

Modelling agents' preferences in complete markets by second order stochastic dominance

R.-A. Dana*, I. Meilijson†

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Abstract

A theory of individual decision and a general equilibrium theory in complete markets are provided, for the case of infinite state space when incomplete preferences are modelled by second order stochastic dominance (SSD). While, unlike the situation in the finite state space case, the demand of a strictly SSD averse agent may not be implementable as a vNM demand nor an SSD equilibrium as a vNM equilibrium, the set of Pareto-optimal allocations for SSD coincides, as in the finite state space case, with the set of Pareto-optimal allocations when agents are EU maximizers with increasing strictly concave utility indices. SSD is also used to give microfoundations to law-invariant risk measures.

Keywords: Second order stochastic dominance, comonotone risk sharing.
Law-invariant risk measures

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*Université Paris-Dauphine, CEREMADE, UMR 7534, dana@ceremade.dauphine.fr.

†Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Israel, isaco@post.tau.ac.il.

1 Introduction

Second order stochastic dominance (henceforth denoted by SSD), first studied by Hardy & Littlewood [15] and generalized by Strassen [24], was introduced in Economics by Rothschild & Stiglitz [21] as a measure of risk. It has been used in a wide variety of economic contexts, such as efficiency pricing (Peleg & Yaari [20], Chew & Zilcha [6]), finance (Ross & Dybvig [12], Dybvig [11], Kim [18], Jouini & Kallal[16]), insurance (Gollier & Schlesinger [14]), risk sharing (Landsberger & Meilijson[19]), measurement of inequality (Atkinson[1]), etc. This list is far from being exhaustive. Most of these papers are concerned with choices in markets of agents with incomplete preferences. However, there has never been a systematic theory of individual decision making nor a general equilibrium theory when incomplete preferences are modeled directly by SSD in contingent or complete markets, rather than by a particular utility model that preserves SSD, whether von-Neumann & Morgenstern (vNM) or a broader class such as *Shur concave* or *SSD risk averse* utilities.

Restricting attention to a discrete model with finitely many states of the world, Dana [8] showed that individual decision making in contingent markets when preferences are modeled by SSD is equivalent to decision making by vNM risk-averse agents:

- **Demand.** The demand correspondence under SSD (in other words, the set of undominated budget feasible contingent claims) is the collection of demands of vNM expected utility (EU) maximizers with strictly concave, increasing utility indices.
- **Pareto optima.** An allocation is Pareto optimal under SSD iff there exists some vNM EU maximizers with strictly concave, increasing utility indices for which it is Pareto optimal.
- **Equilibria.** The set of SSD equilibria is the collection of vNM equilibria.

In particular, she argues that any complete market solution concept obtained by using SSD risk averse utilities may be obtained by using increasing strictly concave vNM utility indices. The main tool of the analysis is a result due to Peleg & Yaari [20] to the effect that any non-increasing function of the pricing density can be rationalized as the demand of some vNM EU maximizer with increasing, strictly concave utility index. More precisely, let there be k states of the world. Let $P \in \mathbb{R}_+^k$ be a pricing density and let $w \in \mathbb{R}$ be a revenue.

If X is a non-increasing function of P fulfilling $E(PX) = w$, then there exists an increasing strictly concave function u such that X solves

$$\begin{cases} \max E[u(C)] \text{ s.t.} \\ E(PC) \leq E(PX) = w \end{cases}$$

The first goal of this paper is to provide a theory of individual decision making and a general equilibrium theory in contingent or complete markets, for the case of infinite state space when (incomplete) preferences are modeled by SSD and (complete) preferences are represented by SSD risk averse utilities.

We first show that a necessary pre-requisite to obtain an infinite version of Peleg & Yaari's result, is to assume that the demand function is a continuous non-increasing function of the pricing-density. In other words, vNM EU maximizers have *continuous* demand functions. We then prove that the demand correspondence under SSD (in other words, the set of budget feasible undominated claims for SSD) is the set of *all* non-increasing functions of the pricing-density. Hence, there is no bijection between the set of vNM demands and the set of budget-feasible SSD-undominated claims. In particular, the demand of a SSD risk averse agent may not be implementable as a vNM demand. Similar results hold for SSD strictly risk averse equilibria. In contrast, the set of Pareto-optimal allocations for SSD coincides with the set of Pareto-optimal allocations when agents are vNM EU maximizers with increasing strictly concave utility indices.

SSD is a partial or "incomplete" measure of risk. *Coherent measures*, recently introduced in the mathematical finance literature (see Delbaen [9] and Föllmer and Shied [13]) circumvent this drawback. The second goal of this paper is to formulate *law-invariant coherent measures* on contingent claims (which were introduced to smoothen the "irregularities" of *value at risk*) in terms of SSD. Building on an idea by Dybvig [11], we show that on a non-atomic space, a contingent claim X 's law-invariant coherent measure is *minus* the minimal expenditure at a given set of prices, to get X among the contingent claims that SSD dominate it.

The paper is organized as follows. Section 2 recalls some basic definitions. Section 3 presents infinite dimensional extensions of two main tools: Peleg & Yaari's lemma [20] and Dybvig's lemma [11]. Section 4 first characterizes the set of budget feasible undominated claims for SSD and discusses to what extent its elements are rationalized as demands of some vNM EU maximizer with increasing strictly concave utility index. The demand of a strongly risk

averse agent is also analyzed. Section 4 proceeds to deal with the link between minimal expenditure among claims that SSD-dominate a given claim and law-invariant coherent measures. Section 5 characterizes feasible allocations that are SSD-undominated by others and shows that any such allocation is implementable as a Pareto-optimal allocation of vNM EU maximizers with increasing strictly concave utility indices. The last section is devoted to the introduction and characterization of the concept of SSD equilibrium.

2 A few basic definitions

Given as primitive is a probability space (Ω, \mathcal{B}, P) . *Contingent claims* are identified to elements of L^∞ . A *price* is defined by a *pricing density* $P \in L^1_+$. Thus, a contingent claim is a bounded random variable and a price is a positive integrable random variable.

Let F_X denote the (right-continuous) distribution function of a contingent claim X and F_X^{-1} a version of its generalized inverse

$$F_X^{-1}(t) = \sup\{z \in \mathbb{R} \mid F_X(z) < t\}, \forall t \in (0, 1)$$

Definition 1 *Let X and Y be contingent claims. X (strictly) dominates Y in the sense of second order stochastic dominance (SSD), denoted $X \succsim_2 Y$ (respectively, $X \succ_2 Y$) if any of the following equivalent conditions is fulfilled:*

1. $\int_{-\infty}^t F_Y(s) ds \geq \int_{-\infty}^t F_X(s) ds, \forall t \in \mathbb{R}$
(resp. with strict inequality for some t)
2. $\int_0^t F_X^{-1}(s) ds \geq \int_0^t F_Y^{-1}(s) ds, \forall t \in]0, 1]$,
(resp. with strict inequality for some t)
3. $E[u(X)] \geq E[u(Y)], \forall u : \mathbb{R} \rightarrow \mathbb{R}$ concave increasing
(resp. with strict inequality for some such u)
4. $Y \sim^d X + \varepsilon$ for some ε such that $E[\varepsilon \mid X] \leq 0$ a.s.
(resp. where in addition $P(\varepsilon \neq 0) > 0$)

X and Y are second-order equivalent, denoted by $X \sim_2 Y$, if 1, 2 or 3 holds with equality throughout. In other words, $X \sim_2 Y$ iff $X \sim^d Y$.

It follows from Definition 1, condition 4 and Jensen's inequality that $X \succ_2 Y$ if and only if $E[u(X)] > E[u(Y)]$ for every increasing and strictly concave $u : \mathbb{R} \rightarrow \mathbb{R}$.

Definition 2 A utility $v : L^\infty \rightarrow \mathbb{R}$ is “ \succsim_2 risk averse” if $X \succsim_2 Y$ implies $v(X) \geq v(Y)$ and \succ_2 risk averse if $X \succ_2 Y$ implies $v(X) > v(Y)$.

We also say that v preserves SSD. Such utilities that have been considered in various settings (decision theory, finance, insurance, etc) need not be in general concave nor quasi-concave (see Chew and Mao [5] for examples).

Definition 3 A family of measurable maps $(X_i)_{i=1}^n$ is comonotone if there exists a subset $B \subset \Omega$ of P -measure one such that

$$[X_i(s) - X_i(s')] [X_j(s) - X_j(s')] \geq 0, \forall s, s' \in B, \forall 1 \leq i \leq j \leq n$$

Similarly, two measurable maps (X, Y) are *antithetic* to each other if the pair $(X, -Y)$ is comonotone.

An alternative characterization of comonotonicity (see Denneberg [10]) expresses each X_i as a continuous non-decreasing function of their sum:

Lemma 1 Denneberg’s Lemma A family of measurable maps $(X_i)_{i=1}^n$ is comonotone if and only if there exist continuous non-decreasing functions $(f_i)_{i=1}^n$ ($f_i : \mathbb{R} \rightarrow \mathbb{R}$) with $\sum_{i=1}^n f_i = Id$, such that $\forall i, X_i = f_i(\sum_{j=1}^n X_j)$ a.s.

It is clear from comonotonicity that each such function is 1-Lipschitz, i.e., $0 \leq f_i(x + |\delta|) - f_i(x) \leq |\delta|$.

Hardy & Littlewood’s inequality:

$$\int_0^1 F_X^{-1}(1-t) F_Y^{-1}(t) dt \leq E(XY) \leq \int_0^1 F_X^{-1}(t) F_Y^{-1}(t) dt$$

The integrals may possibly take infinite values. Thus the covariance between two L^2 random variables X and Y is less than or equal to the covariance between two comonotone random variables X' and Y' distributed like X and Y .

3 Two useful tools

In this section, we formulate and prove infinite dimensional extensions of two lemmas that are central tools in the use of SSD in finite dimensional markets. The first is Peleg & Yaari’s [20] lemma mentioned in the introduction. The second, introduced in Economics by Dybvig [11] in the finite state, uniform probability case, was generalized to incomplete markets by Jouini & Kallal [16].

3.1 Peleg & Yaari's Lemma

3.1.1 The finite state space Lemma

Let $\Omega = \{1, 2, \dots, k\}$, let $X = (X(1), X(2), \dots, X(k))$ denote a contingent claim and $P = (P(1), P(2), \dots, P(k))$ a pricing density. The following lemma by Peleg & Yaari [20] proved to be a fundamental tool for the characterization of demand, expenditure, Pareto optima and equilibria under SSD.

Lemma 2 *Let $(X, P) \in \mathbb{R}^k \times \mathbb{R}_+^k$ be such that $X(\ell) < X(j)$ implies $P(\ell) \geq P(j)$ ($P(\ell) > P(j)$) and let $E(PX) = w$. Then there exists a strictly increasing and concave (resp. strictly concave) function u such that X solves*

$$\begin{cases} \max E[u(C)] \text{ s.t.} \\ E(PC) \leq E(PX) = w \end{cases}$$

Remark The regular (symmetric) assumption in the Lemma is that X and P are antithetic. The stronger (asymmetric) assumption in parenthesis means that X is a non-increasing function of P .

3.1.2 An infinite dimensional extension

We propose the following infinite dimensional extension:

Theorem 1 *Let $X \in L^\infty$ and $P \in L_+^1$ be antithetic random variables. Then*

- *There exists a non-decreasing, right continuous, concave function U from $(-\infty, \infty)$ into $[-\infty, \infty)$ such that X solves*

$$\begin{cases} \max EU(C) \text{ s.t.} \\ E(PC) \leq E(PX) \\ C \in L^\infty \end{cases}$$

- *If $\text{esssup } P < \infty$, then U can be taken to be finite, strictly increasing, continuous and concave.*
- *If $\text{essinf } P > 0$, then U can be defined to be strictly concave on some right ray.*
- *If $\text{esssup } P < \infty$, $\text{essinf } P > 0$ and there exists a continuous function Ψ such that $X = \Psi(P)$ a.e., then there is a U that is finite, increasing, continuous and strictly concave on \mathbb{R} .*

The proof of Theorem 1 appears in the Appendix.

3.2 An extension of the Dybvig-Jouini-Kallal utility price

Let $P \in L^1_+$ and $X \in L^\infty$ be given. Consider the following problem (\mathcal{E})

$$\begin{cases} \min E(PC) \\ C \succsim_2 X \\ C \in L^\infty \end{cases}$$

That is, find a contingent claim C that minimizes expenditure among those that dominate X by SSD. The value function of (\mathcal{E}), $e(P, X)$, is called *utility price* by Jouini & Kallal [16].

Theorem 2

(i) *The utility price is given by $e(P, X) = \int_0^1 F_P^{-1}(1-t)F_X^{-1}(t)dt$ and there exists a non-decreasing function f such that $f(P)$ is a solution to the problem.*

If X is not a function of P , then $f(P) \succ_2 X$.

(ii) *If $g(P) \succ_2 X$ and $e(P, X) = E(Pg(P))$, then $g(P) = f(P)$ a.e.*

The proof of Theorem 2 may be found in the Appendix.

Corollary 1 *If X is a non-increasing function of P , then X is the unique solution to the problem*

$$\begin{cases} \min E(PC) \\ C \succsim_2 X \\ C \in L^\infty \end{cases}$$

Proof: Consider any $X' \succ_2 X$, $X' \neq X$, such that $E(PX') = E(PX) = \min_{C \succsim_2 X} E(PC)$. If X' is not a function of P , then $E(X' | P) \succ_2 X$ and $E(PE(X' | P)) = E(PX)$. Hence, by Theorem 2 (ii), $E(X' | P) = X$, a contradiction. If, on the other hand, X' is a function of P , then Theorem 2 (ii) is contradicted.

4 Demands

This section introduces a concept of demand correspondence for SSD, the set of SSD-undominated contingent claims. We then show that this set contains the demands of vNM EU maximizers with increasing strictly concave utility indices as well as the possibly more general set of demands of agents with utility functions preserving SSD. We show by means of an example that these two sets do not coincide. In particular, the demands of an agent with a utility function preserving SSD may have jumps, which can't be the case for a vNM EU maximizer with an increasing and strictly concave utility index.

4.1 Demand correspondence for SSD

Let $(P, W) \in L^1_+ \times L^\infty$ be a pricing density - endowment pair with $P > 0$ a.e. The demand correspondence for SSD is defined by

$$\xi_2(P, W) = \{X \in L^\infty \mid E(PX) \leq E(PW), \nexists X' \succ_2 X \text{ with } E(PX') \leq E(PW)\}$$

The following proposition characterizes $\xi_2(P, W)$.

Proposition 1

$$\xi_2(P, W) = \{h(P) \in L^\infty \mid h \text{ non-increasing, } E(Ph(P)) = E(PW)\}$$

Proof: Clearly, if $E(PX) \leq E(PW)$ and there doesn't exist $X' \succ_2 X$ with $E(PX') \leq E(PW)$, then $E(PX) = E(PW)$ (if $E(PX) < E(PW)$, choose $\varepsilon > 0$ sufficiently small so that $E(P(X + \varepsilon)) < E(PW)$. Then $X' = X + \varepsilon \succ_2 X$ is a contradiction).

If $X \in \xi_2(P, W)$, then X minimizes $\{E(PC), C \succ_2 X, C \in L^\infty\}$ (for if not, there would exist $C \succ_2 X$ such that $E(PC) < E(PW)$. Then $\frac{C+X}{2} \succ_2 X$ and $E(P(\frac{C+X}{2})) < E(PW)$, contradicting the definition of $\xi_2(P, W)$).

From Theorem 2, there exists a non-decreasing function f such that $X = f(P)$ a.s.

Conversely, if $X = f(P)$ a.s. for some non-increasing function f and there exists $X' \succ_2 X$ with $E(PX') \leq E(PW) = E(PX)$, then $E(X' \mid P)$ is P -measurable with $E(PE(X' \mid P)) = E(PX') \leq E(PW)$ and $E(X' \mid P) \succ_2 X$. Hence, $E(X' \mid P) = X$ by Theorem 2, a contradiction. ■

Having characterized the demand correspondence for SSD, we wish to know whether any contingent claim in the demand correspondence is implementable as the demand of some EU maximizer. By Theorem 1, this is the case if price is bounded below by some positive value and demand is continuous in price. As the next Proposition shows, the general answer is negative: continuity of demand with respect to price is necessary for vNM implementability.

4.2 An expected utility maximizer's demand

Proposition 2 *Let $(P, W) \in L^1_+ \times L^\infty$ be a pricing density - endowment pair. If a claim X is the demand of an EU maximizer with increasing and strictly concave utility index, then X is a continuous function of P . Conversely, if $P \in L^\infty$, $\text{essinf } P > 0$ and X is a continuous function of P , then there*

exists an increasing strictly concave utility index, such that X is the demand at (P, W) of an EU maximizer with that utility index.

Proof. A claim X is the demand of an EU maximizer with increasing and strictly concave utility index iff there exists a multiplier $\alpha > 0$ such that $-P \in \alpha \partial -U(X)$ or equivalently, iff $X \in \partial(-\alpha U)^*(-P)$, where $(-\alpha U)^*$ is the Fenchel transform of the convex function $-\alpha U$. Since U is strictly concave, $(-\alpha U)^*$ is C^1 , thus $X = [(-\alpha U)^*]'(-P)$ which proves that X is a continuous function of P . The converse statement follows from Theorem 1 .

4.3 Demand of a strongly risk averse agent

The next step is to show that the demand of an agent with utility compatible with second order stochastic dominance is an element of the demand correspondence for second order stochastic dominance. We will then discuss whether this demand is implementable as the demand of some EU maximizer with increasing and strictly concave utility index.

Proposition 3 *Let $v : L^\infty \rightarrow \mathbb{R}$ be \succ_2 risk averse and let X solve*

$$\begin{cases} \max v(C) \\ E(PC) \leq E(PW) \\ C \in L^\infty \end{cases}$$

Then $X \in \xi_2(P, W)$. In particular, X is P -measurable.

Proof: Assume that $X \notin \xi_2(P, W)$. Then there exists $X' \succ_2 X$ with $E(PX') \leq E(PW)$. As v is \succ_2 risk averse, $v(X') > v(X)$, a contradiction. ■

4.4 A comparison of the various concepts

The question we now pose is the following: Are there pairs (P, X) with P -measurable $X \in \xi_2(P, W)$ for some W , such that X cannot be rationalized as a demand under the budget constraint $E(PX') \leq E(PW)$ for some vNM utility? The answer is positive. We provide a pair (P^*, X^*) such that X^* is the (discontinuous!) demand under a strictly risk averse utility, satisfying the budget constraint $E(P^*X) \leq E(P^*X^*)$. By Proposition 2 , as X^* is not a continuous function of P^* , it cannot be a vNM demand.

For the sake of completeness, let us first recall a few definitions.

A *capacity* on $(\Omega, 2^\Omega)$ is a monotone set function $\nu : 2^\Omega \rightarrow \mathbb{R}$ (for all $A, B \in 2^\Omega$, $A \subset B$, implies $\nu(A) \leq \nu(B)$) normalized so that $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$. A capacity ν is convex if for all $A, B \in 2^\Omega$, $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$. Let $X : \Omega \rightarrow \mathbb{R}$ be a bounded function. The Choquet integral of X with respect to the capacity ν is defined by:

$$E_\nu(X) = \int_{-\infty}^0 (\nu(X \geq t) - 1)dt + \int_0^\infty \nu(X \geq t)dt \quad (1)$$

Let the non-decreasing $g : [0, 1] \rightarrow [0, 1]$ satisfy $g(0) = 0$, $g(1) = 1$ and let P be a probability on $(\Omega, 2^\Omega)$. Then $\nu = f(P)$ is a capacity that satisfies $\nu(X) = \nu(Y)$ whenever $X = Y$ P a.e.. The Choquet integral with respect to $g(P)$ defines a utility $E_g(X)$, the Yaari utility:

$$E_g(X) = \int_{-\infty}^0 g(P(X \geq t) - 1)dt + \int_0^\infty g(P(X \geq t))dt \quad (2)$$

Moreover, $g(P)$ is a convex capacity iff g is convex. $E_g(X)$ is then concave.

Example 1 Demand for a concave Yaari utility

Let the probability space be $([0, 1], \mathcal{B}, \lambda)$ where λ is Lebesgue measure and \mathcal{B} the Borel sets of $[0, 1]$. Let $X^* = 1$ if $0 \leq t \leq \frac{1}{2}$, $X^* = 2$ if $\frac{1}{2} < t \leq 1$, and let $P^*(t) = -t + \frac{3}{2}$. Claim: there exists a strictly risk averse Yaari utility $E_g(X)$ with convex distortion function g such that

$$E_g(X^*) = \max\{E_g(X) \mid E(P^*X) \leq E(P^*X^*)\} \quad (3)$$

Assume that g is convex C^1 . As a Yaari utility is concave, a necessary and sufficient condition for X^* to be the associated demand is the existence of $\alpha > 0$ such that $P^* \in \alpha \partial E_g(X^*)$. It follows from Carlier & Dana corollary 2, [3], that $P^* \in \partial E_g(X^*)$ iff $P \succeq_2 g'$ and P is antithetic to X^* . Since X^* and P^* are antithetic, we only have to find g such that $P^* \succeq_2 g'$, i.e., such that for all $t \in [0, 1]$,

$$\int_0^t F_{P^*}^{-1}(u)du \geq g(t)$$

Notice that $\int_0^1 F_{P^*}^{-1}(u)du = \int_0^1 (u + \frac{1}{2})du = 1$ and observe that the integral condition can be re-phrased as

$$\int_0^t (u + \frac{1}{2})du \geq g(t), \quad \forall t \in [0, 1[$$

It suffices to take $g(t) = \frac{1}{2}t^2 + \frac{1}{2}t$.

Thus, X^* is the demand under a strictly risk averse utility $v(X) = E_g(X)$ satisfying the budget constraint $E(P^*X) \leq E(P^*X^*)$, that is not a vNM demand. ■

Similarly, let P^* be any price with continuous distribution function satisfying $E(P^*) = 1$. Then any non-increasing function of P^* may be rationalized as the demand at price P^* of a Yaari expected utility maximizer with distortion g . Indeed let $([0, 1], \mathcal{B}, \lambda)$ be the probability space, let the price be $F_{P^*}^{-1}$ and let $g(t) = \int_0^t F_{P^*}^{-1}(u)du$, $\forall t \in [0, 1[$.

Let us now show that a slight modification of the above example yields a demand implementable as a vNM demand.

Example 2 Demand in the RDEU model

On the same standard probability space as in the previous example, let the utility function be $v(C) = E_g(U(C))$ with U increasing and strictly concave C^1 , and $g : [0, 1] \rightarrow [0, 1]$ increasing convex C^1 with $g(0) = 0$, $g(1) = 1$. As v is strictly concave, if the demand problem has a solution X , then there exists $\alpha \in \mathbb{R}_+$ such that $P \in \lambda \partial v(X)$ a.e. It follows from Carlier & Dana, corollary 2, [3] that

$$P \in \alpha [g'(1 - F_X(X))U'(X), g'(1 - F_X(X_-))U'(X)] \quad \text{a.e.}$$

Let $V : \mathbb{R} \rightarrow \mathbb{R}$ be the continuous, increasing and strictly concave utility function satisfying $\partial V(x) = [g'(1 - F_X(x-))U'(x), g'(1 - F_X(x))U'(x)]$ and $V(x_0) = 0$. As $\partial V(X) = [g'(1 - F_X(X-))U'(X), g'(1 - F_X(X))U'(X)]$ a.e., X is also the solution of the vNM demand problem with utility index V ,

$$\begin{cases} \max E(V(C)) \\ E(PC) \leq E(PW) \\ C \in L^\infty \end{cases}$$

We have thus proven that any RDEU demand on the standard $[0, 1]$ probability space can be rationalized as the demand of a vNM agent with increasing and strictly concave utility index ■

5 Expenditure as a measure of risk

In decision theory, a random variable X is more risky than another Y if Y dominates X for some stochastic order modelling risk, for example SSD. The drawback of this approach is its "incomplete" nature. Recently, a class of complete measures of risk called *coherent measures* have been introduced in the mathematical finance literature (see Delbaen [9] and Föllmer and Shied [13]). A subclass of these measures, the *law-invariant coherent measures* smooth the "irregularities" of value at risk (VaR), used by both theorists and practitioners. Building on an idea of Dybvig [11], we show that given a set of prices on a non atomic space, the minimal expenditure to get a contingent claim among those that SSD-dominate it, is a law-invariant coherent measure.

5.1 Minimal expenditure under a single price

Let the non-decreasing $g : [0, 1] \rightarrow [0, 1]$ satisfy $g(0) = 0$, $g(1) = 1$ and let P be a probability on $(\Omega, 2^\Omega)$. We recall that if g is continuous at 1, then

$$E_g(X) = \int_0^1 g'(1-t)F_X^{-1}(t)dt$$

Recall from Section 3 that the minimal expenditure at a given price P with $E(P) = 1$ to get a contingent claim among the contingent claims that SSD-dominate it, is

$$\int_0^1 F_P^{-1}(1-t)F_X^{-1}(t)dt = \min \{E(PC), C \in L^\infty, C \succeq_2 X\} = E_g(X)$$

with $g(x) = \int_0^x F_P^{-1}(t)dt$

5.2 Law-invariant coherent measures

Definition 4 A map $\rho : L^\infty \rightarrow \mathbb{R}$ is *supermodular* if

- $\rho(X) \geq 0$ whenever $X \geq 0$
- $\rho(X_1 + X_2) \geq \rho(X_1) + \rho(X_2)$
- $\rho(\lambda X) = \lambda\rho(X)$ for non negative λ .

The supermodular map ρ is *translation invariant* if moreover

- $\rho(X + c) = \rho(X) + c$ for constant c .

If ρ is a translation invariant supermodular map, then $-\rho$ is called a coherent measure of risk.

The proof of the following lemma may be found in Föllmer and Shied [13].

Lemma 3 *For a supermodular translation invariant function, there is equivalence between:*

1. *there exists a closed convex set \mathcal{Q} of probability measures absolutely continuous with respect to P such that*

$$\rho(X) = \inf \left\{ E^{\mathcal{Q}}(X), Q \in \mathcal{Q} \right\}$$

2. *ρ is $\sigma(L^\infty, L^1)$ upper semi-continuous,*
3. *$\{X \in L^\infty \mid \rho(X) \geq a\}$ is $\sigma(L^\infty, L^1)$ closed for every a .*
4. *$\{X \in L^\infty \mid \rho(X) \geq 0\}$ is $\sigma(L^\infty, L^1)$ closed.*
5. *If (X_n) is a uniformly bounded sequence and $X_n \rightarrow X$ in probability, then $\rho(X) \geq \limsup \rho(X_n)$.*

Let us further describe \mathcal{Q} or the set of its Radon-Nikodym derivatives w.r. to Q . Let

$$\partial\rho(0) =: \left\{ h \in L^1 \mid E(hX) \geq \rho(X) \text{ for all } X \in L^\infty \right\}$$

be the superdifferential of ρ at 0. Since ρ is monotone, $\partial\rho(0) \subseteq L^1_+$. Furthermore, since ρ is translation invariant, $\rho(1) = 1$ and $\rho(-1) = -1$ and therefore for any $h \in \partial\rho(0)$, $-\rho(-1) = 1 \geq E(h) \geq \rho(1)$, hence $E(h) = 1$.

Lemma 4 *If ρ is a translation invariant, $\sigma(L^\infty, L^1)$ upper semi-continuous, supermodular function, then \mathcal{Q} is uniquely defined. Let $\mathcal{H} = \left\{ \frac{dQ}{dP}, Q \in \mathcal{Q} \right\}$ denote the set of its Radon-Nikodym densities. Then*

$$\mathcal{H} = \partial\rho(0) =: \left\{ h \in L^1 \mid E(hX) \geq \rho(X) \text{ for all } X \in L^\infty \right\}$$

Proof: Let us prove that \mathcal{Q} is uniquely defined. Assume that there exists another closed convex set of densities $\mathcal{G} \neq \mathcal{H}$ such that $\rho(X) = \inf\{E(hX), h \in \mathcal{G}\}$. Let $h \in \mathcal{H} \cap \mathcal{G}^c$. By the Hahn-Banach theorem, there exists $X \in L^\infty$ such that

$$\rho(X) \leq E(hX) < \inf_{\tilde{h} \in \mathcal{G}} E(\tilde{h}X) = \rho(X)$$

a contradiction .

To prove the second assertion, clearly $\partial\rho(0)$ is closed and convex and contains any \mathcal{H} such that $\rho = \inf\{E(hX), h \in \mathcal{H}\}$, hence $\mathcal{H} = \partial\rho(0)$.■

Definition 5 *A supermodular map ρ is law-invariant if $\rho(X) = \rho(Y)$ whenever $X \sim^d Y$.*

A Choquet integral with respect to a convex distortion is an example of a $\sigma(L^\infty, L^1)$ upper semi-continuous law-invariant supermodular map. The next result, shows that on a non-atomic space, any law-invariant supermodular map is the infimum of a family of Choquet integrals with respect to a distortion.

We shall make extensive use of a result due to Ryff [22] under the assumption that $(\Omega, \mathcal{A}, \mathcal{Q})$ is non-atomic. For a proof of these results, we refer to Chong and Rice [7].

Lemma 5 *Let Y be a random variable on $(\Omega, \mathcal{A}, \mathcal{P})$ non atomic. Then there exists a random variable U on $(\Omega, \mathcal{A}, \mathcal{P})$ uniformly distributed such that $Y = F_Y^{-1}(U)$ a.e..*

Corollary 2 *Let X and Y be two random variables on $(\Omega, \mathcal{A}, \mathcal{P})$ non atomic. Then there exists $\tilde{X} \sim X$ comonotone (resp anticomonotone) with Y .*

In order to characterize supermodular maps, we introduce the following definition:

Definition 6 *A subset $\mathcal{H} \subseteq L_+^1$ is law-invariant (resp \succsim_2 invariant) if $h \in \mathcal{H}$ and $g \in L_+^1, g \sim h$ (resp if $h \in \mathcal{H}$ and $g \in L_+^1, g \succsim_2 h$) imply $g \in \mathcal{H}$.*

Theorem 3 *Assume (Ω, \mathcal{A}, P) non atomic. The following statements are equivalent:*

1. ρ is a law-invariant $\sigma(L^\infty, L^1)$ upper semi-continuous supermodular map
2. $\rho(X) = \inf_{\mathcal{H}} \int_0^1 F_X^{-1}(1-t)F_h^{-1}(t)dt = \inf\{E(PC), P \in \mathcal{H}, C \in L^\infty, C \succsim_2 X\}$
3. $\partial\rho(0)$ is law-invariant
4. $\partial\rho(0)$ is \succeq_2 invariant

Moreover,

$$\rho(X) = \inf \{\rho(C), C \in L^\infty, C \succsim_2 X\}$$

Proof: To prove that **1 is equivalent to 2**, from Hardy & Littlewood's inequality, we first have that

$$E(hX) \geq \int_0^1 F_X^{-1}(1-t)F_h^{-1}(t)dt, \forall h \in \partial\rho(0)$$

Thus

$$\rho(X) \geq \inf_{\partial\rho(0)} \int_0^1 F_X^{-1}(1-t)F_h^{-1}(t)dt$$

Furthermore, let $h \in \partial\rho(0)$. Since (Ω, \mathcal{A}, P) is non atomic, there exists U uniformly distributed such that $h = F_h^{-1} \circ U$. Let $Y = F_X^{-1} \circ (1-U)$. Then $Y \sim^d X$. Thus

$$\rho(X) = \rho(Y) \leq E(hY) = \int_0^1 F_X^{-1}(1-t)F_h^{-1}(t)dt$$

and

$$\rho(X) \leq \inf_{\partial\rho(0)} \int_0^1 F_X^{-1}(1-t)F_h^{-1}(t)dt$$

which proves that $\rho(X) = \inf_{\partial\rho(0)} \int_0^1 F_X^{-1}(1-t)F_h^{-1}(t)dt$. Conversely if $\rho(X) = \inf_{\partial\rho(0)} \int_0^1 F_X^{-1}(1-t)F_h^{-1}(t)dt$, clearly ρ is law-invariant.

Let us next show that **2 is equivalent to 3**. Let $h \in \partial\rho(0)$ and $\tilde{h} \sim^d h$. From Hardy & Littlewood's inequality,,

$$E(\tilde{h}X) \geq \int_0^1 F_X^{-1}(1-t)F_{\tilde{h}}^{-1}(t)dt \geq \rho(X)$$

thus $\tilde{h} \in \partial\rho(0)$. Conversely, since (Ω, \mathcal{A}, P) is non atomic, if $\partial\rho(0)$ is law-invariant,

$$\inf_{\partial\rho(0)} E(hX) \leq \int_0^1 F_X^{-1}(1-t)F_h^{-1}(t)dt$$

The reverse inequality being true from Hardy & Littlewood's inequality, we get

$$\inf_{\partial\rho(0)} E(hX) = \int_0^1 F_X^{-1}(1-t)F_h^{-1}(t)dt$$

4 implies 3 being obvious, let us show that **2 implies 4**. Let $h \in \partial\rho(0)$ and $\tilde{h} \succeq_2 h$. Since

$$\int_0^x F_{\tilde{h}}^{-1}(t)dt \geq \int_0^x F_h^{-1}(t)dt, \forall x \in [0, 1],$$

$$\int_0^1 g(t)F_h^{-1}(t)dt \geq \int_0^1 g(t)F_{\tilde{h}}^{-1}(t)dt$$

for any g non-increasing, non negative step function. By monotone convergence, this also holds true for any non-increasing $g \in L_+^\infty$. Thus, for any $X \in L_+^\infty$,

$$E(\tilde{h}X) \geq \int_0^1 F_X^{-1}(1-t)F_{\tilde{h}}^{-1}(t)dt \geq \int_0^1 F_X^{-1}(1-t)F_h^{-1}(t)dt \geq \rho(X) \quad (4)$$

for any $X \in L_+^\infty$. Let $X \in L^\infty$. As $X + \|X\|_\infty \geq 0$, using translation invariance, we get (4) for any $X \in L^\infty$, which implies that $\tilde{h} \in \partial\rho(0)$. ■

Part of this result has been proved by Kusuoka [17]. The proof that we provide is much shorter.

6 Pareto optimality

Throughout this section, we consider an n -agent economy with initial endowments $\mathcal{W} = \{W_i\}_{i=1}^n \in (L^\infty)^n$ of an aggregate endowment $W = \sum_{i=1}^n W_i$. Any $\mathcal{X} = \{X_i\}_{i=1}^n \in (L^\infty)^n$ adding up a.s. to W is a *feasible allocation*, i.e., $\mathcal{X} \in \mathcal{FA}$.

Definition 7 A feasible allocation $\tilde{\mathcal{X}}$ is a \succsim_2 Pareto optimum if there doesn't exist $\mathcal{X} \in \mathcal{FA}$ such that $X_i \succsim_2 \tilde{X}_i$ for every i , with strict inequality for some i .

Definition 8 A pair $(P^*, \mathcal{X}^*) \in L^1 \times \mathcal{FA}$ is a \succsim_2 equilibrium if $E(P^*X_i^*) \leq E(P^*W_i)$ for every i , and for no i there exists in L^∞ an $X_i \succ_2 X_i^*$ with $E(P^*X_i) \leq E(P^*W_i)$.

Definition 9 A pair $(P^*, \mathcal{X}^*) \in L^1 \times \mathcal{FA}$ is a \succsim_2 equilibrium with transfer payments if for no i there exists in L^∞ an $X_i \succ_2 X_i^*$ with $E(P^*X_i) \leq E(P^*X_i^*)$.

The following result is easy to ascertain:

Proposition 4 (i) If there exist $\{U_i\}_{i=1}^n$ increasing and strictly concave such that \mathcal{X} is a Pareto optimum of the associated vNM economy, then \mathcal{X} is a \succsim_2 Pareto optimum.

(ii) If there exist $\{U_i\}_{i=1}^n$ increasing and strictly concave such that $(P, \mathcal{X}) \in L^1 \times \mathcal{FA}$ is an equilibrium (resp. an equilibrium with transfer payments) of the associated economy, then (P, \mathcal{X}) is a \succsim_2 equilibrium (resp. a \succsim_2 equilibrium with transfer payments).

We shall prove that any \succsim_2 Pareto optimum is a Pareto optimum of a vNM economy with strictly concave utility functions. The proof uses a second welfare theorem for second order stochastic dominance and the generalization of Peleg & Yaari's Lemma that we proposed previously.

6.1 Characterization of \succsim_2 Pareto optimality

Proposition 5 *Let $\mathcal{X} \in \mathcal{FA}$. The following conditions are equivalent*

1. \mathcal{X} is a \succsim_2 Pareto optimum
2. The family of random variables $(X_i)_{i=1}^n$ is comonotone
3. There exist $(U_i)_{i=1}^n$ increasing and strictly concave such that \mathcal{X} is a Pareto optimum of the associated vNM economy.

Proof:

1 implies 2. Let us first prove that every $\mathcal{X} \in \mathcal{FA}$ is weakly SSD dominated by a comonotone allocation. Landsberger & Meilijson [19] stated that every $\mathcal{X} \in \mathcal{FA}$ is weakly SSD dominated by a comonotone allocation, but restricted the formal proof to the case of finite probability space. We provide here a limiting argument that extends the proof to $W \in \mathcal{L}^\infty$ on any probability space.

Let $W \in \mathcal{L}^\infty$ and $\mathcal{X} \in \mathcal{FA}$. Let \mathcal{F}_m be the sigma algebra generated by the partition into events of the form $\{k2^{-m} \leq W < (k+1)2^{-m}\}$ and let $W^{(m)} = E[W|\mathcal{F}_m]$ and $X_i^{(m)} = E[X_i|\mathcal{F}_m]$, $i = 1, \dots, n$. By the Martingale convergence theorem, $W^{(m)} \rightarrow W$ and $X_i^{(m)} \rightarrow X_i$ a.s. By Jensen's inequality $X_i^{(m)} \succsim_2 X_i$ for every i and every m .

By Landsberger & Meilijson [19] and Lemma 1, there exist non-decreasing, 1-Lipschitz functions g_{im} such that $\sum_{m=1}^n g_{im} = \text{Id}$ and such that $X_i^{(m)*} = g_{im}(W_m) \succsim_2 X_i^{(m)}$ for every i and every m . By Ascoli's theorem, for every i , there is a subsequence of the g_{im} 's that converges uniformly on $[-\|W\|_\infty, \|W\|_\infty]$, to a function g_i , which is non-decreasing and 1-Lipschitz. Hence $\sum_{m=1}^n g_m = \text{Id}$

on $[-\|W\|_\infty, \|W\|_\infty]$. Now consider the allocation $(X_i^*) = (g_i(W))$. Since X_i^* is the uniformly bounded a.s. limit of a sequence of r.v.'s that \succ_2 dominate X_i , as such, it dominates X_i too. We have thus now proven that every allocation is weakly SSD dominated by a comonotone allocation.

Let us finally prove that every Pareto allocation is comonotone. Let \mathcal{X} be a PO allocation. As we have just shown, it is weakly SSD dominated by a comonotone allocation \mathcal{X}^* . If $X_l \neq X_l^*$ for some l , then $\frac{X_l + X_l^*}{2} \succ X_l$. As $(\frac{X_i + X_i^*}{2})_{i=1}^n \in \mathcal{FA}$, this contradicts the Pareto-optimality of \mathcal{X} . Hence $\mathcal{X} = \mathcal{X}^*$ and \mathcal{X} is comonotone.

2 implies 3. Let $J = [-\|W\|_\infty, \|W\|_\infty]$. If the family of random variables $(X_i)_{i=1}^n$ is comonotone, by Lemma 1, there exist $f_i : J \rightarrow \mathbb{R}$ continuous and non-decreasing such that $X_i = f_i(W)$ and $\sum_{i=1}^n f_i = \text{Id}$. Choose any strictly decreasing continuous function P of W with $\inf_{t \in J} P(t) > 0$ and $\sup_{t \in J} P(t) < \infty$, and compose $\psi_i = f_i \circ P^{-1}$. Then $X_i = \psi_i(P)$ a.e. for each i . By Theorem 1, there exists $U_i : \mathbb{R} \rightarrow \mathbb{R}$ increasing and strictly concave, such that X_i solves

$$\begin{cases} \max EU_i(C) \text{ s.t.} \\ E(PC) \leq E(PX_i) \\ C \in L^\infty \end{cases}$$

The pair (P, \mathcal{X}) is therefore an equilibrium with transfer payments of the associated vNM economy, hence a Pareto optimum of that economy.

3 implies 1 follows from Proposition 4. ■

Let us remark that while proving that **2 implies 3**, we have proven a second welfare theorem for second order stochastic dominance:

Corollary 3 *The feasible allocation \mathcal{X} is a \succ_2 Pareto optimum iff there exists $P \gg 0$ such that (P, \mathcal{X}) is a \succ_2 equilibrium with transfer payments.*

6.2 Risk averse Pareto optima

Let us finally state:

Proposition 6 *A feasible allocation \mathcal{X} is a Pareto optimum of an economy with \succ_2 risk averse utilities iff it fulfills any of the following equivalent conditions:*

1. *The family of random variables $(X_i)_{i=1}^n$ is comonotone*

2. There exist $(U_i)_{i=1}^n$ increasing and strictly concave such that \mathcal{X} is a Pareto optimum of the associated vNM economy.

Proof: From the previous proposition, the two assertions are clearly equivalent. Also, any Pareto optimum of an economy with \succ_2 risk averse utilities is obviously a \succeq_2 Pareto optimum. Hence, from Proposition 5, the allocation has to be comonotone. The converse is obvious. ■

In particular, it follows from Proposition 6 that in insurance models with symmetric information, a model with \succ_2 risk averse utilities will not give rise to a behavior that may not be explained by EU maximizers. In particular, Chew and Zilcha [6] show that in the situation where a \succ_2 risk averse agent buys a contract from a risk-neutral insurer, only contracts with a deductible are efficient. Gollier & Schlesinger [14] show that when a \succ_2 risk averse agent buys a contract from a \succ_2 risk averse insurer, only contracts under which the insured position is a 1-Lipschitz function of the risk, are efficient.

7 Second Order Stochastic Dominance Equilibria

In this section, we characterize \succeq_2 equilibria, risk-averse equilibria and strictly concave vNM equilibria.

Proposition 7 *A pair (P, \mathcal{X}) of price and feasible allocation is a \succeq_2 equilibrium iff*

1. $P \gg 0$ a.e. and for every i , $E(PX_i) = E(PW_i)$
2. For every i , X_i is a non-increasing function of P .

Equivalently (P, \mathcal{X}) fulfills

1. For every i , $E(PX_i) = E(PW_i)$
2. The family of random variables $(X_i)_{i=1}^n$ is comonotone and W is a non-increasing function of P .

Proof. The first assertion follows from Proposition 1. If for every i , X_i is a non-increasing function of P , it follows from Denneberg chapter four [10] that the family of random variables $(X_i)_{i=1}^n$ is comonotone and obviously W is a non-increasing function of P . Conversely if the family of random variables

$(X_i)_{i=1}^n$ is comonotone and W is a non-increasing function of P , as X_i is a non-decreasing continuous function of W , X_i is a non-increasing function of P , which proves the equivalence between the two assertions.■

The next result follows from Proposition 3:

Proposition 8 *If a pair (P, \mathcal{X}) is an equilibrium of an economy with monotone strictly strongly risk averse utilities, then it is a \succ_2 equilibrium.*

Finally,

Proposition 9 *If (P, \mathcal{X}) is a vNM equilibrium of an economy where agents have increasing strictly concave utility indices, then*

1. $P \gg 0$ a.e. and for every i , $E(PX_i) = E(PW_i)$.
2. For every i , X_i is a non-increasing continuous function of P .

Equivalently (P, \mathcal{X}) fulfills

1. For every i , $E(PX_i) = E(PW_i)$
2. The family of random variables $(X_i)_{i=1}^n$ is comonotone and W is a continuous non-increasing function of P .

Conversely, if $P \in L_+^\infty$ and $\text{essinf } P > 0$, if assertion 1 is fulfilled and for every i , X_i is a continuous non-increasing function of P , then there exist strictly concave, increasing utility indices U_i , $i = 1, \dots, n$ such that (P, \mathcal{X}) is an equilibrium of the associated vNM economy.

This Proposition follows from Proposition 2.■

8 Appendix

Proof of Theorem 1: Since X and P are antithetic, there exist by Lemma 1 two continuous functions f_1 and f_2 with f_1 (non negative and) non-decreasing and f_2 non-increasing, such that

$$P = f_1(P - X) \text{ a.e. } , X = f_2(P - X) \text{ a.e.}$$

Assume first that $\text{esssup } P < \infty$. Then by modifying f_1 and f_2 outside the range of $P - X$, we may assume without loss of generality that $f_1 \circ f_2^{-1}$ is a non-negative-valued correspondence defined on the whole real line, that is constant on some right ray.

The correspondence f_2^{-1} is closed-interval valued, strictly decreasing and a.e. single valued. Let $x_0 \in (\text{essinf}(X), \text{esssup}(X))$. The function

$$U(x) = \int_{x_0}^x f_1 \circ f_2^{-1}(t) dt, \quad (5)$$

well defined on \mathbb{R} , is non-decreasing since $f_1 \circ f_2^{-1} \geq 0$ and concave since $f_1 \circ f_2^{-1}$ is non-increasing. Furthermore, U is differentiable at points where $f_1 \circ f_2^{-1}$ is single valued, with $U'(x) = f_1 \circ f_2^{-1}(x)$. At points where $f_1 \circ f_2^{-1}$ is multi-valued, concavity of U implies that

$$\partial U(x) = \left[\lim_{t \uparrow x} U'(t), \lim_{t \downarrow x} U'(t) \right] = f_1 \circ f_2^{-1}(x), \quad (6)$$

since f_2 is continuous and non-decreasing.

Let us finally remark that X solves

$$\begin{cases} \max EU(C) \text{ s.t.} \\ E(PC) \leq E(PX) \\ C \in L^\infty \end{cases}$$

since the first order condition for optimality $P \in \partial U(X) = f_1 \circ f_2^{-1}(X)$ is satisfied at almost all X .

If $\text{esssup } P = \infty$, then f_2 necessarily has a finite limit at infinity and this limit is $\underline{x} = \text{essinf}(X)$. This means that $f_2^{-1}(x) = +\infty$ and $f_1 \circ f_2^{-1}(x) = +\infty$ for all $x < \underline{x}$. The function U is thus finite and concave on (\underline{x}, ∞) and equals $-\infty$ on $(-\infty, \underline{x})$.

Lastly, if $\text{esssup } P < \infty$, $\text{essinf } P > 0$ and there exists a continuous function Ψ such that $X = \Psi(P)$ a.e., we may assume without loss of generality that Ψ^{-1} is affine on $(-\infty, \underline{x})$ and is asymptotic to zero on $(\text{esssup}(X), \infty)$. Then (5) takes the form

$$U(x) = \int_{x_0}^x \Psi^{-1}(t) dt$$

with U well defined on \mathbb{R} , increasing since $\Psi^{-1} > 0$, and strictly concave since the correspondence Ψ^{-1} is strictly decreasing everywhere. As in (6),

$\partial U(x) = \Psi^{-1}(x)$ and X solves

$$\begin{cases} \max EU(C) \\ E(PC) \leq E(PX) \\ C \in L^\infty \end{cases}$$

since the first order condition for optimality $P \in \partial U(X) = \Psi^{-1}(X)$ a.e. is fulfilled. ■

To summarize, U may have the following pathologies:

- If $\text{esssup } P = \infty$, then U is equal to $-\infty$ on some left ray.
- If $\text{essinf } P = 0$, then U is constant on some right ray.

Proof of Theorem 2. Let us first prove that $e(P, X) \geq \int_0^1 F_P^{-1}(1-t)F_X^{-1}(t)dt$. It follows from Hardy & Littlewood's inequality that for any $C \in L^\infty$,

$$E(PC) \geq \int_0^1 [F_P^{-1}(1-t)F_C^{-1}(t)]dt$$

Assume further that $C \succ_2 X$. For any non-increasing, non-negative step function $g(t) = \sum_{i=1}^k a_i 1_{[0, b_i]}(t)$ with $0 < b_1 < \dots < b_k = 1$ and $a_i > 0$ for all i , we have by definition of second order stochastic dominance,

$$\begin{aligned} \int_0^1 g(t)F_C^{-1}(t)dt &= \sum_{i=1}^k a_i \int_0^{b_i} F_C^{-1}(t)dt \geq \sum_{i=1}^k a_i \int_0^{b_i} F_X^{-1}(t)dt \\ &= \int_0^1 g(t)F_X^{-1}(t)dt \end{aligned} \quad (7)$$

It follows from the Lebesgue dominated convergence theorem that the result holds true for any non-increasing, non-negative integrable function g . In particular,

$$\int_0^1 [F_P^{-1}(1-t)F_C^{-1}(t)]dt \geq \int_0^1 [F_P^{-1}(1-t)F_X^{-1}(t)]dt ,$$

whence

$$e(P, X) = \inf_{C \succ_2 X} E(PC) \geq \int_0^1 [F_P^{-1}(1-t)F_X^{-1}(t)]dt$$

To prove the reverse inequality, it suffices to prove that there exists a contingent claim $C \succsim_2 X$ such that $E(PC) = \int_0^1 F_P^{-1}(1-t)F_X^{-1}(t)dt$. If $C \succsim_2 X$, then $E(C | P) \succsim_2 X$. Hence, we may assume w.l.o.g. that $C = f(P)$ with f non-increasing. We need to find f such that

$$E(Pf(P)) = \int_0^1 [F_P^{-1}(t)f(F_P^{-1}(t))]dt = \int_0^1 [F_X^{-1}(1-t)F_P^{-1}(t)]dt$$

Let f be defined by $f(F_P^{-1}) = E(F_X^{-1}(1-Id) | F_P^{-1})$ (conditional expectation being taken with respect to the probability space $([0, 1], \mathcal{B}, \lambda)$ with λ Lebesgue measure). Since at continuity points x of F_P , F_P^{-1} is uniquely determined,

$$f(x) = F_X^{-1}(1 - F_P(x))$$

At discontinuity points x of F_P ,

$$f(x) = \frac{1}{F_P(x) - F_P(x_-)} \int_{F_P(x_-)}^{F_P(x)} F_X^{-1}(1-t)dt$$

It follows from Jensen's inequality that if X is not a function of P , then $f(P) \succ_2 X$.

To prove the second assertion of the theorem, observe that F_P^{-1} is non-increasing and express

$$F_P^{-1}(1-t) = F_P^{-1}(0) + \int_0^1 I_{[0,1-x]}(y)dF_P^{-1}(y)$$

to evaluate

$$\begin{aligned} & \int_0^1 F_P^{-1}(1-t)f(F_P^{-1}(1-t))dt = \\ & F_P^{-1}(0) \int_0^1 f(F_P^{-1}(1-t))dt + \int_0^1 f(F_P^{-1}(1-t)) \left[\int_0^{1-t} dF_P^{-1}(y) \right] dt = \\ & F_P^{-1}(0) \int_0^1 f(F_P^{-1}(1-t))dt + \int_0^1 \left[\int_0^{1-y} f(F_P^{-1}(1-t))dt \right] dF_P^{-1}(y) = \\ & F_P^{-1}(0) \int_0^1 F_X^{-1}(1-t)dt + \int_0^1 \left[\int_0^{1-y} F_X^{-1}(1-t)dt \right] dF_P^{-1}(y) \end{aligned}$$

The second equality follows from Fubini's Theorem, the last from the minimizing property of $f(P)$. Since $f(P) \succsim_2 X$,

$$\int_0^1 f(F_P^{-1}(1-t))dt \geq \int_0^1 F_X^{-1}(1-t)dt$$

and for every y ,

$$\int_0^{1-y} f(F_P^{-1}(1-t))dt \geq \int_0^{1-y} F_X^{-1}(1-t)dt$$

Hence,

$$\int_0^1 f(F_P^{-1}(1-t))dt = \int_0^1 F_X^{-1}(1-t)dt$$

and

$$\int_0^{1-y} f(F_P^{-1}(1-t))dt = \int_0^{1-y} F_X^{-1}(1-t)dt \quad F_P^{-1} \text{ a.e.}$$

Assume that $g(P) \succsim_2 X$ and $e(P, X) = E(g(P)P)$. Then

$$\int_0^{1-y} f(F_P^{-1}(1-t))dt = \int_0^{1-y} g(F_P^{-1}(1-t))dt \quad F_P^{-1} \text{ a.e.}$$

Equivalently,

$$\int_y^1 f(F_P^{-1}(t))dt = \int_y^1 g(F_P^{-1}(t))dt, \quad F_P^{-1} \text{ a.e.}$$

We now use the following lemma:

Lemma 6 *If $\int_y^1 f(F_P^{-1}(t))dt = \int_y^1 g(F_P^{-1}(t))dt \quad F_P^{-1} \text{ a.e.}$, then*

$$\int_{F_P(x)}^1 f(F_P^{-1}(t))dt = \int_{F_P(x)}^1 g(F_P^{-1}(t))dt = \int_x^\infty g(y)dF_P(y) \quad \forall x$$

This implies that $\int_x^\infty f(y)dF_P(y) = \int_x^\infty g(y)dF_P(y)$, so f and g are versions of the Radon-Nikodym derivative of the measure $\mu(A) = \int_A f(y)dF_P(y)$ with respect to the measure dF_P . Hence, $f = g \quad F_P$ a.e. Equivalently, $f(P) = g(P)$ a.e. ■

Proof of Lemma 6: Assume that $\int_y^1 f(F_P^{-1}(t))dt = \int_y^1 g(F_P^{-1}(t))dt, \quad F_P^{-1}$ a.e. and let x be such that

$$\int_{F_P(x)}^1 f(F_P^{-1}(t))dt \neq \int_{F_P(x)}^1 g(F_P^{-1}(t))dt$$

Since $\int_y^1 f(F_P^{-1}(t))dt$ is continuous, there exists an open interval I containing $F_P(x)$ such that

$$\int_y^1 f(F_P^{-1}(t))dt \neq \int_y^1 g(F_P^{-1}(t))dt \quad \forall y \in I$$

But an open interval with zero F_P^{-1} measure corresponds to a discontinuity of F_P and no point in its interior can be of the form $F_P(x)$.

Using the monotone convergence theorem, to prove that

$$\int_{F_P(x)}^1 f(F_P^{-1}(t))dt = \int_x^\infty f(y)dF_P(y)$$

it suffices to prove the result for step functions. By linearity, it is enough to prove it for f of the form $f(x) = 1_{[0,q]}(x)$. But then each side is equal to $(F_P(q) - F_P(x))^+$. ■

9 References

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