

# ENTROPY-ENERGY INEQUALITIES AND IMPROVED CONVERGENCE RATES FOR NONLINEAR PARABOLIC EQUATIONS

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ABSTRACT. In this paper, we prove new functional inequalities of Poincaré type on the one-dimensional torus  $S^1$  and explore their implications for the long-time asymptotics of periodic solutions of nonlinear singular or degenerate parabolic equations of second and fourth order. We generically prove a global algebraic decay of an *entropy* functional, faster than exponential for short times, and an asymptotically exponential convergence of positive solutions towards their average. The asymptotically exponential regime is valid for a larger range of parameters for all relevant cases of application: porous medium/fast diffusion, thin film and logarithmic fourth order nonlinear diffusion equations. The techniques are inspired by direct entropy-entropy production methods and based on appropriate Poincaré type inequalities.

**1. Introduction.** One of the classical methods to study the convergence to equilibrium of solutions of both linear and nonlinear PDEs is the analysis of the decay of appropriate Lyapunov functionals. In the context of statistical physics and probability or information theory, some of such Lyapunov functionals can be interpreted as *entropies*. Following a recent trend, we will call them generalized entropies, or simply entropies (see [2] for a review).

Our analysis of the decay rates of the entropies associated to nonlinear diffusion equations will be guided by the entropy-entropy production method. A functional inequality relating the entropy to the energy, or entropy production, is the essential ingredient. This strategy is easily described in the case of the linear Fokker-Planck equation, or Ornstein-Uhlenbeck stochastic process:

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (x u), \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad (1)$$

with an initial condition  $u_0$  in, for instance,  $C^2(\mathbb{R}^n) \cap L_+^1(\mathbb{R}^n)$ . In a classical approach, two entropies are widely used:

$$\int_{\mathbb{R}^n} \left| \frac{u}{u_\infty} - 1 \right|^2 u_\infty dx \quad \text{and} \quad \int_{\mathbb{R}^n} u \log \left( \frac{u}{u_\infty} \right) dx,$$

where  $u_\infty(x) := C \exp(-|x|^2/2)$  is the limit of  $u$  as  $t \rightarrow \infty$ , and  $C = (2\pi)^{-n/2} \int u_0 dx$ . The corresponding functional inequalities are respectively the Poincaré inequality

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in the first case,

$$\forall f \in \mathcal{C}^1(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \left| f - \int_{\mathbb{R}^n} f u_\infty dx \right|^2 u_\infty dx \leq \int_{\mathbb{R}^n} |\nabla f|^2 u_\infty dx, \quad (2)$$

and the logarithmic Sobolev inequality introduced by Gross [28],

$$\forall f \in \mathcal{C}^1(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} f^2 \log \left( \frac{f^2}{\int_{\mathbb{R}^n} f^2 u_\infty dx} \right) u_\infty dx \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 u_\infty dx, \quad (3)$$

in the second case. The right hand side of both inequalities is the energy and coincides with the entropy production with  $f = u/u_\infty$  in the first case and  $f = \sqrt{u/u_\infty}$  in the second case. Indeed, equation (1) can conveniently be rewritten as

$$u_\infty \frac{\partial v}{\partial t} = \nabla \cdot (u_\infty v), \quad t \geq 0, \quad x \in \mathbb{R}^n,$$

with  $v(x, t) := u(x, t)/u_\infty(x)$ . Integrating by parts and employing (2) and (3), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} \left| \frac{u}{u_\infty} - 1 \right|^2 u_\infty dx &= \frac{d}{dt} \int_{\mathbb{R}^n} |v - 1|^2 u_\infty dx = -2 \int_{\mathbb{R}^n} |\nabla v|^2 u_\infty dx \\ &\leq -2 \int_{\mathbb{R}^n} |v - 1|^2 u_\infty dx, \\ \frac{d}{dt} \int_{\mathbb{R}^n} u \log \left( \frac{u}{u_\infty} \right) dx &= \frac{d}{dt} \int_{\mathbb{R}^n} v \log v u_\infty dx = -4 \int_{\mathbb{R}^n} |\nabla \sqrt{v}|^2 u_\infty dx \\ &\leq -2 \int_{\mathbb{R}^n} v \log v u_\infty dx. \end{aligned}$$

By Gronwall's lemma, we infer the following exponential convergence to equilibrium measured in entropy sense, *i.e.*,

$$\forall t \geq 0, \quad \int_{\mathbb{R}^n} \left| \frac{u}{u_\infty} - 1 \right|^2 u_\infty dx \leq \int_{\mathbb{R}^n} \left| \frac{u_0}{u_\infty} - 1 \right|^2 u_\infty dx \cdot e^{-2t},$$

and

$$\forall t \geq 0, \quad \int_{\mathbb{R}^n} u \log \left( \frac{u}{u_\infty} \right) dx \leq \int_{\mathbb{R}^n} u_0 \log \left( \frac{u_0}{u_\infty} \right) dx \cdot e^{-2t}.$$

Beckner [8] introduced a family of inequalities which interpolates between the Poincaré inequality and the logarithmic Sobolev inequality: For any  $p \in (1, 2]$ ,

$$\forall f \in \mathcal{C}^1(\mathbb{R}^n), \quad \frac{1}{p-1} \left[ \int_{\mathbb{R}^n} |f|^2 u_\infty dx - \left( \int_{\mathbb{R}^n} |f|^{2/p} u_\infty dx \right)^p \right] \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 u_\infty dx.$$

The logarithmic Sobolev inequality corresponds to the limit  $p \rightarrow 1$ , and the Poincaré inequality is achieved for  $p = 2$ ; see [1, 3, 4, 7] for more details. For a solution of (1), the above results on entropy decay can be generalized as follows: Let

$$\psi_p(v) := \frac{v^p - 1 - p(v-1)}{p-1}$$

and compute with, again,  $v(x, t) := u(x, t)/u_\infty(x)$ ,

$$\frac{d}{dt} \int_{\mathbb{R}^n} \psi_p(v) u_\infty dx = - \int_{\mathbb{R}^n} \psi_p''(v) |\nabla v|^2 u_\infty dx = -\frac{4}{p} \int_{\mathbb{R}^n} |\nabla (v^{p/2})|^2 u_\infty dx.$$

With  $\int_{\mathbb{R}^n} v u_\infty dx = 1$  and  $f = v^{p/2}$ , we derive

$$\frac{1}{p-1} \int_{\mathbb{R}^n} \left[ \left( \frac{u}{u_\infty} \right)^p - 1 \right] u_\infty dx \leq \frac{1}{p-1} \int_{\mathbb{R}^n} \left[ \left( \frac{u_0}{u_\infty} \right)^p - 1 \right] u_\infty dx \cdot e^{-2t}.$$

Similar results can be obtained on the torus  $S^1 \equiv [0, 1)$ , except that the asymptotic state  $u_\infty$  is now a constant; see, for instance, [27] for some results in this direction. In this paper, we will also work on  $S^1$  for two main reasons: Because of the periodicity of the boundary conditions, integrations by parts are simple, and by Sobolev's embeddings, we have an  $L^\infty(S^1)$  control on the functions as soon as they are in  $H^1(S^1)$ . As in  $\mathbb{R}^n$ , the Poincaré inequality, the logarithmic Sobolev inequality and all interpolating Beckner inequalities also hold, but with other optimal constants. Our goal is to prove that exactly as for the linear heat equation on  $S^1$ , which replaces the Fokker-Planck equation in  $\mathbb{R}^n$ , there exists a one-parameter family of entropies associated to nonlinear diffusion equations. This is a first step for the understanding of rates of decay of generalized entropies associated to general nonlinear diffusion equations and related functional inequalities which generalize Beckner's inequalities.

We will use as guiding examples the one dimensional porous medium/fast diffusion equation

$$\frac{\partial u}{\partial t} = (u^m)_{xx}, \quad x \in S^1, \quad t > 0, \quad (4)$$

the thin film equation

$$u_t = -(u^m u_{xxx})_x, \quad x \in S^1, \quad t > 0, \quad (5)$$

and the Derrida-Lebowitz-Speer-Spohn (DLSS) equation [26],

$$u_t = -(u(\log u)_{xx})_{xx}, \quad x \in S^1, \quad t > 0, \quad (6)$$

with an initial condition  $u(\cdot, 0) = u_0 \geq 0$  in  $S^1 \equiv [0, 1)$ . The last two equations are particular cases of a more general family of fourth-order equations that will be treated in Section 3.2. In the special case of equation (4) with  $m = 1$  (heat equation), the inequalities relating the entropies and the energies are given by the family of Beckner's inequalities.

In Section 2, we define a family of entropies  $\Sigma_{p,q}[v]$ , for  $p \in (0, +\infty)$  and  $q \in \mathbb{R}$ , and prove that for any  $v \geq 0$  in  $H^1(S^1)$ ,

$$\Phi(\Sigma_{p,q}[v]) \leq J[v]$$

for some nonlinear function  $\Phi$  which generalizes the left hand side of Beckner's inequalities. Here  $J[v]$  is either  $J_1[v] := \int_{S^1} |v'|^2 dx$  for second order problems, or  $J_2[v] := \int_{S^1} |v''|^2 dx$  for fourth order problems. As a special case, we work with

$$\Phi(s) = s^{2/q}, \quad q \in (0, 2),$$

which is natural in view of the homogeneity of  $\Sigma_{p,q}[v]$  (see Section 2.2).

Some other examples are already known. Weisler [39] considered in  $\mathbb{R}^n$

$$\Phi(s) = \frac{\pi}{2} n e^{1+2s/n}$$

for the scale invariant form of the logarithmic Sobolev inequality in the Euclidean space ( $p = 1/2$ ,  $q = 2$ ). The case

$$\Phi(s) = \frac{p}{2-p} \left( 1 + s - (1+s)^{(2-p)/p} \right), \quad s > 0, \quad p \in (1, 2),$$

has recently been studied in [3] for entropies taking the form  $\Sigma_{p/2,2}$ , and inequalities involving more general functions  $\Phi$  have been studied in [1]. Also see [6] for a review of other related results. Analogous decay results for the  $H^1$ -norm of the solutions in the particular case of (5) with  $m = 1$  have recently been reported in [21]. In [31], a systematic study of entropies, without rates of decay, has been carried out. Some partial results have already been obtained in [35]. We will give more details on these results in Section 3.3 and explain in which sense we improve them. The main novelty is that we systematically prove entropy-energy inequalities for the whole family  $\Sigma_{p,q}$ .

We also consider the linearization of the above inequality around constants, which provides asymptotic inequalities for solutions approaching their limits for large times. Because we work in dimension one, remainder terms can be uniformly controlled.

In Section 3, we use the functional inequalities to compute the decay rates of the entropies of the solutions of the nonlinear diffusion equations (4), (5) and (6). In particular, we generalize exponential decay results given in [33, 35]; see Section 3.3 for further details.

The entropies  $\Sigma_{p,q}$  decay at least algebraically for all times, and exponentially for the large time asymptotic regime. For large initial entropies, the algebraic decay is faster than the exponential decay in the following sense. Assume that  $y$  is a nonnegative function on  $\mathbb{R}^+ \ni t$ , which satisfies simultaneously the two differential inequalities

$$\frac{dy}{dt} \leq -C_1 y^\alpha \quad \text{and} \quad \frac{dy}{dt} \leq -C_2 y, \quad t > 0. \quad (7)$$

Here  $C_1$  and  $C_2$  are two positive constants, and we assume that  $\alpha > 1$ . By a Gronwall argument, it follows that

$$y(t) \leq \min\{y_1(t), y_2(t)\},$$

where  $y_1$  and  $y_2$  are the solutions of the two ODEs

$$\frac{dy_1}{dt} = -C_1 y_1^\alpha \quad \text{and} \quad \frac{dy_2}{dt} = -C_2 y_2$$

with the same initial data:  $y_1(0) = y_2(0) = y(0) := y_0 > 0$ . It is an elementary exercise to check that if  $C_1 y_0^\alpha > C_2 y_0$ , then there exists a  $t_* > 0$  such that

$$0 < y_1(t) < y_2(t) \quad \forall t \in (0, t_*). \quad (8)$$

Such a condition is satisfied for  $y_0$  large enough, whatever  $C_1$  and  $C_2$  are. Although for large times,  $y_2(t) = o(y_1(t))$ , the solution of the first equation

$$y_1(t) = \left[ y_0^{-(\alpha-1)} + (\alpha-1) C_1 t \right]^{-1/(\alpha-1)}$$

has initially a faster decay than the solution of the second ODE,  $y_2(t) = y_0 e^{-C_2 t}$ , in the sense of (8). This is the reason why we claim that the algebraic inequality may initially provide a *faster decay* of the entropy or an *improved rate* of convergence towards the equilibrium state. It is however not obvious that we can choose  $C_2$  independently of  $y_0$  and we will come back to this point in Remark 4 (see Section 3.1).

**2. Functional inequalities.** The two fundamental tools of this paper are the entropy and energy functionals that we are going to relate through functional inequalities. We will denote by  $H_+^1(S^1)$  the set of non-negative 1-periodic functions in the space of measurable functions on  $\mathbb{R}$  with derivatives in  $L_{\text{loc}}^2(\mathbb{R})$ . We will in particular make the identification  $S^1 \equiv [0, 1)$  by imposing periodic boundary conditions on  $[0, 1)$ . The measure induced by Lebesgue's measure is then a probability measure:  $\int_{S^1} dx = 1$ .

**2.1. Entropies and energies.** Denote by  $\overline{\mu_p[v]}$  and  $\bar{v}$  the following averages of a non-negative function  $v$  on  $S^1$ :

$$\mu_p[v] := \left( \int_{S^1} v^{1/p} dx \right)^p \quad \text{and} \quad \bar{v} := \int_{S^1} v dx .$$

Notice that  $\bar{v} = \mu_1[v]$ .

**Definition 1.** Let  $p \in (0, +\infty)$  and  $q \in \mathbb{R}$ . On  $\{v \in H_+^1(S^1) : v \neq 0 \text{ a.e.}\}$ , we define a family of *entropies* depending on  $(p, q)$  by

$$\begin{aligned} \Sigma_{p,q}[v] &:= \frac{1}{pq(pq-1)} \left[ \int_{S^1} v^q dx - (\mu_p[v])^q \right] && \text{if } pq \neq 1 \text{ and } q \neq 0 , \\ \Sigma_{1/q,q}[v] &:= \int_{S^1} v^q \log \left( \frac{v^q}{\int_{S^1} v^q dx} \right) dx && \text{if } pq = 1 \text{ and } q \neq 0 , \\ \Sigma_{p,0}[v] &:= -\frac{1}{p} \int_{S^1} \log \left( \frac{v}{\mu_p[v]} \right) dx && \text{if } q = 0 . \end{aligned}$$

We claim that  $\Sigma_{p,q}[v]$  is non-negative for all  $p \in (0, +\infty)$  and  $q \in \mathbb{R}$ . Indeed, use the fact that if  $pq \neq 1$  and  $q \neq 0$ , the function

$$u \mapsto \frac{u^{pq} - 1 - pq(u-1)}{pq(pq-1)} =: \sigma_{p,q}(u)$$

is strictly convex on  $(0, +\infty)$  and, by Jensen's inequality,

$$\begin{aligned} \Sigma_{p,q}[v] &= \mu_p[v]^q \int_{S^1} \sigma_{p,q} \left( \frac{v^{1/p}}{(\mu_p[v])^{1/p}} \right) dx \\ &\geq \mu_p[v]^q \sigma_{p,q} \left( \int_{S^1} \frac{v^{1/p}}{(\mu_p[v])^{1/p}} dx \right) = \mu_p[v]^q \sigma_{p,q}(1) = 0 . \end{aligned}$$

If  $pq = 1$  or  $q = 0$ ,  $\Sigma_{p,q}[v]$  is also non-negative because of the convexity of the functions  $u^q \mapsto \sigma_{1/q,q}(u) := u^q \log(u^q)$  and  $u \mapsto \sigma_{p,0}(u) := -\log(u^{1/p})$ , respectively. Because of the strict convexity of  $\sigma_{p,q}$ ,  $\Sigma_{p,q}[v] = 0$  holds if and only if  $v \equiv \mu_p[v]$  a.e. in  $S^1$ . The definitions of the limit cases  $pq = 1$  and  $q = 0$  are coherent in the sense that

$$\begin{aligned} \lim_{p \rightarrow 1/q} \Sigma_{p,q}[v] &= \Sigma_{1/q,q}[v] \quad \text{for } q > 0 , \\ \lim_{q \rightarrow 0} \Sigma_{p,q}[v] &= \Sigma_{p,0}[v] \quad \text{for } p > 0 . \end{aligned}$$

Notice also that we can define an entropy in the limit case  $p = q = 0$  by

$$-\int_{S^1} \log \left( \frac{v}{\|v\|_\infty} \right) dx ,$$

although we are not aware of any application of such functional. Beckner's inequalities involve the entropies  $\Sigma_{p/2,2}$ ,  $p \in (1, 2]$ , and the limit as  $p \rightarrow 1$  corresponds to the logarithmic Sobolev inequality.

**Definition 2.** The *energy functional* corresponding to second order equations, or Dirichlet integral, is defined by

$$J_1[v] := \int_{S^1} |v'|^2 dx \quad \forall v \in H^1(S^1).$$

The *energy functional* corresponding to fourth order equations is given by

$$J_2[v] = \int_{S^1} |v''|^2 dx \quad \forall v \in H^2(S^1).$$

**2.2. Global functional inequalities.** By “global” we mean that there is no restriction on the values of the entropies. Our first main result relies the entropies  $\Sigma_{p,q}$  with the first type of energy,  $J_1[v]$ , through a functional inequality.

**Theorem 1.** *For all  $p \in (0, +\infty)$  and  $q \in (0, 2)$ , there exists a positive constant  $\kappa_{p,q}$  such that, for any  $v \in H_+^1(S^1)$ ,*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{\kappa_{p,q}} J_1[v].$$

In other words, the existence of  $\kappa_{p,q} > 0$  is equivalent to the minimization problem:

$$\kappa_{p,q} := \inf_{v \in H_+^1(S^1), v \neq \mu_p[v] \text{ a.e.}} \frac{J_1[v]}{\Sigma_{p,q}[v]^{2/q}} > 0. \quad (9)$$

*Proof.* We first prove the result in the case  $pq \neq 1$ . Let  $v \in H^1(S^1) \subset C^0(S^1)$ . Without loss of generality we may assume that  $\mu_p[v] = 1$ , by homogeneity. Then there exists  $x_0 \in S^1$  such that  $v(x_0) = 1$ . Employing

$$|v(x) - 1| = |v(x) - v(x_0)| = \left| \int_{x_0}^x v'(y) dy \right| \leq \sqrt{|x - x_0|} \left| \int_{x_0}^x |v'(y)|^2 dy \right|^{1/2},$$

we obtain

$$\|v - 1\|_{L^\infty(S^1)} \leq \frac{1}{\sqrt{2}} J_1[v]^{1/2}, \quad \|v\|_{L^\infty(S^1)} \leq 1 + \frac{1}{\sqrt{2}} J_1[v]^{1/2}, \quad (10)$$

since either  $|x - x_0| < 1/2$  or  $|x - x_0 \pm 1| < 1/2$  because of the 1-periodicity. As a consequence,  $\Sigma_{p,q}[v]$  is well defined on  $H^1(S^1)$ .

Consider a minimizing sequence  $(v_n)_{n \in \mathbb{N}}$  for (9) such that  $\mu_p[v_n] = 1$ .

If  $\lim_{n \rightarrow \infty} J_1[v_n] = \infty$ , then, as  $n \rightarrow \infty$ ,

$$\frac{J_1[v_n]}{\Sigma_{p,q}[v_n]^{2/q}} \geq \frac{(pq |pq - 1|)^{2/q} J_1[v_n]}{\left[ \left(1 + \frac{1}{\sqrt{2}} J_1[v_n]^{1/2}\right)^q - 1 \right]^{2/q}} \rightarrow 2(pq |pq - 1|)^{2/q} > 0,$$

and we are done. Otherwise, the sequence  $(v_n)_{n \in \mathbb{N}}$  is bounded in  $H^1(S^1)$ , and, by compactness, there exists a function  $v \in H^1(S^1)$  satisfying  $\mu_p[v] = 1$  and, up to the extraction of a subsequence,

$$v_n \rightharpoonup v \text{ in } H^1(S^1) \text{ and } \Sigma_{p,q}[v_n] \rightarrow \Sigma_{p,q}[v] \text{ as } n \rightarrow \infty. \quad (11)$$

There are two possibilities: either  $\Sigma_{p,q}[v] \neq 0$  or  $\Sigma_{p,q}[v] = 0$ . The former case implies that  $J_1[v] > 0$  since otherwise,  $v$  would be constant and thus  $\Sigma_{p,q}[v] = 0$  which contradicts our assumption  $\Sigma_{p,q}[v] \neq 0$ . By semi-continuity, we have  $\lim_{n \rightarrow \infty} J_1[v_n] \geq J_1[v]$  and, employing (11), we arrive at

$$\lim_{n \rightarrow \infty} \frac{J_1[v_n]}{\Sigma_{p,q}[v_n]^{2/q}} \geq \frac{J_1[v]}{\Sigma_{p,q}[v]^{2/q}} > 0,$$

which proves our result. In the second case,  $\Sigma_{p,q}[v] = 0$ , we have  $\lim_{n \rightarrow \infty} J_1[v_n] = 0$  since otherwise, the quotient  $J_1[v_n]/\Sigma_{p,q}[v_n]^{2/q}$  would diverge and  $(v_n)_{n \in \mathbb{N}}$  would not be a minimizing sequence. We claim that this case leads to a contradiction.

Let

$$\varepsilon_n := J_1[v_n], \quad w_n := \frac{v_n - 1}{\sqrt{\varepsilon_n}}$$

and assume with no restriction that  $\varepsilon_0 \geq \varepsilon_n$  is uniformly small for all  $n \in \mathbb{N}$ . By definition,  $J_1[w_n] = 1$ . Using (10), we infer  $\|w_n\|_{L^\infty(S^1)} \leq 1/\sqrt{2}$ . A Taylor expansion shows that there exists a positive constant  $r(\varepsilon_0, p)$  such that

$$\left| (1 + \sqrt{\varepsilon} x)^{1/p} - 1 - \frac{\sqrt{\varepsilon}}{p} x \right| \leq \frac{1}{p} r(\varepsilon_0, p) \varepsilon \quad \forall (x, \varepsilon) \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \times (0, \varepsilon_0).$$

The condition  $\mu_p[v_n] = 1$  now means that

$$\int_{S^1} (1 + \sqrt{\varepsilon_n} w_n)^{1/p} dx - 1 = 0$$

and hence, by the above Taylor expansion,

$$\left| \int_{S^1} w_n dx \right| = \frac{p}{\sqrt{\varepsilon_n}} \left| \int_{S^1} \left( (1 + \sqrt{\varepsilon_n} w_n)^{1/p} - 1 - \frac{\sqrt{\varepsilon_n}}{p} w_n \right) dx \right| \leq r(\varepsilon_0, p) \sqrt{\varepsilon_n}.$$

The Taylor expansion also gives, with  $p$  replaced by  $1/q$ ,

$$\left| \int_{S^1} (1 + \sqrt{\varepsilon_n} w_n)^q dx - 1 - q \sqrt{\varepsilon_n} \int_{S^1} w_n dx \right| \leq q r(\varepsilon_0, 1/q) \varepsilon_n,$$

and therefore,

$$\left| \int_{S^1} (1 + \sqrt{\varepsilon_n} w_n)^q dx - 1 \right| \leq q [r(\varepsilon_0, 1/q) + r(\varepsilon_0, p)] \varepsilon_n =: p q (p q - 1) c(\varepsilon_0, p, q) \varepsilon_n$$

for  $p q \neq 1$ , where  $c(\varepsilon_0, p, q) > 0$  is an explicit constant. This proves that

$$\Sigma_{p,q}[v_n] \leq c(\varepsilon_0, p, q) \varepsilon_n, \tag{12}$$

and immediately amounts to, since  $q < 2$ ,

$$\frac{J_1[v_n]}{\Sigma_{p,q}[v_n]^{2/q}} = \frac{\varepsilon_n J_1[w_n]}{\Sigma_{p,q}[v_n]^{2/q}} \geq [c(\varepsilon_0, p, q)]^{-2/q} \varepsilon_n^{1-2/q} \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

which contradicts the fact that  $(v_n)_{n \in \mathbb{N}}$  is a minimizing sequence.

If  $p q = 1$ , the proof follows the same lines using the estimate

$$(2 \log 2 - 1) x^2 \leq (1 + x) \log(1 + x) - x \leq x^2 \quad \forall x \in (-1, 1),$$

which again proves that  $\varepsilon_n \Sigma_{1/q,q}[v_n]^{-2/q} = O(\varepsilon_n^{1-2/q})$ .  $\square$

As far as we know, the optimal constants  $\kappa_{p,q}$  have no explicit expression, but can be computed numerically without major difficulties.

Our second main result relates the entropies with the second energy functional,  $J_2$ , through another functional inequality.

**Corollary 1.** *Assume that  $p \in (0, +\infty)$  and  $q \in (0, 2)$  are such that  $p q \neq 1$ . Then*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{4\pi^2 \kappa_{p,q}} J_2[v] \quad \forall v \in H_+^2(S^1). \tag{13}$$

*Proof.* The proof of (13) is a consequence of (9) and the classical and optimal Poincaré inequality

$$(2\pi)^2 \|v - \bar{v}\|_{L^2(S^1)}^2 \leq J_1[v]. \quad (14)$$

Applied to  $v'$ , this inequality gives  $J_1[v] \leq (2\pi)^{-2} J_2[v]$  since  $\bar{v}' = 0$ .  $\square$

**2.3. Asymptotic functional inequalities.** We consider the regime of small entropies, i.e., we restrict the set of admissible functions to

$$\mathcal{X}_\varepsilon^{p,q} := \{v \in H_+^1(S^1) : \Sigma_{p,q}[v] \leq \varepsilon \text{ and } \mu_p[v] = 1\},$$

for which we establish a linear relation between the entropies and the energies.

**Theorem 2.** *For any  $p > 0$ ,  $q \in \mathbb{R}$  and  $\varepsilon_0 > 0$ , there exists a positive constant  $C$  such that, for any  $\varepsilon \in (0, \varepsilon_0]$ ,*

$$\Sigma_{p,q}[v] \leq \frac{1 + C\sqrt{\varepsilon}}{8p^2\pi^2} J_1[v] \quad \forall v \in \mathcal{X}_\varepsilon^{p,q}. \quad (15)$$

*Proof.* If  $v$  satisfies  $J_1[v] > 8p^2\pi^2\varepsilon$ , then Inequality (15) is clear. Assume therefore that

$$J_1[v] \leq (\kappa_p^\infty)^2 \varepsilon \quad \text{with} \quad \kappa_p^\infty := \sqrt{8p^2\pi^2}$$

and define  $w := (v - 1)/(\kappa_p^\infty \sqrt{\varepsilon})$  which implies that  $J_1[w] \leq 1$ .

The proof of Theorem 2 is based on Taylor expansions. We give first a heuristic argument which is easier to understand than the rigorous proof which will be given below. We have

$$\begin{aligned} & \int_{S^1} (1 + \kappa_p^\infty \sqrt{\varepsilon} w)^q dx \\ &= 1 + q \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx + \frac{q(q-1)(\kappa_p^\infty)^2}{2} \varepsilon \int_{S^1} w^2 dx + O(\varepsilon^{3/2}), \\ & \left( \int_{S^1} (1 + \kappa_p^\infty \sqrt{\varepsilon} w)^{1/p} dx \right)^{pq} \\ &= \left( 1 + \frac{1}{p} \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx - \frac{(p-1)(\kappa_p^\infty)^2}{2p^2} \varepsilon \int_{S^1} w^2 dx + O(\varepsilon^{3/2}) \right)^{pq} \\ &= 1 + q \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx - \frac{q(p-1)(\kappa_p^\infty)^2}{2p} \varepsilon \int_{S^1} w^2 dx \\ & \quad + \frac{q(pq-1)(\kappa_p^\infty)^2}{2p} \varepsilon \left( \int_{S^1} w dx \right)^2 + O(\varepsilon^{3/2}). \end{aligned}$$

Taking the difference, we obtain

$$\begin{aligned} \Sigma_{p,q}[v] &= \frac{1}{pq(pq-1)} \left[ \int_{S^1} (1 + \kappa_p^\infty \sqrt{\varepsilon} w)^q dx - \left( \int_{S^1} (1 + \kappa_p^\infty \sqrt{\varepsilon} w)^{1/p} dx \right)^{pq} \right] \\ &= \varepsilon \frac{(\kappa_p^\infty)^2}{2p^2} \left[ \int_{S^1} w^2 dx - \left( \int_{S^1} w dx \right)^2 \right] + O(\varepsilon^{3/2}) \\ &= \varepsilon \frac{(\kappa_p^\infty)^2}{2p^2} \int_{S^1} (w - \bar{w})^2 dx + O(\varepsilon^{3/2}) \\ &\leq \varepsilon \frac{(\kappa_p^\infty)^2}{2p^2} \frac{J_1[w]}{(2\pi)^2} + O(\varepsilon^{3/2}) = \frac{J_1[v]}{8p^2\pi^2} + O(\varepsilon^{3/2}), \end{aligned} \quad (16)$$



using Poincaré's inequality, which shows the result. In order to make the above argument rigorous, the only difficulty is to take the constraint  $\mu_p[v] = 1$  into account and to control the remainder terms uniformly. This is what we are going to do next.

By Inequality (10),  $\|w\|_{L^\infty(S^1)} \leq J_1[w]^{1/2}/\sqrt{2} \leq 1/\sqrt{2}$ . The same computation as in the proof of Theorem 1, Inequality (12), shows that

$$\Sigma_{p,q}[v] \leq c(\varepsilon_0, p, q) J_1[v].$$

Expanding  $\Sigma_{p,q}[v]$  to the next order gives a more precise expression of  $c(\varepsilon_0, p, q)$ . Using a Taylor expansion up to second order in  $\varepsilon$ , we can write

$$\left| (1 + \sqrt{\varepsilon} x)^{1/p} - 1 - \frac{\sqrt{\varepsilon}}{p} x + \frac{p-1}{2p^2} \varepsilon x^2 \right| \leq s(\varepsilon_0, p) \varepsilon^{3/2} x^2 \quad (17)$$

for all  $(x, \varepsilon) \in (-1/\sqrt{2}, 1/\sqrt{2}) \times (0, \varepsilon_0)$ . Thus

$$\left| \int_{S^1} (1 + \kappa_p^\infty \sqrt{\varepsilon} w)^{1/p} dx - 1 - \frac{1}{p} \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx + \frac{(p-1)(\kappa_p^\infty)^2}{2p^2} \varepsilon \int_{S^1} w^2 dx \right| \leq s(\varepsilon_0, p) \varepsilon^{3/2} \|w\|_{L^\infty(S^1)}^2.$$

The quantity  $(\int_{S^1} (1 + \kappa_p^\infty \sqrt{\varepsilon} w)^{1/p} dx)^{pq}$  is bounded from below and above by, respectively,

$$\begin{aligned} & \left( 1 + \frac{1}{p} \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx - \frac{p-1}{2p^2} (\kappa_p^\infty)^2 \varepsilon \int_{S^1} w^2 dx - s(\varepsilon_0, p) \varepsilon^{3/2} \right)^{pq} \\ & \geq 1 + q \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx - \frac{q(p-1)}{2p} (\kappa_p^\infty)^2 \varepsilon \int_{S^1} w^2 dx \\ & \quad + \frac{q(pq-1)}{2p} (\kappa_p^\infty)^2 \varepsilon \left( \int_{S^1} w dx \right)^2 - a(\varepsilon_0, p) \varepsilon^{3/2} \|w\|_{L^\infty(S^1)}^2 \end{aligned}$$

and

$$\begin{aligned} & \left( 1 + \frac{1}{p} \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx - \frac{p-1}{2p^2} (\kappa_p^\infty)^2 \varepsilon \int_{S^1} w^2 dx + s(\varepsilon_0, p) \varepsilon^{3/2} \right)^{pq} \\ & \leq 1 + q \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx - \frac{q(p-1)}{2p} (\kappa_p^\infty)^2 \varepsilon \int_{S^1} w^2 dx \\ & \quad + \frac{q(pq-1)}{2p} (\kappa_p^\infty)^2 \varepsilon \left( \int_{S^1} w dx \right)^2 + b(\varepsilon_0, p) \varepsilon^{3/2} \|w\|_{L^\infty(S^1)}^2, \end{aligned}$$

where  $a(\varepsilon_0, p)$  and  $b(\varepsilon_0, p)$  are two explicit constants. The condition  $\mu_p[v] = 1$  means

$$0 = \left( \int_{S^1} (1 + \kappa_p^\infty \sqrt{\varepsilon} w)^{1/p} dx \right)^{pq} - 1$$

which, using the above Taylor expansion, gives

$$\left| q \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx - \frac{q(p-1)}{2p} (\kappa_p^\infty)^2 \varepsilon \int_{S^1} w^2 dx + \frac{q(pq-1)}{2p} (\kappa_p^\infty)^2 \varepsilon \left( \int_{S^1} w dx \right)^2 \right| \leq c(\varepsilon_0, p) \varepsilon^{3/2} \|w\|_{L^\infty(S^1)}^2 \quad (18)$$

with  $c(\varepsilon_0, p) := \max\{a(\varepsilon_0, p), b(\varepsilon_0, p)\}$ . Exactly as in the proof of Theorem 1, by the Taylor expansion (17) with  $p$  replaced by  $1/q$ , we also obtain

$$\left| \int_{S^1} (1 + \kappa_p^\infty \sqrt{\varepsilon} w)^q dx - 1 - q \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx - \frac{1}{2} q(q-1) (\kappa_p^\infty)^2 \varepsilon \int_{S^1} w^2 dx \right| \leq s(\varepsilon_0, 1/q) \varepsilon^{3/2} \|w\|_{L^\infty(S^1)}^2,$$

and therefore,

$$\left| \int_{S^1} (1 + \kappa_p^\infty \sqrt{\varepsilon} w)^q dx - 1 \right| \leq \left| q \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx - \frac{1}{2} q(q-1) (\kappa_p^\infty)^2 \varepsilon \int_{S^1} w^2 dx \right| + s(\varepsilon_0, 1/q) \varepsilon^{3/2} \|w\|_{L^\infty(S^1)}^2.$$

Using (18) to compute  $q \kappa_p^\infty \sqrt{\varepsilon} \int_{S^1} w dx$  and inequality (10),  $2\|w\|_{L^\infty(S^1)}^2 \leq J_1[w]$ , we conclude that

$$\left| \int_{S^1} (1 + \kappa_p^\infty \sqrt{\varepsilon} w)^q dx - 1 \right| \leq \frac{q(pq-1)}{2p} (\kappa_p^\infty)^2 \varepsilon \left| \left( \int_{S^1} w dx \right)^2 - \int_{S^1} w^2 dx \right| + ((c(\varepsilon_0, p) + s(\varepsilon_0, 1/q))/2) \varepsilon^{3/2} J_1[w],$$

which proves that, for  $pq \neq 1$  and  $q \neq 0$  (see (16)),

$$\begin{aligned} \Sigma_{p,q}[v] &\leq \varepsilon \frac{(\kappa_p^\infty)^2}{2p^2} \frac{J_1[w]}{(2\pi)^2} + ((c(\varepsilon_0, p) + s(\varepsilon_0, 1/q))/2) \varepsilon^{3/2} J_1[w] \\ &\leq \frac{1 + C(p, q, \varepsilon_0) \sqrt{\varepsilon}}{8p^2 \pi^2} J_1[v], \end{aligned}$$

where

$$C = C(p, q, \varepsilon_0) := \frac{c(\varepsilon_0, p) + s(\varepsilon_0, 1/q)}{2(\kappa_p^\infty)^2}.$$

This shows Theorem 2 if  $pq \neq 1$  and  $q \neq 0$ . If  $pq = 1$  or  $q = 0$ , the proofs are similar using the appropriate Taylor expansions.  $\square$

**Remark 1.** The condition  $\mu_p[v] = 1$  in Theorem 2 breaks the homogeneity. If we only assume that  $v \in H_+^1(S^1)$  and  $\Sigma_{p,q}[v] \leq \varepsilon (\mu_p[v])^q$ , then we obtain the more general inequality

$$\Sigma_{p,q}[v] \leq \frac{1 + C\sqrt{\varepsilon}}{8p^2 \pi^2} (\mu_p[v])^{q-2} J_1[v] \quad (19)$$

for any  $\varepsilon \in (0, \varepsilon_0]$ , where  $C$ ,  $\varepsilon_0$ ,  $p$  and  $q$  are as in Theorem 2.

**Corollary 2.** For any  $p > 0$ ,  $q \in (0, 2)$  and  $\varepsilon_0 > 0$ , there exists a positive constant  $C$  such that, for any  $\varepsilon \in (0, \varepsilon_0]$ ,

$$\Sigma_{p,q}[v] \leq \frac{1 + C\sqrt{\varepsilon}}{32p^2 \pi^4} J_2[v] \quad \forall v \in \mathcal{X}_\varepsilon^{p,q} \cap H^2(S^1).$$

**Remark 2.** According to [27], for any  $v \in H^2(S^1)$ ,

$$\Sigma_{1/2,2}[v] = \int_{S^1} v^2 \log \left( \frac{v^2}{\int_{S^1} v^2 dx} \right) dx \leq \frac{1}{2\pi^2} \int_{S^1} |v'|^2 dx = \frac{1}{2\pi^2} J_1[v].$$

Moreover the constant  $(2\pi^2)^{-1}$  is optimal; i.e., the constant in Theorem 2 cannot be improved for  $p = 1/2$ ,  $q = 2$ . This is also true in the other cases. The constants  $(8p^2 \pi^2)^{-1}$  in Theorem 2 and  $(32p^2 \pi^4)^{-1}$  in Corollary 2 are also optimal, as shown

by taking as test functions  $1 + \eta w$  in the limit  $\eta \rightarrow 0$ , where  $w$  is the optimal function for, respectively, the optimal Poincaré inequalities (14) and

$$(2\pi)^4 \|v - \bar{v}\|_{L^2(S^1)}^2 \leq J_2[v] \quad \forall v \in H^2(S^1),$$

such that, additionally,  $\bar{w} = 0$ . In other words, if

$$\mathcal{K}_{p,q}^1(\varepsilon) := \inf_{v \in \mathcal{Y}_{\varepsilon}^{p,q}} \frac{J_1[v]}{\Sigma_{p,q}[v]} \quad \text{and} \quad \mathcal{K}_{p,q}^2(\varepsilon) := \inf_{v \in \mathcal{Y}_{\varepsilon}^{p,q} \cap H^2(S^1)} \frac{J_2[v]}{\Sigma_{p,q}[v]} \quad (20)$$

where

$$\mathcal{Y}_{\varepsilon}^{p,q} := \{v \in H_+^1(S^1) : \Sigma_{p,q}[v] = \varepsilon \text{ and } \mu_p[v] = 1\},$$

then for  $\varepsilon_0 > 0$  fixed and  $C = C(\varepsilon_0, p, q)$ , we have the estimates

$$\frac{8p^2\pi^2}{1 + C\sqrt{\varepsilon}} \leq \inf_{v \in \mathcal{X}_{\varepsilon}^{p,q}} \frac{J_1[v]}{\Sigma_{p,q}[v]} \leq \mathcal{K}_{p,q}^1(\varepsilon),$$

$$\frac{32p^2\pi^4}{1 + C\sqrt{\varepsilon}} \leq \inf_{v \in \mathcal{X}_{\varepsilon}^{p,q} \cap H^2(S^1)} \frac{J_2[v]}{\Sigma_{p,q}[v]} \leq \mathcal{K}_{p,q}^2(\varepsilon),$$

$$\lim_{\varepsilon \rightarrow 0} \mathcal{K}_{p,q}^1(\varepsilon) = 8p^2\pi^2 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \mathcal{K}_{p,q}^2(\varepsilon) = 32p^2\pi^4.$$

Notice that the set  $\mathcal{Y}_{\varepsilon}^{p,q}$  for sufficiently small  $\varepsilon > 0$  is not empty since for any non-constant function  $v \in H_+^1(S^1)$  with  $\Sigma_{p,q}[v] = \varepsilon_0$ , the range of the mapping  $[0, 1] \ni \theta \mapsto \Sigma_{p,q}[1 + \theta(v - 1)]$  is  $[0, \varepsilon_0]$ . To construct functions  $v$  with arbitrarily large entropies and  $\mu_p[v] = 1$  is easy. For instance, take  $u = v^{1/p}$  to be any  $H^1$ -regularized function of  $\zeta^{-1}\chi_{A_{\zeta}}$ ,  $\zeta \rightarrow 0_+$ , where  $\chi_{A_{\zeta}}$  is the characteristic function of a union of intervals of total length  $\zeta$ . Notice that  $\|v\|_{H^1(S^1)}$  is large.

**Remark 3.** Away from the limiting regime  $\varepsilon \rightarrow 0$ , we can also compare the results of Theorems 1 and 2. It is straightforward to check that for any  $\varepsilon \in (0, \varepsilon_0)$ , these two results provide the two terms of the following lower bound:

$$\mathcal{K}_{p,q}^1(\varepsilon) = \inf_{v \in \mathcal{Y}_{\varepsilon}^{p,q}} \frac{J_1[v]}{\Sigma_{p,q}[v]} \geq \max \left\{ \frac{8p^2\pi^2}{1 + C\sqrt{\varepsilon}}, \frac{\varepsilon^{(2-q)/q}}{\kappa_{p,q}} \right\}.$$

**2.4. Optimal functional inequalities.** Theorems 1 and 2 and Corollaries 1 and 2 provide examples of various linear and nonlinear inequalities relating the entropy  $\Sigma_{p,q}[v]$  and the energies  $J_1[v]$  and  $J_2[v]$ . The constants have been estimated in Theorem 2, which are optimal in some cases at leading order in  $\varepsilon$ . However, the question of the optimal relation between the entropies and the energies is essentially open: What is the largest function  $\Phi$  such that

$$\Phi(\Sigma_{p,q}[v]) \leq J_1[v] \quad (21)$$

holds for all  $v \in H_+^1(S^1)$  satisfying  $\mu_p[v] = 1$ ? With the notations of (20), we may define  $\Phi$  by  $\Phi(x) := \mathcal{K}_{p,q}^1(x)$ , but this gives no explicit estimate of  $\Phi$ .

A less formal approach goes as follows. An interpolation between the nonlinear inequality of Theorem 1 and the linear inequality of Theorem 2 is easy to achieve: For any  $\varepsilon_0 > 0$ , define

$$\Phi_{\varepsilon_0}(x) := \begin{cases} 8p^2\pi^2(1 + C\sqrt{x})x & \text{if } x \in [0, \varepsilon_0] \\ 0 & \text{if } x > \varepsilon_0 \end{cases} \quad \text{and} \quad \Phi_0(x) := \kappa_{p,q}x^{2/q},$$

and consider

$$\Phi(x) := \sup_{\varepsilon_0 \geq 0} \{\Phi_{\varepsilon_0}(x)\} \quad (22)$$

for any  $x > 0$ . Inequality (21) holds for such a function  $\Phi$ , which is not identically equal to  $\Phi_0$  at least for  $x$  small, and which is lower semi-continuous as the supremum of a family of lower semi-continuous functions.

**3. Decay rates of entropies associated to nonlinear diffusions.** In this section, we apply the variational inequalities of Section 2 to derive decay rates of various entropies and proving the convergence towards equilibrium of the solutions of several nonlinear diffusion equations such as the porous medium, thin film and more general fourth order nonlinear diffusion equations. We will prove that for all of these models, some entropies have at least a global algebraic decay. Asymptotically for large times, these entropies decay exponentially. This exponential decay for small values of the entropy is intuitively explained by the fact that for solutions which are close enough to their equilibrium values, the linearized version of the entropy-entropy dissipation inequality becomes relevant. Both regimes, short-time fast algebraic decay and asymptotically exponential decay, are direct consequences of Theorems 1 and 2.

For simplicity, we consider only smooth solutions. Extension to more general classes of solutions will be mentioned whenever an approximation procedure is known, but we will not give details to avoid unnecessary technicalities. We keep the same framework as in Theorems 1 and 2 and consider only the case of periodic boundary conditions.

**3.1. Porous medium/fast diffusion equation: an illuminating example.** We start by applying the method to the simple example of the one dimensional porous medium equation. For any  $m > 0$ , let  $u$  be a solution of

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \quad x \in S^1, \quad t > 0, \quad (23)$$

with initial condition  $u(\cdot, 0) = u_0$  in  $S^1$ . Global unique solutions to the Cauchy problem in the whole space have been obtained in [9] for locally integrable initial data. More informations and references on the subject can be found in [5, 37, 38]. For non-negative periodic integrable initial data,  $u_0 \in L^1_+(S^1)$ , solutions to the Cauchy problem become positive and smooth after a finite time.

In order to study the long time asymptotics, consider the entropies

$$\Sigma_k[u] := \begin{cases} \frac{1}{k(k+1)} \int_{S^1} (u^{k+1} - \bar{u}^{k+1}) dx & \text{if } k \in \mathbb{R} \setminus \{-1, 0\}, \\ \int_{S^1} u \log\left(\frac{u}{\bar{u}}\right) dx & \text{if } k = 0, \\ - \int_{S^1} \log\left(\frac{u}{\bar{u}}\right) dx & \text{if } k = -1. \end{cases} \quad (24)$$

We recall that  $\bar{u}$  is the usual average of  $u$ ,  $\bar{u} := \int_{S^1} u dx$ . Define

$$v := u^p, \quad p := \frac{m+k}{2}, \quad q := \frac{k+1}{p} = 2 \frac{k+1}{m+k}.$$

Then

$$\bar{u} = \int_{S^1} u dx = \int_{S^1} v^{1/p} dx = (\mu_p[v])^{1/p}.$$

For  $k \in \mathbb{R} \setminus \{0, -1\}$ , the entropy functionals  $\Sigma_k[u]$  can be written in terms of the entropy functionals  $\Sigma_{p,q}[v]$  of Section 2 as:

$$\begin{aligned}\Sigma_k[u] &= \frac{1}{k(k+1)} \int_{S^1} (u^{k+1} - \bar{u}^{k+1}) dx \\ &= \frac{1}{pq(pq-1)} \int_{S^1} (v^q - (\mu_p[v])^q) dx = \Sigma_{p,q}[v],\end{aligned}$$

A similar relation holds in the limit cases  $k = 0$  and  $k = -1$ . We have the following basic properties.

**Lemma 1.** *For any  $k \in \mathbb{R}$ , the functional  $u \mapsto \Sigma_k[u]$  is convex non-negative on  $L^1_+(S^1)$ , and  $\Sigma_k$  reaches its minimum value, 0, if and only if  $u$  coincides with  $\bar{u}$  a.e.*

The proof is straightforward: For  $k+1 = q$ ,  $p = 1$ , i.e.  $m = 2 - k$ , we can write:  $\Sigma_k[u] = \Sigma_{p,q}[u]$ . Notice additionally that for  $k \in [0, 1]$ ,  $\Sigma_k[u]$  controls the  $L^{k+1}(S^1)$ -norm of  $u - \bar{u}$  by a generalized Csiszár-Kullback inequality, see [20].

Irreversibility in Equation (23) is measured by  $\Sigma_k$ , as shown by the following lemma.

**Lemma 2.** *Let  $k \in \mathbb{R}$ . If  $u$  is a smooth positive solution of (23), then*

$$\frac{d}{dt} \Sigma_k[u(\cdot, t)] + \lambda \int_{S^1} \left| (u^{(k+m)/2} \right)_x \right|^2 dx = 0$$

with  $\lambda := 4m/(m+k)^2$  whenever  $k+m \neq 0$ , and

$$\frac{d}{dt} \Sigma_k[u(\cdot, t)] + \lambda \int_{S^1} |(\log u)_x|^2 dx = 0$$

with  $\lambda := m$  for  $k+m = 0$ .

A direct application of Theorems 1 and 2 gives the following result.

**Proposition 1.** *Let  $m \in (0, +\infty)$ ,  $k \in \mathbb{R} \setminus \{-m\}$ ,  $q = 2(k+1)/(m+k)$ ,  $p = (m+k)/2$  and  $u$  be a smooth positive solution of (23).*

i) Short-time Algebraic Decay: *If  $m > 1$  and  $k > -1$ , then*

$$\Sigma_k[u(\cdot, t)] \leq \left[ \Sigma_k[u_0]^{-(2-q)/q} + \frac{2-q}{q} \lambda \kappa_{p,q} t \right]^{-q/(2-q)} \quad \forall t \in \mathbb{R}^+.$$

ii) Asymptotically Exponential Decay: *If  $m > 0$  and  $m+k > 0$ , there exists  $C > 0$  and  $t_1 > 0$  such that*

$$\Sigma_k[u(\cdot, t)] \leq \Sigma_k[u(\cdot, t_1)] \exp \left( -\frac{8p^2 \pi^2 \lambda \bar{u}^{p(2-q)} (t-t_1)}{1 + C \sqrt{\Sigma_k[u(\cdot, t_1)]}} \right) \quad \forall t \geq t_1.$$

*Proof.* Applying Theorem 1 with  $p \in (0, +\infty)$  and  $q \in (0, 2)$ , i.e.,  $m > 1$  and  $k > -1$ , to  $v(\cdot, t) = u(\cdot, t)^p$ , we obtain

$$\Sigma_{p,q}[v(\cdot, t)]^{2/q} \leq \kappa_{p,q}^{-1} J_1[v(\cdot, t)],$$

By Lemma 2, we infer

$$\frac{d}{dt} \Sigma_k[u(\cdot, t)] = \frac{d}{dt} \Sigma_{p,q}[v(\cdot, t)] \leq -\lambda \kappa_{p,q} \Sigma_{p,q}[v(\cdot, t)]^{2/q} = -\lambda \kappa_{p,q} \Sigma_k[u(\cdot, t)]^{2/q},$$

and i) follows from an integration.

To prove ii), we first claim that

$$\lim_{t \rightarrow \infty} \Sigma_k[u(\cdot, t)] = 0.$$

Indeed, by a Sobolev-Poincaré inequality and the entropy estimate of Lemma 2, for some constant  $c > 0$ ,

$$\begin{aligned} \int_0^\infty \left\| u^{(k+m)/2}(\cdot, s) - \bar{u}^{(m+k)/2} \right\|_{L^\infty(S^1)}^2 ds \\ \leq c \int_0^\infty \int_{S^1} \left| (u^{(k+m)/2})_x(x, s) \right|^2 dx ds < +\infty. \end{aligned}$$

Thus, there exists an increasing sequence  $(t_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} t_n = +\infty$  such that  $\|u^{(k+m)/2}(\cdot, t_n) - \bar{u}^{(m+k)/2}\|_{L^\infty(S^1)} \rightarrow 0$  as  $n \rightarrow \infty$  and, consequently,  $u(\cdot, t_n) - \bar{u} \rightarrow 0$  in  $L^\infty(S^1)$ . This shows that

$$\Sigma_k[u(\cdot, t)] \leq \Sigma_k[u(\cdot, t_{n(t)})] \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where  $n(t) := \inf\{n \in \mathbb{N} : t_n \geq t\}$ .

Hence, we can choose  $t_1$  large enough such that  $\Sigma_k[u(\cdot, t_1)] = \epsilon$ , and then  $\Sigma_k[u(\cdot, t)] \leq \epsilon$  for all  $t \geq t_1$ . A direct application of Theorem 2 with  $p = (m+k)/2 > 0$ , or, to be precise, of Inequality (19), implies that

$$\frac{\kappa_p^\infty}{1 + C\sqrt{\epsilon}} \Sigma_{p,q}[v(\cdot, t)] (\mu_p[v])^{2-q} \leq J_1[v(\cdot, t)],$$

for all  $t \geq t_1$ . The average  $\bar{u}^p = \mu_p[v]$  is preserved by the evolution according to (23). By Lemma 2, we conclude that

$$\frac{d}{dt} \Sigma_{p,q}[v(\cdot, t)] \leq -\frac{\lambda \kappa_p^\infty}{1 + C\sqrt{\epsilon}} \Sigma_{p,q}[v(\cdot, t)] (\mu_p[v])^{2-q},$$

which can be written in terms of  $u$  as

$$\frac{d}{dt} \Sigma_k[u(\cdot, t)] \leq -\frac{\lambda \kappa_p^\infty}{1 + C\sqrt{\epsilon}} \Sigma_k[u(\cdot, t)] \bar{u}^{p(2-q)},$$

for all  $t \geq t_1$ . Integrating this differential inequality completes the proof of ii).  $\square$

**Remark 4.** We come back to the comparison of the rates of decay i) and ii) of Proposition 1. With the notations of the introduction,  $y(t) := \Sigma_k[u(\cdot, t)]$  satisfies (7) with  $\alpha = 2/q$ ,  $C_1 = \lambda \kappa_{p,q}$  and  $C_2 = 8p^2 \pi^2 \lambda \bar{u}^{p(2-q)} / (1 + C\sqrt{\epsilon})$ , under the additional condition  $y_0 \leq \epsilon (\mu_p[v_0])^q = \epsilon \bar{u}^{pq}$  (cf. Remark 1). Notice that the initial value of the entropy  $y_0$  and the average  $\bar{u}$  can be chosen independently. The algebraic decay i) is therefore initially faster than the exponential decay given by ii) if  $C_1 y_0^{2/q} > C_2 y_0$  and  $y_0 \leq \epsilon \bar{u}^{pq}$ , which is the case if

$$y_0^{(2-q)/q} (1 + C\sqrt{\epsilon}) > \frac{8p^2 \pi^2}{\kappa_{p,q}} \bar{u}^{p(2-q)} \quad \text{and} \quad 0 < y_0 \leq \epsilon \bar{u}^{pq}$$

or, equivalently,

$$K(\epsilon) \bar{u}^{pq} < y_0 \leq \epsilon \bar{u}^{pq} \quad \text{with} \quad K(\epsilon) = \left( \frac{8p^2 \pi^2}{\kappa_{p,q}(1 + C\sqrt{\epsilon})} \right)^{1/(p(2-q))}.$$

Thus, the algebraic decay is initially faster than the exponential decay if  $\epsilon$  is sufficiently large. This justifies the words ‘‘improved decay rates’’ in the title of this paper. Such a situation is illustrated below in Fig. 1.

We can also state a formal result corresponding to the general functional inequality (21)

$$\Phi(\Sigma_{p,q}[v]) \leq J_1[v]$$

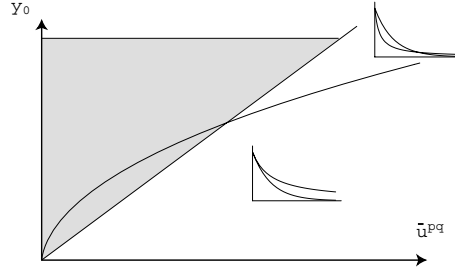


FIGURE 1. In the domain  $0 < y_0 \leq \varepsilon \bar{u}^{p,q}$ , we distinguish two subdomains: the lower one in which the exponential decay is always faster than the algebraic decay, and the upper one in which the algebraic decay is initially faster.

stated in Section 2.4. Let  $\Psi$  be an antiderivative of  $1/\Phi$ . By Lemma 2, we have the differential inequality

$$\frac{d}{dt} \Psi(\Sigma_k[u(\cdot, t)]) \leq -\lambda,$$

if we assume that  $\bar{u} = 1$ . Otherwise, by homogeneity we have

$$\Phi\left(\frac{\Sigma_{p,q}[v]}{(\mu_p[v])^q}\right) \leq \frac{J_1[v]}{(\mu_p[v])^2}.$$

If  $\bar{u} = 1$ , we can use a Gronwall estimate to conclude

$$\Sigma_k[u(\cdot, t)] \leq \Psi^{-1}(\Psi(\Sigma_k[u_0]) - \lambda t) \quad \forall t \in \mathbb{R}^+. \quad (25)$$

As a consequence, we have the following result, which improves Proposition 1.

**Corollary 3.** *If  $m > 1$  and  $k > -1$ , any smooth positive solutions of (23) with initial data  $u_0$  such that  $\bar{u}_0 = 1$  satisfies (25) with  $\Phi$  defined by (22).*

*Proof.* The only difficulty arises from the fact that  $\Phi$  is just a lower semi-continuous function and then the ODE has to be understood in the distribution sense.  $\square$

According to Remark 2, from  $\lim_{\varepsilon \rightarrow 0} \mathcal{K}_{p,q}^1(\varepsilon) = 8p^2\pi^2$ , we deduce that for any  $\eta > 0$ , there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon \in (0, \varepsilon_0)$ ,

$$\frac{J_1[v]}{\Sigma_{p,q}[v]} > 8p^2\pi^2 - \eta \quad \forall v \in \mathcal{X}_\varepsilon^{p,q}.$$

Let  $t_* > 0$  be large enough so that  $\Sigma_k[u(\cdot, t_*)] < \varepsilon_0$ . Then, for any  $t > t_1 > t_*$ ,

$$\Sigma_k[u(\cdot, t)] \leq \Sigma_k[u(\cdot, t_1)] e^{-(8p^2\pi^2 - \eta)(t - t_1)}.$$

Our method shows that in the long-time range, the asymptotic decay of the entropies is exponential and corresponds to the decay given by the linearized equation. Although such a property is natural in the context of the porous medium equations, it is as far as we know entirely new. For earlier results on the linearized equation and its spectral properties, we refer to [23, 24, 25].

**3.2. Fourth order nonlinear diffusions: entropy decay.** In this section we apply the entropy-entropy production inequality to the following class of fourth order equations:

$$u_t = -\left(u^m (u_{xxx} + a u^{-1} u_x u_{xx} + b u^{-2} u_x^3)\right)_x, \quad x \in S^1, t > 0, \quad (26)$$

with  $m, a, b \in \mathbb{R}$  and the initial condition  $u(\cdot, 0) = u_0 \in L^1_+(S^1)$ . This class of equations contains two classical examples.

*Example 1.* The *thin film equation* [13, 36],

$$u_t = -(u^m u_{xxx})_x, \quad (27)$$

corresponding to  $a = b = 0$ .

*Example 2.* The *Derrida-Lebowitz-Speer-Spohn (DLSS) equation* [26],

$$u_t = -\left(u (\log u)_{xx}\right)_{xx}, \quad (28)$$

corresponding to  $m = 0$ ,  $a = -2$ , and  $b = 1$ .

We notice that (26) can be written in a form which is more convenient for a weak formulation:

$$u_t = -\frac{1}{\beta} \left(u^\alpha (u^\beta)_{xx}\right)_{xx} + \gamma \left(u^{\alpha+\beta-3} u_x^3\right)_x.$$

Equation (26) is recovered by choosing  $m = \alpha + \beta - 1$ ,  $a = \alpha + 3(\beta - 1)$ , and  $b = (\beta - 1)(\alpha + \beta - 2) - \gamma$ . Both formulations are equivalent for smooth positive solutions of (26).

The study of the global existence of weak solutions for the Cauchy problem to Equation (27) was initiated in [11] and further developed in, for instance, [10, 15] (also see the references therein). The first asymptotic results for the thin film equation in the periodic case were obtained in [15] (also see [17] for a more recent reference). Bertozzi and Pugh proved that the constructed solutions converge in  $L^\infty$  towards their average exponentially as  $t \rightarrow \infty$ . These results were further analyzed and complemented in [35]. The only asymptotic result known for the Cauchy problem in the whole line [22] shows in the particular case of  $m = 1$  that, as  $t \rightarrow \infty$ , solutions behave like certain particular self-similar solutions of the problem, by exploiting analogies with the porous medium equation for this particular exponent.

Regarding the DLSS equation (28), the first local-in-time existence result for periodic positive solutions has been given in [18]. The existence of global-in-time non-negative weak solutions has been shown in [32] in the case where both the function and its derivative are prescribed at each end of the interval and later in [27] for periodic solutions. Decay estimates measuring the convergence of the solutions towards their mean for periodic boundary conditions were studied in [19, 27]. Decay rates for different boundary conditions have been proved in [29, 33].

In this paper, we recover and generalize the known results for periodic boundary conditions. Our main contribution is to prove the results for a whole family of entropies, for which we have proved an entropy-entropy production inequality, which generalizes Beckner's inequalities for the heat equation, and to distinguish two regimes, corresponding to a global algebraic decay and asymptotically for large time to an exponential decay. In some cases, approximation procedures already available in the literature allow to extend our results to more general classes of solutions. To avoid technicalities or delicate considerations on the existence and weak formulations of the equations, which are out of the scope of this paper, we will deal only with smooth positive solutions of the equations.



We start with some decay estimates of the entropies  $\Sigma_k[u]$  which are defined as in (24). It is convenient to introduce

$$\begin{aligned} L_{\pm} &:= \frac{1}{4}(3a+5) \pm \frac{3}{4}\sqrt{(a-1)^2 - 8b}, \\ A &:= (k+m+1)^2 - 9(k+m-1)^2 + 12a(k+m-2) - 36b, \end{aligned} \quad (29)$$

where  $a$ ,  $b$  and  $m$  are the coefficients in (26).

**Theorem 3.** *Let  $u$  be a smooth positive solution to (26) and let  $(a-1)^2 \geq 8b$ .*

i) Entropy dissipation: *Let  $k, m \in \mathbb{R}$  such that  $L_- \leq k+m \leq L_+$ . Then*

$$\frac{d}{dt}\Sigma_k[u(\cdot, t)] \leq 0 \quad \forall t > 0.$$

ii) Entropy production: *Let  $k, m \in \mathbb{R}$  such that  $k+m+1 \neq 0$  and  $L_- < k+m < L_+$ . Then  $A$  is positive and*

$$\frac{d}{dt}\Sigma_k[u(\cdot, t)] + \mu \int_{S^1} \left| (u^{(k+m+1)/2})_{xx} \right|^2 dx \leq 0 \quad \forall t > 0,$$

where

$$\mu := \frac{4}{(k+m+1)^4} \min\{(k+m+1)^2, A\}. \quad (30)$$

If  $k+m+1 = 0$  and  $a+b+2 - \mu \leq 0$  for some  $0 < \mu < 1$ , then

$$\frac{d}{dt}\Sigma_k[u(\cdot, t)] + \mu \int_{S^1} |(\log u)_{xx}|^2 dx \leq 0 \quad \forall t > 0.$$

*Example 1.* In the case of the *thin film equation*,  $L_- = 1/2$ ,  $L_+ = 2$ , and

$$\mu = \frac{16}{(k+m+1)^4} (-k-m+2)(2k+2m-1).$$

*Example 2.* In the case of the *DLSS equation*,  $L_- = -1$ ,  $L_+ = 1/2$ ,  $m = 0$ , and

$$\mu = \begin{cases} \frac{4}{(k+1)^2} & \text{if } -1 < k \leq 1/3, \\ \frac{16(1-2k)}{(k+1)^3} & \text{if } 1/3 \leq k < 1/2. \end{cases}$$

For the proof of Theorem 3 we employ the algorithmic entropy construction method recently developed in [31]. This method is based on a reformulation of the task of proving entropy dissipation as a decision problem for polynomial systems.

*Proof.* Formal differentiation of  $\Sigma_k[u(\cdot, t)]$ , employing (26) and integration by parts, leads to

$$\frac{d}{dt}\Sigma_k[u(\cdot, t)] + \int_{S^1} u^{k+m+1} \frac{u_x}{u} \left( -\frac{u_{xxx}}{u} - a \frac{u_x}{u} \frac{u_{xx}}{u} - b \frac{u_x^3}{u^3} \right) dx = 0. \quad (31)$$

In order to prove that the above integral is non-negative for an appropriate choice of the parameters, we use again integration by parts. The possible integration-by-parts

formula are as follows:

$$\begin{aligned} \left( u^{k+m+1} \left( \frac{u_x}{u} \right)^3 \right)_x &= u^{k+m+1} \left( (k+m-2) \left( \frac{u_x}{u} \right)^4 + 3 \left( \frac{u_x}{u} \right)^2 \frac{u_{xx}}{u} \right), \\ \left( u^{k+m+1} \frac{u_x}{u} \frac{u_{xx}}{u} \right)_x &= u^{k+m+1} \left( (k+m-1) \left( \frac{u_x}{u} \right)^2 \frac{u_{xx}}{u} + \left( \frac{u_{xx}}{u} \right)^2 + \frac{u_x}{u} \frac{u_{xxx}}{u} \right), \\ \left( u^{k+m+1} \frac{u_{xxx}}{u} \right)_x &= u^{k+m+1} \left( (k+m) \frac{u_{xxx}}{u} \frac{u_x}{u} + \frac{u_{xxxx}}{u} \right). \end{aligned}$$

Integrating these expressions over  $S^1$  and taking into account the boundary conditions gives

$$\begin{aligned} J_1 &= \int_{S^1} u^{k+m+1} \left( (k+m-2) \left( \frac{u_x}{u} \right)^4 + 3 \left( \frac{u_x}{u} \right)^2 \frac{u_{xx}}{u} \right) dx = 0, \\ J_2 &= \int_{S^1} u^{k+m+1} \left( (k+m-1) \left( \frac{u_x}{u} \right)^2 \frac{u_{xx}}{u} + \left( \frac{u_{xx}}{u} \right)^2 + \frac{u_x}{u} \frac{u_{xxx}}{u} \right) dx = 0, \\ J_3 &= \int_{S^1} u^{k+m+1} \left( (k+m) \frac{u_{xxx}}{u} \frac{u_x}{u} + \frac{u_{xxxx}}{u} \right) dx = 0. \end{aligned}$$

Therefore, we can write the production term

$$P := \int_{S^1} u^{k+m+1} \frac{u_x}{u} \left( -\frac{u_{xxx}}{u} - a \frac{u_x}{u} \frac{u_{xx}}{u} - b \frac{u_x^3}{u^3} \right) dx$$

in (31) as  $P = P + c_1 J_1 + c_2 J_2 + c_3 J_3$  with arbitrary constants  $c_1, c_2$ , and  $c_3 \in \mathbb{R}$ . We wish to find  $c_1, c_2$ , and  $c_3$  such that  $P \geq 0$  or, for some constant  $\mu > 0$ ,

$$P \geq \mu I, \quad \text{where } I := \int_{S^1} \left| (u^{(k+m+1)/2})_{xx} \right|^2 dx.$$

For this task we identify the derivative  $\partial_x^j u/u$  with the variable  $\xi_j$  and deal with the polynomials

$$\begin{aligned} S_0(\xi) &= \xi_1(-\xi_3 - a \xi_1 \xi_2 - b \xi_1^3), \quad \text{which corresponds to } P, \\ T_1(\xi) &= (k+m-2) \xi_1^4 + 3 \xi_1^2 \xi_2, \quad \text{which corresponds to } J_1, \\ T_2(\xi) &= (k+m-1) \xi_1^2 \xi_2 + \xi_2^2 + \xi_1 \xi_3, \quad \text{which corresponds to } J_2, \\ T_3(\xi) &= (k+m) \xi_1 \xi_3 + \xi_4, \quad \text{which corresponds to } J_3, \\ E(\xi) &= \left( \frac{k+m+1}{2} \right)^2 \left( \left( \frac{k+m-1}{2} \right)^2 \xi_1^4 + (k+m-1) \xi_1^2 \xi_2 + \xi_2^2 \right), \\ &\quad \text{which corresponds to } I. \end{aligned}$$

Thus, we need to find constants  $c_i \in \mathbb{R}$  and  $\mu > 0$  such that

$$(S_0 + c_1 T_1 + c_2 T_2 + c_3 T_3)(\xi) \geq \mu E(\xi) \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \xi_4)^\top \in \mathbb{R}^4,$$

which corresponds to a pointwise estimate of the integrand of the production term. The determination of all parameters such that the above inequality is true is called a quantifier elimination problem. In this situation it can be explicitly solved.

In [31] it has been shown that it is sufficient to study polynomials being in *normal form* which leads to the following formulation: Find  $c \in \mathbb{R}$  and  $\mu > 0$  such that for all  $\xi \in \mathbb{R}^4$ ,

$$S(\xi) := (S_0 + c \cdot T_1 + 1 \cdot T_2 + 0 \cdot T_3)(\xi) \geq \mu E(\xi).$$

This inequality is equivalent to

$$\begin{aligned} S_0 + cT_1 + T_2 - \mu E &= \left( c(k+m-2) - b - \frac{\mu}{16}(k+m+1)^2(k+m-1)^2 \right) \xi_1^4 \\ &\quad + \left( 3c + k + m - 1 - a - \frac{\mu}{4}(k+m+1)^2(k+m-1) \right) \xi_1^2 \xi_2 \\ &\quad + \left( 1 - \frac{\mu}{4}(k+m+1)^2 \right) \xi_2^2 \geq 0. \end{aligned}$$

First, let  $k+m+1 \neq 0$ . Setting  $\delta = \mu(k+m+1)^2/4$ , by [31, Lemma 11], the above inequality holds true if and only if  $1 - \delta \geq 0$  and

$$\begin{aligned} 0 &\leq 4(1-\delta) \left( c(k+m-2) - b - \frac{\delta}{4}(k+m-1)^2 \right) \\ &\quad - (3c + k + m - 1 - a - \delta(k+m-1))^2 \\ &= -9 \left( c + \frac{1}{9}((1-\delta)(k+m+1) - 3a) \right)^2 + \frac{1}{9}((1-\delta)(k+m+1) - 3a)^2 \\ &\quad - (1-\delta)(k+m-1)^2 - a^2 + 2a(1-\delta)(k+m-1) - 4(1-\delta)b, \end{aligned}$$

Choosing the minimizing value  $c = -((1-\delta)(k+m+1) - 3a)/9$ , we obtain, after some elementary computations,

$$(1-\delta) \left( (1-\delta)(k+m+1)^2 - 9(k+m-1)^2 + 12a(k+m-2) - 36b \right) \geq 0.$$

Since  $1 - \delta \geq 0$ , this inequality is satisfied if

$$(1-\delta)(k+m+1)^2 - 9(k+m-1)^2 + 12a(k+m-2) - 36b \geq 0$$

or

$$\delta \leq \frac{1}{(k+m+1)^2} \left( (k+m+1)^2 - 9(k+m-1)^2 + 12a(k+m-2) - 36b \right) = \frac{A}{(k+m+1)^4}$$

and hence

$$\mu = \frac{4\delta}{(k+m+1)^2} \leq \frac{4A}{(k+m+1)^2}.$$

Now, let  $k+m+1 = 0$ . The polynomial  $E$ , corresponding to the entropy production term

$$\int_{S^1} |(\log u)_{xx}|^2 dx = \int_{S^1} \left( \left( \frac{u_{xx}}{u} \right)^2 - 2 \frac{u_{xx} u_x^2}{u^2} + \left( \frac{u_x}{u} \right)^4 \right) dx$$

reads as  $E(\xi) = \xi_2^2 - 2\xi_2\xi_1^2 + \xi_1^4$ . Then, again by Lemma 11 of [31], the inequality  $S(\xi) - \mu E(\xi) \geq 0$  is equivalent to  $1 - \mu > 0$  and

$$-9 \left( c - \frac{a}{3} - \frac{2}{3}(1-\mu) \right)^2 - 4(1-\mu)(b + 3c + \mu) \geq 0,$$

and choosing the minimizing value  $c = a/3 + 2(1-\mu)/3$  leads to

$$(1-\mu)(a + b + 2 - \mu) \leq 0.$$

Thus, the condition  $1 - \mu > 0$  implies that  $a + b + 2 - \mu \leq 0$  has to be satisfied. This completes the proof of Theorem 3.  $\square$

**3.3. Fourth-order nonlinear diffusions: decay rates.** Now, we can use again the variational inequalities proved in Section 2 to obtain decay rates for smooth positive solutions. We proceed similarly as in the porous medium case and define for  $k + m + 1 \neq 0$ :

$$v := u^p, \quad p := \frac{k + m + 1}{2}, \quad q := \frac{k + 1}{p} = 2 \frac{k + 1}{k + m + 1}.$$

Then

$$\bar{u} = \int_{S^1} u \, dx = \int_{S^1} v^{1/p} \, dx = (\mu_p[v])^{1/p}.$$

Notice that for  $k > -1$  and  $m > 0$ ,  $q$  takes values in  $(0, 2)$ . As a consequence of Corollaries 1, 2 and Theorem 3, we have the following result.

**Theorem 4.** *Let  $k, m \in \mathbb{R}$  such that  $L_- \leq k + m \leq L_+$  and consider a smooth positive solution of (26).*

i) Short-time Algebraic Decay: *If  $k > -1$  and  $m > 0$ , then*

$$\Sigma_k[u(\cdot, t)] \leq \left[ \Sigma_k[u_0]^{-(2-q)/q} + 4\pi^2 \mu \kappa_{p,q} \left( \frac{2}{q} - 1 \right) t \right]^{-q/(2-q)} \quad \forall t \in \mathbb{R}^+.$$

ii) Asymptotically Exponential Decay: *If  $m + k + 1 > 0$ , then there exists  $C > 0$  and  $t_1 > 0$  such that*

$$\Sigma_k[u(\cdot, t)] \leq \Sigma_k[u(\cdot, t_1)] \exp \left( - \frac{32 p^2 \pi^4 \mu \bar{u}^{p(2-q)} (t - t_1)}{1 + C \sqrt{\Sigma_k[u(\cdot, t_1)]}} \right) \quad \forall t \geq t_1,$$

where  $\mu$  is defined in (30).

*Proof.* Applying Corollary 1 with  $p \in (0, +\infty)$  and  $q \in (0, 2)$ , i.e.  $m > 0$  and  $k > -1$ , we obtain

$$\Sigma_{p,q}[v(\cdot, t)]^{2/q} \leq (4\pi^2 \kappa_{p,q})^{-1} J_2[v(\cdot, t)],$$

which proves i) by using Theorem 3 and integrating the ODE

$$\frac{d}{dt} \Sigma_{p,q}[v(\cdot, t)] \leq -4\pi^2 \mu \kappa_{p,q} \Sigma_{p,q}[v(\cdot, t)]^{2/q}.$$

To prove ii), we claim that  $\lim_{t \rightarrow \infty} \Sigma_k[u(\cdot, t)] = 0$ . As in the proof of Proposition 1 we employ Sobolev-Poincaré inequalities and the entropy estimate of Theorem 3 to obtain, for some constant  $c > 0$ ,

$$\begin{aligned} \int_0^\infty \left\| u^{(k+m+1)/2}(\cdot, s) - \bar{u}^{(k+m+1)/2} \right\|_{L^\infty(S^1)}^2 &\leq c \int_0^\infty \int_{S^1} \left( u^{(k+m+1)/2} \right)_x^2 dx ds \\ &\leq c \int_0^\infty \int_{S^1} \left( u^{(k+m+1)/2} \right)_{xx}^2 dx ds < +\infty. \end{aligned}$$

Now, the argument is as in the proof of Proposition 1, yielding the claim.

Take  $t_1 > 0$  large enough such that  $\Sigma_k[u(\cdot, t_1)] = \epsilon$ . Because of the decay of the entropy,  $\Sigma_k[u(\cdot, t)] \leq \epsilon$  for all  $t \geq t_1$ . A direct application of Corollary 2, with  $p = (k + m + 1)/2 > 0$ , implies that

$$\frac{32 p^2 \pi^4 \mu \bar{u}^{p(2-q)}}{1 + C \sqrt{\epsilon}} \Sigma_{p,q}[v(\cdot, t)] \leq J_2[v(\cdot, t)] \quad \forall t \geq t_1,$$

and thus, using Lemma 2, we conclude that

$$\frac{d}{dt} \Sigma_{p,q}[v(\cdot, t)] \leq - \frac{32 p^2 \pi^4 \mu \bar{u}^{p(2-q)}}{1 + C \sqrt{\epsilon}} \Sigma_{p,q}[v(\cdot, t)] \quad \forall t \geq t_1.$$

Integrating this differential inequality, we obtain ii).  $\square$

A result similar to Corollary 3 can be formulated for the fourth-order equation (26). A global existence result of a suitable concept of solution to the general equation (26) is still lacking. However, Theorem 4 applies to the two examples, for which existence and approximation results of general solutions are known.

As a conclusion, we discuss previously known results and show how we improve them in the case of the two examples.

*Example 1: Thin film equation.* The results of Theorem 4 hold for any  $m > 0$  and  $k \in \mathbb{R}$  such that  $1/2 < k + m < 2$  (asymptotically exponential decay), with the additional restriction  $k > -1$  for the algebraic decay estimate.

In Theorem 4, the solutions were assumed to be positive and smooth. In the case of the thin film equation, these results can be extended to general solutions using the approximation procedure introduced in [15] and also used in [35] to which we refer for more details.

To illustrate our results, let us recall the known results. In [15, 35] global exponential decay of entropies were established in the following range (see Figure 2): (i)  $k = 1 - m/2$ ,  $m \in (0, 2)$ , (ii)  $-1 < k < 2 - m$ ,  $m \in [2, 3)$ , (iii)  $1 - m < k < 2 - m$ ,  $m \in [3, +\infty)$ , and an explicit lower estimate of the global exponential rate was given. The method relies on the regularization procedure of [15], some entropy-entropy dissipation estimates which have been generalized in Section 3.2, and various estimates of the entropies based on Poincaré inequalities. The range  $m \geq 3$  corresponds to entropies with negative exponents,  $k < -1$ ; as in our approach, solutions also need to be bounded away from zero.

For completeness, Laugesen in [34] (see [14, 12, 16] for other references) considered various cases corresponding to  $k \in [-1, 0]$ . However, in [34], decay of energies are primarily considered, like in [21] in the case of  $m = 1$ . Concerning the comparison of the thin film equation with the porous media equation, one has to mention [30] (mostly in view of self-similar solutions on the line). The results of Theorem 4 can then be recovered using Corollaries 1 and 2. The regions in the  $(m, k)$ -plane for which our results apply are shown in Figure 3.

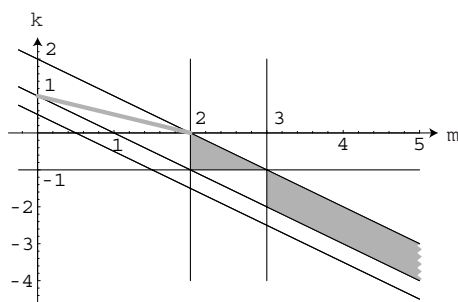


FIGURE 2. Region of parameters for which global exponential decay of the entropy has been shown in [15, 35] for the thin film equation.

*Example 2: The DLSS equation* is not a limit case (when  $m$  goes to 0) of the thin film equation but a special case of (26). The result of asymptotically exponential decay of

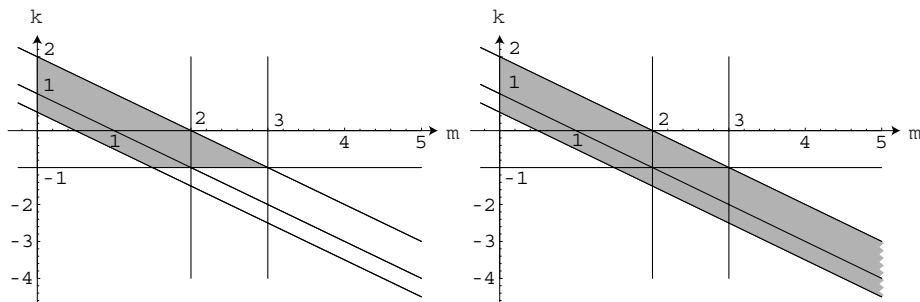


FIGURE 3. Region of parameters for which global algebraic decay of the entropy (left) and asymptotic exponential decay of the entropy (right) is shown by Theorem 4 for the thin film equation.

Theorem 4 (with  $m = 0$ ) holds for any  $k \in (-1, 1/2)$ . For  $k = 0$ , the corresponding asymptotic rate is  $32\pi^4$ , which is the optimal global rate found in [27].

An approximation based on semidiscretization [27] can be used to extend the results for smooth solutions to a larger class of solutions. We refer to [27, 32] for further details. Some partial results are already known. In [19, 27] a global exponential time decay has been proved (with an optimal rate based on a direct entropy-entropy production method in [27]). In [19] convergence was obtained in  $H^1$  under a smallness condition, and in  $\Sigma_k$ -entropies with  $0 < k \leq 1/3$  for general initial data.

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