Capture-recapture experiments

Inference in finite populations

Problem of estimating an unknown population size, N, based on partial observation of this population: domain of *survey sampling*

Warning

We do not cover the official Statistics/stratified type of survey based on a preliminary knowledge of the structure of the population

Numerous applications

- *Biology & Ecology* for estimating the size of herds, of fish or whale populations, etc.
- *Sociology & Demography* for estimating the size of populations at risk, including homeless people, prostitutes, illegal migrants, drug addicts, etc.
- *official Statistics* in the U.S. and French census undercount procedures
- *Economics & Finance* in credit scoring, defaulting companies, etc.,
- *Fraud detection* phone, credit card, etc.
- *Document authentication* historical documents, forgery, etc.,
- *Software debugging*

Setup

Size N of the whole population is unknown but samples (with fixed or random sizes) can be extracted from the population.

The binomial capture model

Simplest model of all: joint capture of n^+ individuals from a population of size N.

Population size $N \in \mathbb{N}^*$ is the parameter of interest, but there exists a nuisance parameter, the probability $p \in [0,1]$ of capture [under assumption of independent captures]

Sampling model

$$n^+ \sim \mathscr{B}(N, p)$$

and corresponding likelihood

$$\ell(N, p|n^+) = \binom{N}{n^+} p^{n^+} (1-p)^{N-n^+} \mathbb{I}_{N \geq n^+} \, .$$

Bayesian inference (1)

Under vague prior

$$\pi(N,p) \propto N^{-1}\mathbb{I}_{\mathbb{N}^*}(N)\mathbb{I}_{[0,1]}(p)\,,$$

posterior distribution of N is

$$\begin{aligned}\pi(N|n^+) &\propto \frac{N!}{(N-n^+)!}\,N^{-1}\mathbb{I}_{N\geq n^+}\mathbb{I}_{\mathbb{N}^*}(N)\int_0^1 p^{n^+}(1-p)^{N-n^+}dp\\ &\propto \frac{(N-1)!}{(N-n^+)!}\,\frac{(N-n^+)!}{(N+1)!}\,\mathbb{I}_{N\geq n^+\vee 1}\\ &= \frac{1}{N(N+1)}\mathbb{I}_{N\geq n^+\vee 1}\,.\end{aligned}$$

where $n^+ \vee 1 = \max(n^+,1)$

Bayesian inference (2)

If we use the uniform prior

$$\pi(N,p) \propto \mathbb{I}_{\{1,\dots,S\}}(N)\mathbb{I}_{[0,1]}(p)\,,$$

the posterior distribution of N is

$$\pi(N|n^+) \propto \frac{1}{N+1}\mathbb{I}_{\{n^+\vee 1,\dots,S\}}(N)\,.$$

Capture-recapture data

European dippers

Birds closely dependent on streams, feeding on underwater invertebrates Capture-recapture data on dippers over years 1981–1987 in 3 zone of 200 km^2 in eastern France with markings and recaptures of breeding adults each year, during the breeding period from early March to early June.

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Each row of 7 digits corresponds to a capture-recapture story: 0 stands for absence of capture and, else, $1, 2$ or 3 represents the zone of capture.

E.g.

1 0 0 0 0 0 0
1 3 0 0 0 0 0
0 2 2 2 1 2 2

means: first dipper only captured the first year [in zone 1], second dipper captured in years 1981–1982 and moved from zone 1 to zone 3 between those years, third dipper captured in years 1982–1987 in zone 2

The two-stage capture-recapture experiment

Extension to the above with two capture periods plus a marking stage:

1. n_1 individuals from a population of size N captured *[sampled without replacement]*
2. captured individuals marked and released
3. n_2 individuals captured during second identical sampling experiment
4. m_2 individuals out of the n_2's bear the identification mark *[captured twice]*

The two-stage capture-recapture model

For *closed populations* [fixed population size N throughout experiment, constant capture probability p for all individuals, and independence between individuals/captures], binomial models:

$$n_1 \sim \mathscr{B}(N, p)\,, \quad m_2|n_1 \sim \mathscr{B}(n_1, p) \quad \text{and}$$

$$n_2 - m_2|n_1, m_2 \sim \mathscr{B}(N - n_1, p)\,.$$

The two-stage capture-recapture likelihood

Corresponding likelihood $\ell(N, p|n_1, n_2, m_2)$

$$\binom{N-n_1}{n_2-m_2} p^{n_2-m_2}(1-p)^{N-n_1-n_2+m_2} \mathbb{I}_{\{0,\ldots,N-n_1\}}(n_2-m_2)$$
$$\times \binom{n_1}{m_2} p^{m_2}(1-p)^{n_1-m_2} \binom{N}{n_1} p^{n_1}(1-p)^{N-n_1} \mathbb{I}_{\{0,\ldots,N\}}(n_1)$$
$$\propto \frac{N!}{(N-n_1-n_2+m_2)!} p^{n_1+n_2}(1-p)^{2N-n_1-n_2} \mathbb{I}_{N\geq n^+}$$
$$\propto \binom{N}{n^+} p^{n^c}(1-p)^{2N-n^c} \mathbb{I}_{N\geq n^+}$$

where $n^c = n_1 + n_2$ and $n^+ = n_1 + (n_2 - m_2)$ are total number of captures/captured individuals over both periods

Bayesian inference (1)

Under prior $\pi(N,p) = \pi(N)\pi(p)$ where $\pi(p)$ is $\mathscr{U}([0,1])$, conditional posterior distribution on p is

$$\pi(p|N,n_1,n_2,m_2) = \pi(p|N,n^c) \propto p^{n^c}(1-p)^{2N-n^c}\,,$$

that is,

$$p|N,n^c \sim \mathscr{B}e(n^c+1, 2N-n^c+1).$$

Marginal posterior distribution of N more complicated. If $\pi(N) = \mathbb{I}_{\mathbb{N}^*}(N)$,

$$\pi(N|n_1,n_2,m_2) \propto \binom{N}{n^+} B(n^c+1, 2N-n^c+1)\mathbb{I}_{N\geq n^+\vee 1}$$

[Beta-Pascal distribution]

Bayesian inference (2)

Same problem if $\pi(N) = N^{-1}\mathbb{I}_{\mathbb{N}^*}(N)$.

Computations

Since $N \in \mathbb{N}$, always possible to approximate the missing normalizing factor in $\pi(N|n_1, n_2, m_2)$ by summing in N. Approximation errors become a problem when N and n^+ are large.

Under proper uniform prior,

$$\pi(N) \propto \mathbb{I}_{\{1,\ldots,S\}}(N)\,,$$

posterior distribution of N proportional to

$$\pi(N|n^+) \propto \binom{N}{n^+} \frac{\Gamma(2N - n^c + 1)}{\Gamma(2N + 2)}\, \mathbb{I}_{\{n^+ \vee 1,\ldots,S\}}(N)\,.$$

and can be computed with no approximation error.

The Darroch model

Simpler version of the above: conditional on both samples sizes n_1 and n_2,

$$m_2|n_1, n_2 \sim \mathscr{H}(N, n_2, n_1/N)\,.$$

Under uniform prior on $N \sim \mathscr{U}(\{1, \ldots, S\})$, posterior distribution of N

$$\pi(N|m_2) \propto \binom{n_1}{m_2}\binom{N-n_1}{n_2-m_2}\bigg/\binom{N}{n_2}\mathbb{I}_{\{n^+\vee 1,\ldots,S\}}(N)$$

and posterior expectations computed numerically by simple summations.

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For the two first years and $S = 400$, posterior distribution of N for the Darroch model given by

$$\pi(N|m_2) \propto (n-n_1)!(N-n_2)!/\{(n-n_1-n_2+m_2)!N!\}\,\mathbb{I}_{\{71,\ldots,400\}}(N)\,,$$

with inverse normalization factor

$$\sum_{k=71}^{400} (k-n_1)!(k-n_2)!/\{(k-n_1-n_2+m_2)!k!\}\,.$$

Influence of prior hyperparameter S (for $m_2 = 11$):

S	100	150	200	250	300	350	400	450	500
$\mathbb{E}[N\|m_2]$	95	125	141	148	151	151	152	152	152

Gibbs sampler for 2-stage capture-recapture

If $n^+ > 0$, both conditional posterior distributions are standard, since

$$\begin{aligned} p|n^c, N &\sim \mathscr{B}e(n^c+1, 2N-n^c+1) \\ N-n^+|n^+, p &\sim \mathscr{N}eg(n^+, 1-(1-p)^2)\,. \end{aligned}$$

Therefore, joint distribution of (N, p) can be approximated by a Gibbs sampler

T-stage capture-recapture model

Further extension to the two-stage capture-recapture model: series of T consecutive captures.
n_t individuals captured at period $1 \le t \le T$, and m_t recaptured individuals (with the convention that $m_1 = 0$)

$$n_1 \sim \mathscr{B}(N, p)$$

and, conditional on earlier captures/recaptures $(2 \le j \le T)$,

$$m_j \sim \mathscr{B}\left(\sum_{t=1}^{j-1}(n_t - m_t), p\right) \quad \text{and}$$
$$n_j - m_j \sim \mathscr{B}\left(N - \sum_{t=1}^{j-1}(n_t - m_t), p\right) \,.$$

T-stage capture-recapture likelihood

Likelihood $\ell(N, p|n_1, n_2, m_2 \ldots, n_T, m_T)$ given by

$$\binom{N}{n_1} p^{n_1} (1-p)^{N-n_1} \prod_{j=2}^{T} \left[\binom{N - \sum_{t=1}^{j-1}(n_t - m_t)}{n_j - m_j} p^{n_j - m_j} \right.$$
$$\times (1-p)^{N - \sum_{t=1}^{j}(n_t - m_t)} \binom{\sum_{t=1}^{j-1}(n_t - m_t)}{m_j}$$
$$\left. \times\ p^{m_j} (1-p)^{\sum_{t=1}^{j-1}(n_t - m_t) - m_j} \right] .$$

Sufficient statistics

Simplifies into

$$\ell(N,p|n_1,n_2,m_2\ldots,n_T,m_T) \propto \frac{N!}{(N-n^+)!} p^{n^c}(1-p)^{TN-n^c}\mathbb{I}_{N\geq n^+}$$

with the sufficient statistics

$$n^+ = \sum_{t=1}^{T}(n_t - m_t) \quad \text{and} \quad n^c = \sum_{t=1}^{T} n_t\,,$$

total number of captured individuals/captures over the T periods

Bayesian inference (1)

Under noninformative prior $\pi(N,p) = 1/N$, joint posterior

$$\pi(N,p|n^+,n^c) \quad \propto \quad \frac{(N-1)!}{(N-n^+)!}\, p^{n^c}(1-p)^{TN-n^c}\mathbb{I}_{N\geq n^+\vee 1}\,.$$

leads to conditional posterior

$$p|N,n^+,n^c \sim \mathscr{B}e(n^c+1, TN-n^c+1)$$

and marginal posterior

$$\pi(N|n^+,n^c) \propto \frac{(N-1)!}{(N-n^+)!}\,\frac{(TN-n^c)!}{(TN+1)!}\,\mathbb{I}_{N\geq n^+\vee 1}$$

which is computable *[under previous provisions]*.

Alternative Gibbs sampler also available.

Bayesian inference (2)

Under prior $N \sim \mathscr{U}(\{1, \ldots, S\})$ and $p \sim \mathscr{U}([0, 1])$,

$$\pi(N|n^+) \propto \binom{N}{n^+} \frac{(TN - n^c)!}{(TN + 1)!} \mathbb{I}_{\{n^+ \vee 1, \ldots, S\}}(N).$$

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For the whole set of observations, $T = 7$, $n^+ = 294$ and $n^c = 519$.
For $S = 400$, the posterior expectation of N is equal to 372.89.
For $S = 2500$, it is 373.99.

Computational difficulties

E.g., heterogeneous capture–recapture model where individuals are captured at time $1 \leq t \leq T$ with probability p_t with both N and the p_t's are unknown.

Corresponding likelihood

$$\ell(N, p_1, \ldots, p_T | n_1, n_2, m_2 \ldots, n_T, m_T) \propto \frac{N!}{(N-n^+)!} \prod_{t=1}^{T} p_t^{n_t} (1-p_t)^{N-n_t} .$$

Computational difficulties (cont'd)

Associated prior $N \sim \mathscr{P}(\lambda)$ and

$$\alpha_t = \log\left(p_t/1-p_t\right) \sim \mathscr{N}(\mu_t, \sigma^2),$$

where the μ_t's and σ are known.
Posterior

$$\pi(\alpha_1,\ldots,\alpha_T,N|,n_1,\ldots,n_T) \propto \frac{N!}{(N-n^+)!}\,\frac{\lambda^N}{N!}\,\prod_{t=1}^{T}(1+e^{\alpha_t})^{-N} \times \prod_{t=1}^{T}\exp\left\{\alpha_t n_t - \frac{1}{2\sigma^2}(\alpha_t-\mu_t)^2\right\}.$$

much less manageable computationally.

Open populations

More realistically, population size does not remain fixed over time: probability q for each individual to leave the population at each time [or between each capture episode]

First occurrence of missing variable model.

Simplified version where only individuals captured during the first experiment are marked and their subsequent recaptures are registered.

Working example

Three successive capture experiments with

$$\begin{aligned} n_1 &\sim \mathscr{B}(N, p), \\ r_1|n_1 &\sim \mathscr{B}(n_1, q), \\ c_2|n_1, r_1 &\sim \mathscr{B}(n_1 - r_1, p), \\ r_2|n_1, r_1 &\sim \mathscr{B}(n_1 - r_1, q) \\ c_3|n_1, r_1, r_2 &\sim \mathscr{B}(n_1 - r_1 - r_2, p) \end{aligned}$$

where only n_1, c_2 and c_3 are observed.

Variables r_1 and r_2 not available and therefore part of unknowns like parameters N, p and q.

Bayesian inference

Likelihood

$$\binom{N}{n_1} p^{n_1}(1-p)^{N-n_1} \binom{n_1}{r_1} q^{r_1}(1-q)^{n_1-r_1} \binom{n_1-r_1}{c_2} p^{c_2}(1-p)^{n_1-r_1-c_2} \binom{n_1-r_1}{r_2} q^{r_2}(1-q)^{n_1-r_1-r_2} \binom{n_1-r_1-r_2}{c_3} p^{c_3}(1-p)^{n_1-r_1-r_2-c_3}$$

and prior

$$\pi(N,p,q) = N^{-1}\mathbb{I}_{[0,1]}(p)\mathbb{I}_{[0,1]}(q)$$

Full conditionals for Gibbs sampling

$$\begin{aligned}
\pi(p|N,q,\mathcal{D}^*) &\propto p^{n_+}(1-p)^{u_+} \\
\pi(q|N,p,\mathcal{D}^*) &\propto q^{c_1+c_2}(1-q)^{2n_1-2r_1-r_2} \\
\pi(N|p,q,\mathcal{D}^*) &\propto \frac{(N-1)!}{(N-n_1)!}(1-p)^N \mathbb{I}_{N\geq n_1} \\
\pi(r_1|p,q,n_1,c_2,c_3,r_2) &\propto \frac{(n_1-r_1)!\, q^{r_1}(1-q)^{-2r_1}(1-p)^{-2r_1}}{r_1!(n_1-r_1-r_2-c_3)!(n_1-c_2-r_1)!} \\
\pi(r_2|p,q,n_1,c_2,c_3,r_1) &\propto \frac{q^{r_2}[(1-p)(1-q)]^{-r_2}}{r_2!(n_1-r_1-r_2-c_3)!}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D}^* &= (n_1,c_2,c_3,r_1,r_2) \\
u_1 &= N-n_1,\; u_2 = n_1-r_1-c_2,\; u_3 = n_1-r_1-r_2-c_3 \\
n_+ &= n_1+c_2+c_3,\; u_+ = u_1+u_2+u_3
\end{aligned}$$

Full conditionals (2)

Therefore,

$$\begin{aligned} p|N,q,\mathcal{D}^* &\sim \mathcal{B}e(n_+ + 1, u_+ + 1) \\ q|N,p,\mathcal{D}^* &\sim \mathcal{B}e(r_1 + r_2 + 1, 2n_1 - 2r_1 - r_2 + 1) \\ N - n_1|p,q,\mathcal{D}^* &\sim \mathcal{N}eg(n_1, p) \\ r_2|p,q,n_1,c_2,c_3,r_1 &\sim \mathcal{B}\left(n_1 - r_1 - c_3, \frac{q}{1+(1-q)(1-p)}\right) \end{aligned}$$

r_1 has a less conventional distribution, but, since n_1 not extremely large, possible to compute the probability that r_1 is equal to one of the values in $\{0, 1, \ldots, \min(n_1 - r_2 - c_3, n_1 - c_2)\}$.

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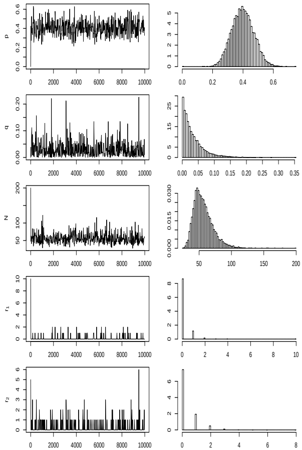
$n_1 = 22$, $c_2 = 11$ and $c_3 = 6$
MCMC approximations to the posterior expectations of N and p equal to 57 and 0.40



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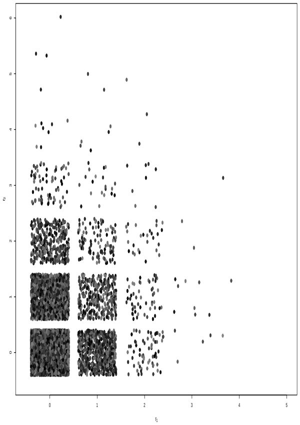


Accept-Reject methods

- Many distributions from which it is difficult, or even impossible, to directly simulate.
- Technique that only require us to know the functional form of the target π of interest up to a multiplicative constant.
- Key to this method is to use a proposal density g *[as in Metropolis-Hastings]*

Principle

Given a target density π, find a density g and a constant M such that

$$\pi(x) \leq Mg(x)$$

on the support of π.

Accept-Reject algorithm is then

1. `Generate` $X \sim g$, $U \sim \mathcal{U}_{[0,1]}$;
2. `Accept` $Y = X$ `if` $U \leq \frac{f(X)}{Mg(X)}$;
3. `Return to 1. otherwise.`

Validation of Accept-Reject

This algorithm produces a variable Y distributed according to f

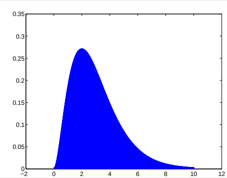
Fundamental theorem of simulation

Simulating

$$X \sim f(x)$$

is equivalent to simulating

$$(X, U) \sim \mathcal{U}\{(x, u) : 0 < u < \pi(x)\}$$



Two interesting properties:

- First, Accept-Reject provides a generic method to simulate from any density π that is known *up to a multiplicative factor* Particularly important for Bayesian calculations since

$$\pi(\theta|x) \propto \pi(\theta)\; f(x|\theta)\;.$$

is specified up to a normalizing constant

- Second, the probability of acceptance in the algorithm is $1/M$, e.g., expected number of trials until a variable is accepted is M

Application to the open population model

Since full conditional distribution of r_1 non-standard, rather than using exhaustive enumeration of all probabilities $\mathbb{P}(m_1 = k) = \pi(k)$ and then sampling from this distribution, try to use a proposal based on a binomial upper bound.

Take g equal to the binomial $\mathscr{B}(n_1, q_1)$ with

$$q_1 = q/(1-q)^2(1-p)^2$$

Proposal bound

$\pi(k)/g(k)$ proportional to

$$\frac{\binom{n_1-c_2}{k}(1-q_1)^k\binom{n_1-k}{r_2+c_3}}{\binom{n_1}{k}} = \frac{(n_1-c_2)!}{(r_2+c_3)!n_1!}\,\frac{((n_1-k)!)^2(1-q_1)^k}{(n_1-c_2-k)!(n_1-r_2-c_3-k)!}$$

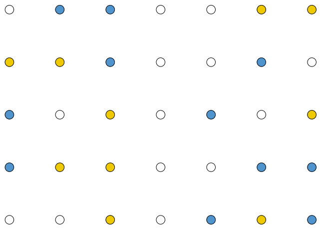
decreasing in k, therefore bounded by

$$\frac{(n_1-c_2)!}{(r_2+c_3)!}\,\frac{n_1!}{(n_1-c_2)!(n_1-r_2-c_3)!} = \binom{n_1}{r_2+c_3}.$$

- ↯ This is *not* the constant M because of unnormalised densities [M may also depend on q_1]. Therefore the average acceptance rate is undetermined and requires an extra Monte Carlo experiment

Arnason–Schwarz Model

Representation of a capture recapture experiment as a collection of individual histories: for each individual captured at least once, individual characteristics of interest (location, weight, social status, &tc.) registered at each capture.



Possibility that individuals vanish from the *[open]* population between two capture experiments.

Parameters of interest

Study the movements of individuals between zones/strata rather than estimating population size.

Two types of variables associated with each individual $i = 1, \ldots, n$

1. a variable for its location *[partly observed]*,

$$\mathbf{z}_i = (z_{(i,t)}, t = 1, .., \tau)$$

   where τ is the number of capture periods,

2. a binary variable for its capture history *[completely observed]*,

$$\mathbf{x}_i = (x_{(i,t)}, t = 1, .., \tau)\,.$$

Migration & deaths

$z_{(i,t)} = r$ when individual i is alive in stratum r at time t and denote $z_{(i,t)} = \dagger$ for the case when it is dead at time t.

Variable $\mathbf{z}_i$ sometimes called *migration* process of individual i as when animals moving between geographical zones.

E.g.,

$$\mathbf{x}_i = 1\,1\,0\,1\,1\,1\,0\,0\,0 \quad \text{and} \quad \mathbf{z}_i = 1\,2\,\cdot\,3\,1\,1\,\cdot\,\cdot\,\cdot$$

for which a possible completed $\mathbf{z}_i$ is

$$\mathbf{z}_i = 1\,2\,1\,3\,1\,1\,2\,\dagger\,\dagger$$

meaning that animal died between 7th and 8th captures

No tag recovery

We assume that

- $\dagger$ is absorbing
- $z_{(i,t)} = \dagger$ always corresponds to $x_{(i,t)} = 0$.
- the $(\mathbf{x}_i, \mathbf{z}_i)$'s $(i = 1, \ldots, n)$ are independent
- each vector $\mathbf{z}_i$ is a Markov chain on $\mathfrak{K} \cup \{\dagger\}$ with uniform initial probability on $\mathfrak{K}$.

Reparameterisation

Parameters of the Arnason–Schwarz model are

1. capture probabilities

$$p_t(r) = \mathbb{P}\left(x_{(i,t)} = 1 | z_{(i,t)} = r\right)$$

2. transition probabilities

$$q_t(r,s) = \mathbb{P}\left(z_{(i,t+1)} = s | z_{(i,t)} = r\right) \quad r \in \mathfrak{K}, s \in \mathfrak{K} \cup \{\dagger\}, \quad q_t(\dagger,\dagger) = 1$$

3. *survival* probabilities $\phi_t(r) = 1 - q_t(r,\dagger)$
4. inter-strata *movement* probabilities $\psi_t(r,s)$ such that

$$q_t(r,s) = \phi_t(r) \times \psi_t(r,s) \quad r \in \mathfrak{K}, s \in \mathfrak{K}\,.$$

Modelling

Likelihood

$$\ell((\mathbf{x}_1,\mathbf{z}_1),\ldots,(\mathbf{x}_n,\mathbf{z}_n)) \propto \prod_{i=1}^{n} \left[\prod_{t=1}^{\tau} p_t(z_{(i,t)})^{x_{(i,t)}} (1 - p_t(z_{(i,t)}))^{1-x_{(i,t)}} \times \prod_{t=1}^{\tau-1} q_t(z_{(i,t)}, z_{(i,t+1)}) \right].$$

Conjugate priors

Capture and survival parameters

$$p_t(r) \sim \mathscr{B}e(a_t(r), b_t(r))\,, \qquad \phi_t(r) \sim \mathscr{B}e(\alpha_t(r), \beta_t(r))\,,$$

where $a_t(r), \ldots$ depend on both time t and location r,
For movement probabilities/Markov transitions
$\psi_t(r) = (\psi_t(r,s); s \in \mathfrak{K})$,

$$\psi_t(r) \sim \mathscr{D}ir(\gamma_t(r))\,,$$

since

$$\sum_{s \in \mathfrak{K}} \psi_t(r,s) = 1\,,$$

where $\gamma_t(r) = (\gamma_t(r,s); s \in \mathfrak{K})$.

lizards

Capture–recapture experiment on the migrations of lizards between three adjacent zones, with are six capture episodes.

Prior information provided by biologists on p_t (which are assumed to be zone independent) and $\phi_t(r)$, in the format of prior expectations and prior confidence intervals.

Differences in prior on p_t due to differences in capture efforts differences between episodes $1, 3, 5$ and $2, 4$ due to different mortality rates over winter.

Prior information

	Episode	2	3	4	5	6
p_t	Mean	0.3	0.4	0.5	0.2	0.2
	95% int.	[0.1,0.5]	[0.2,0.6]	[0.3,0.7]	[0.05,0.4]	[0.05,0.4]

	Site	A		B,C	
	Episode	t=1,3,5	t=2,4	t=1,3,5	t=2,4
$\phi_t(r)$	Mean	0.7	0.65	0.7	0.7
	95% int.	[0.4,0.95]	[0.35,0.9]	[0.4,0.95]	[0.4,0.95]

Prior equivalence

Prior information that can be translated in a collection of beta priors

Episode	2	3	4	5	6
Dist.	$\mathscr{B}e(6,14)$	$\mathscr{B}e(8,12)$	$\mathscr{B}e(12,12)$	$\mathscr{B}e(3.5,14)$	$\mathscr{B}e(3.5,14)$

Site	A		B	
Episode	t=1,3,5	t=2,4	t=1,3,5	t=2,4
Dist.	$\mathscr{B}e(6.0,2.5)$	$\mathscr{B}e(6.5,3.5)$	$\mathscr{B}e(6.0,2.5)$	$\mathscr{B}e(6.0,2.5)$

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Prior belief that the capture and survival rates should be constant over time

$$p_t(r) = p(r) \quad \text{and} \quad \phi_t(r) = \phi(r)$$

Assuming in addition that movement probabilities are time-independent,

$$\psi_t(r) = \psi(r)$$

we are left with $3[p(r)] + 3[\phi(r)] + 3 \times 2[\phi_t(r)] = 12$ parameters.

Use non-informative priors with

$$a(r) = b(r) = \alpha(r) = \beta(r) = \gamma(r, s) = 1$$

Gibbs sampling

Needs to account for the missing parts in the $\mathbf{z}_i$'s, in order to simulate the parameters from the full conditional distributions

$$\pi(\theta|\mathbf{x},\mathbf{z}) \propto \ell(\theta|\mathbf{x},\mathbf{z}) \times \pi(\theta)\,,$$

where $\mathbf{x}$ and $\mathbf{z}$ are the collections of the vectors of capture indicators and locations.

Particular case of *data augmentation*, where the missing data $\mathbf{z}$ is simulated at each step t in order to reconstitute a complete sample $(\mathbf{x},\mathbf{z}^{(t)})$ with two steps:

- Parameter simulation
- Missing location simulation

Arnason–Schwarz Gibbs sampler

Algorithm

Iteration l $(l \geq 1)$

1. **Parameter simulation**
   simulate $\theta^{(l)} \sim \pi(\theta|\mathbf{z}^{(l-1)}, \mathbf{x})$ as $(t = 1, \ldots, \tau)$

$$p_t^{(l)}(r)|\mathbf{x}, \mathbf{z}^{(l-1)} \sim \mathscr{B}e\left(a_t(r) + u_t(r), b_t(r) + v_t^{(l)}(r)\right)$$

$$\phi_t^{(l)}(r)|\mathbf{x}, \mathbf{z}^{(l-1)} \sim \mathscr{B}e\left(\alpha_t(r) + \sum_{j \in \mathfrak{K}} w_t^{(l)}(r, j), \beta_t(r) + w_t^{(l)}(r, \dagger)\right)$$

$$\psi_t^{(l)}(r)|\mathbf{x}, \mathbf{z}^{(l-1)} \sim \mathscr{D}ir\left(\gamma_t(r, s) + w_t^{(l)}(r, s); s \in \mathfrak{K}\right)$$

Arnason–Schwarz Gibbs sampler (cont'd)

where

$$w_t^{(l)}(r,s) = \sum_{i=1}^{n} \mathbb{I}_{\left(z_{(i,t)}^{(l-1)}=r,\, z_{(i,t+1)}^{(l-1)}=s\right)}$$

$$u_t^{(l)}(r) = \sum_{i=1}^{n} \mathbb{I}_{\left(x_{(i,t)}=1,\, z_{(i,t)}^{(l-1)}=r\right)}$$

$$v_t^{(l)}(r) = \sum_{i=1}^{n} \mathbb{I}_{\left(x_{(i,t)}=0,\, z_{(i,t)}^{(l-1)}=r\right)}$$

Arnason–Schwarz Gibbs sampler (cont'd)

2. **Missing location simulation**
generate the unobserved $z_{(i,t)}^{(l)}$'s from the full conditional distributions

$$\mathbb{P}(z_{(i,1)}^{(l)} = s | x_{(i,1)}, z_{(i,2)}^{(l-1)}, \theta^{(l)}) \propto q_1^{(l)}(s, z_{(i,2)}^{(l-1)})(1 - p_1^{(l)}(s)),$$

$$\mathbb{P}(z_{(i,t)}^{(l)} = s | x_{(i,t)}, z_{(i,t-1)}^{(l)}, z_{(i,t+1)}^{(l-1)}, \theta^{(l)}) \propto q_{t-1}^{(l)}(z_{(i,t-1)}^{(l)}, s) \times q_t(s, z_{(i,t+1)}^{(l-1)})(1 - p_t^{(l)}(s)),$$

$$\mathbb{P}(z_{(i,\tau)}^{(l)} = s | x_{(i,\tau)}, z_{(i,\tau-1)}^{(l)}, \theta^{(l)}) \propto q_{\tau-1}^{(l)}(z_{(i,\tau-1)}^{(l)}, s)(1 - p_\tau(s)^{(l)}).$$

Gibbs sampler illustrated

Take $\mathfrak{K} = \{1, 2\}$, $n = 4$, $m = 8$ and ,for $\mathbf{x}$,

1	1	1	·	·	1	·	·	·
2	1	·	1	·	1	·	2	1
3	2	1	·	1	2	·	·	1
4	1	·	·	1	2	1	1	2

Take all hyperparameters equal to 1

Gibbs sampler illust'd (cont'd)

One instance of simulated $\mathbf{z}$ is

1	1	1	2	1	1	2	†
1	1	1	2	1	1	1	2
2	1	2	1	2	1	1	1
1	2	1	1	2	1	1	2

which leads to the simulation of the parameters:

$$\begin{aligned} p_4^{(l)}(1)|\mathbf{x},\mathbf{z}^{(l-1)} &\sim \mathscr{B}e(1+2,1+0) \\ \phi_7^{(l)}(2)|\mathbf{x},\mathbf{z}^{(l-1)} &\sim \mathscr{B}e(1+0,1+1) \\ \psi_2^{(l)}(1,2)|\mathbf{x},\mathbf{z}^{(l-1)} &\sim \mathscr{B}e(1+1,1+2) \end{aligned}$$

in the Gibbs sampler.

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Fast convergence