.8, dividing by C(f), and rearranging gives: $C(f) \leqslant C(f*) + \sum_{e \in E} a_e f_e f_e^* \leqslant C(f*) + \sqrt{C(f)} . \sqrt{(C(f^*))} \implies \frac{C(f)}{C(f*)} \leqslant 1 + \sqrt{\frac{C(f).C(f*)}{(C(f*))^2}} = 1 + \sqrt{\frac{C(f)}{C(f*)}} \implies \frac{C(f)}{C(f*)} - 1 \leqslant \sqrt{\frac{C(f)}{C(f*)}}$. Set $\frac{C(f)}{C(f*)} = X \geqslant 0$, hence, $X - 1 \leqslant \sqrt{X} \implies (X-1)^2 \leqslant X \implies X^2 - 3X + 1 \leqslant 0$. A simple study in $\mathbb{R}^+$ of the 2nd degree polynomial $X^2 - 3X + 1$ gives us : $X^2 - 3X + 1 \leqslant 0 \implies X \leqslant \frac{3+\sqrt{5}}{2} \approx 2.618$. Since $X = \frac{C(f)}{C(f*)} = POA$, this ends the proof of the theorem.*

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