Variational and viscosity solutions of the Hamilton-Jacobi equation

Patrick Bernard

June 19, 2014

The Equations

$$\partial_t u(t,q) + H(q, \partial_q u(t,q)) = 0 \qquad \text{(HJ)}$$

unknown : $u(t,q) : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$
initial condition $u(0,q) = u_0(q)$.

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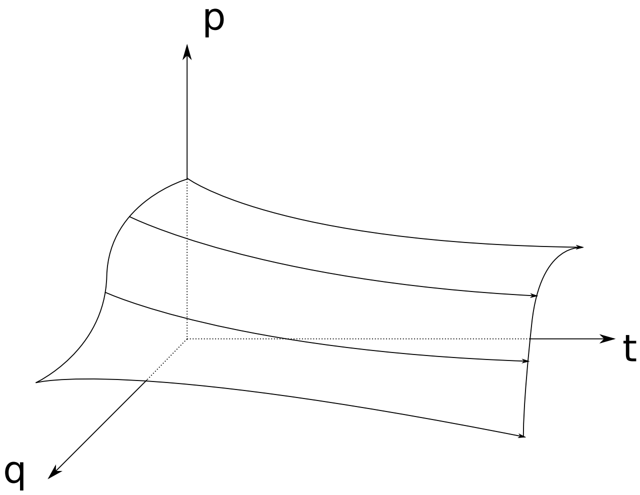
Hamiltonian action of the curve $(Q(t), P(t))$:

$$\mathcal{A}_0^t(Q,P) := \int_0^t P(s) \cdot \dot{Q}(s) - H(s, Q(s), P(s)) ds$$

Geometry of the Equations



graph of $\partial_q u(t, q)$.

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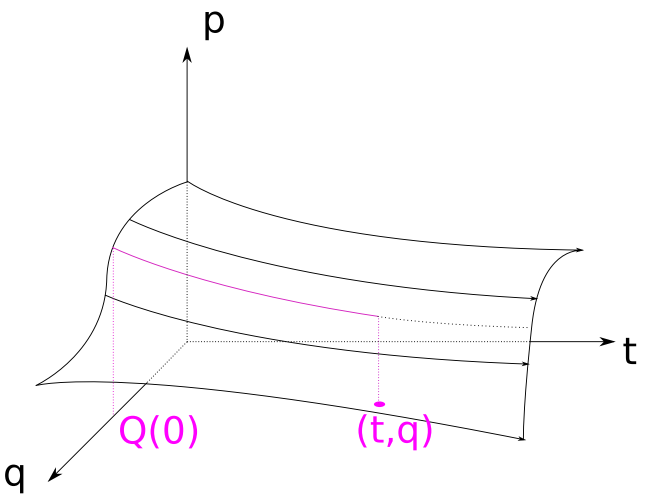

$$u(t,q) = u(0, Q(0)) + \mathcal{A}_0^t(Q, P)$$

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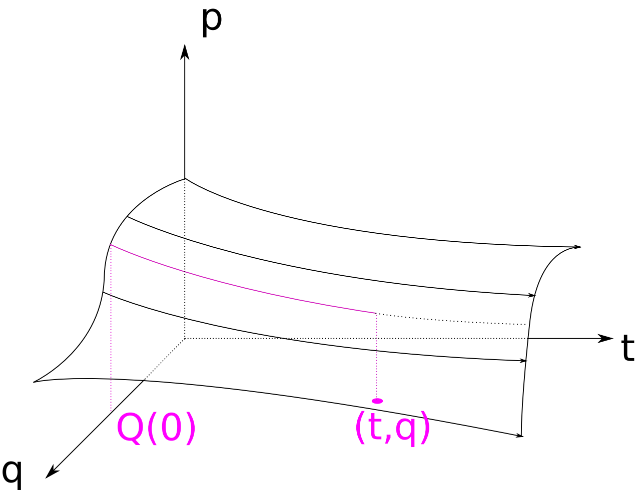

$$Q(t) = q \quad , \quad P(t) = \partial_q u(t, Q(t))$$

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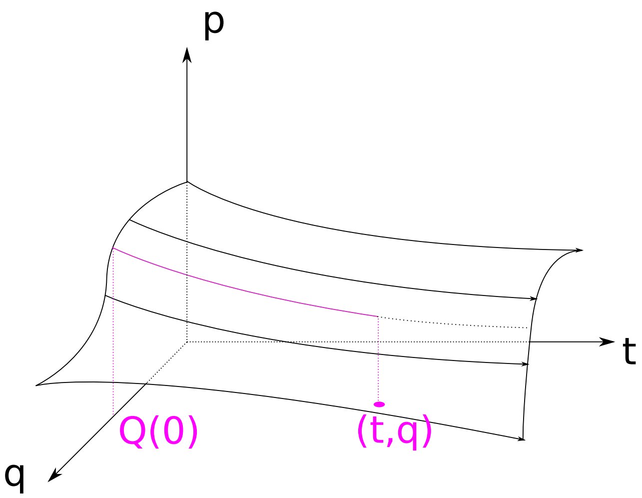

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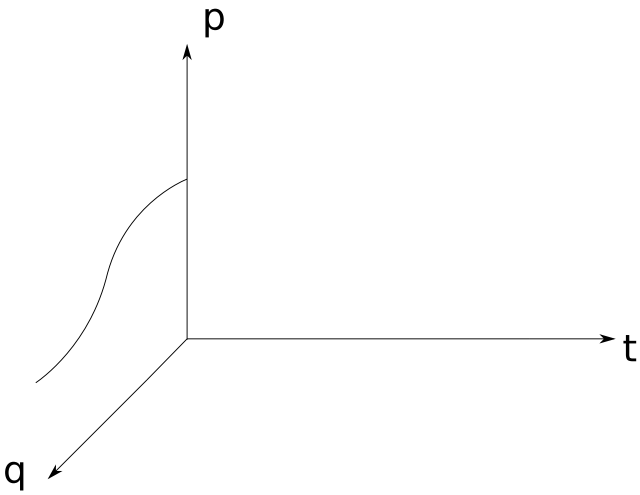
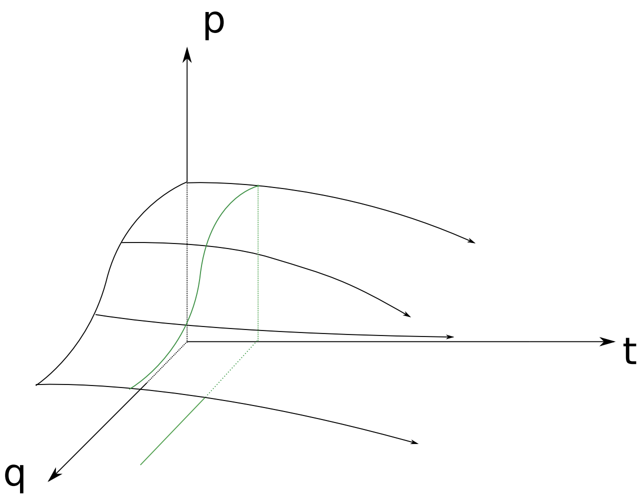
smooth solutions



graph of $q \longmapsto \partial_q u_0(q)$

smooth solutions



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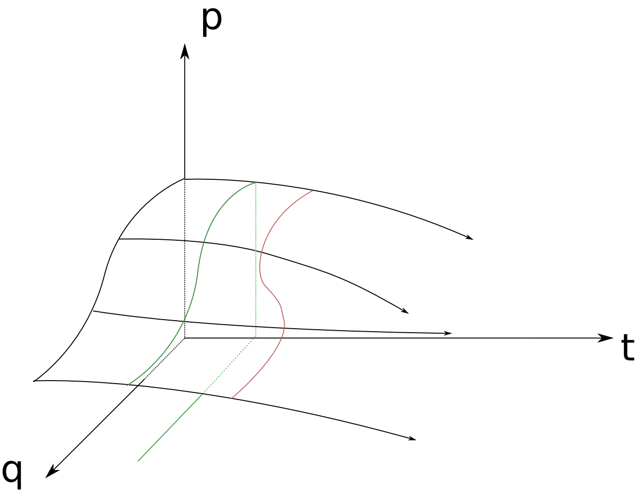

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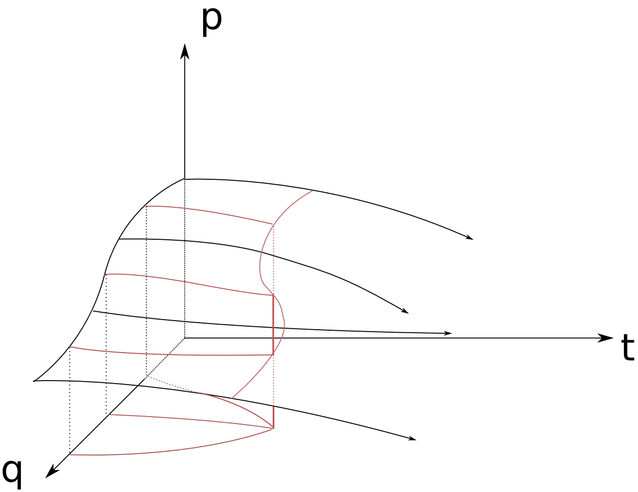

shock!

Quantitative statement

We assume the existence of $A > 0$ such that

$$|H(q,p)| \leqslant A(1+|p|^2), \quad |dH(q,p)| \leqslant A(1+|p|), \quad |d^2H(q,p)| \leqslant A.$$

Theorem

If f_0 is C^2 and $|d^2 f_0| \leqslant K$, then there exists $T > 0$, which depends only on A and K, such that (HJ) has a C^2 solution f on $[0,T] \times \mathbb{R}^d$ with initial condition f_0.

Variational solutions

Definition (Variational solution)

A variational solution of (HJ) (with smooth initial condition u_0) is a function $g(t,q)$ such that, for each (t,q), the real $g(t,q)$ is a critical value of the functional

$$(Q,P) \mapsto u_0(Q(0)) + \mathcal{A}_0^t(Q,P)$$

on the space of curves such that $Q(t) = q$. In other words, for each (t,q), there exists an orbit (Q,P) of the Hamiltonian system such that

$$Q(t) = q \quad , \quad P(0) = du_0(Q(0))$$

$$g(t,q) = u_0(Q(0)) + \mathcal{A}_0^t(Q,P)$$

Variational solutions exist

Theorem (Chaperon, Viterbo)

If u_0 is a C^2 initial condition, then there exists a Lipschitz variational solution $g(t,x) : [0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ of (HJ). This function is also a solution almost everywhere of the equation.

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More precisely, there exists a family $G^t, t \geqslant 0$ of maps from $C^2(\mathbb{R}^d)$ to $Lip(\mathbb{R}^d)$ such that the function $(t,q) \longmapsto G^t u(q)$ is a Lipschitz variational solution and such that

(1) $u \leqslant v \Rightarrow G^t u \leqslant G^t v$.

(2) $G^t(c+u) = c + G^t u$ *for all constant* c.

(3) *If* $u(t,q)$ *is a* C^2 *solution, then* $G^s u_t = u_{t+s}$.

(1) and (2) imply that $\|G^t u - G^t v\|_{C^0} \leqslant \|u - v\|_{C^0}$.

Nonsmooth initial condition

The maps G^t extend to $C^0(\mathbb{R}^d)$, and take values in $C^0(\mathbb{R}^d)$.

We shall rather consider its restriction

$$G^t : Lip(\mathbb{R}^d) \longrightarrow Lip(\mathbb{R}^d).$$

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Proposition

For each (t,q) there exists a trajectory (Q,P) of (HS) such that

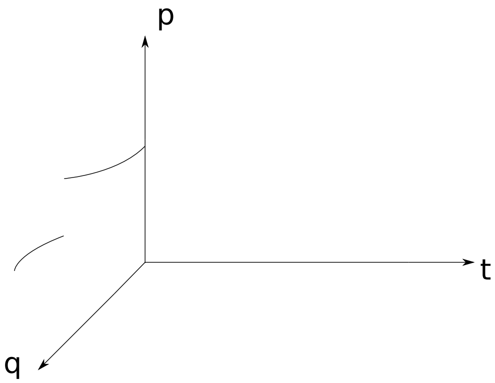
$$Q(t) = q \quad , \quad P(0) \in \partial u_0(Q(0))$$

$$g(t,q) = u_0(Q(0)) + \mathcal{A}_0^t(Q,P)$$

where $\partial u_0(Q(0))$ is the Clarke differential of u_0.

Clarke differential

The Clarke differential $\partial u(x)$ of a Lipschitz function on $\mathbb{R}^d$ at a point x is the compact subset of $\mathbb{R}^d$ generated by limits of sequences of the form $du(q_n)), q_n \longrightarrow q$.



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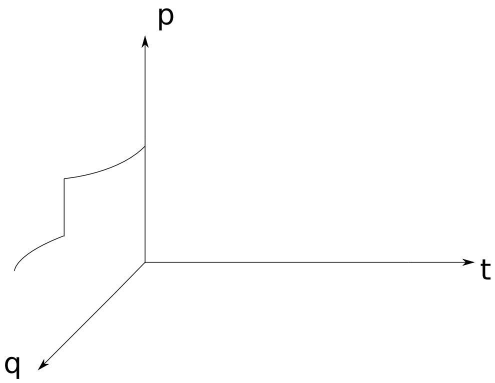

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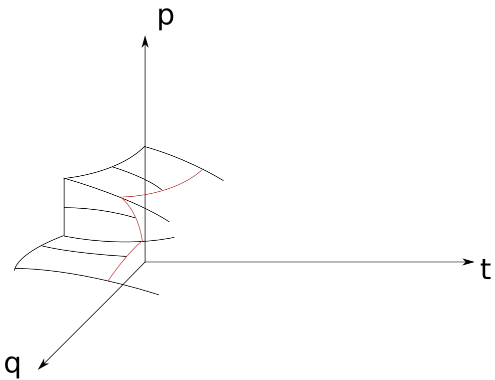

Viscosity solutions

Does the semi-group property $G^{t+s} = G^t \circ G^s$ hold?

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Theorem

There exists a unique family $V^t, t \geqslant 0$ of maps of $Lip(\mathbb{R}^d)$ such that

(1) $u \leqslant v \Rightarrow V^t u \leqslant V^t v$.

(2) $V^t(c + u) = c + G^t u$ *for all constant* c.

(3) *If* $u(t, q)$ *is a* C^2 *solution, then* $V^s u_t = u_{t+s}$.

(4) $V^{t+s} = V^t \circ V^s$

For each u_0, the functions $(t, q) \longmapsto V^t u_0(q)$ is the viscosity solution of (HJ).

Equality

The following properties are equivalent :

- "The" variational resolution G satisfies the semi-group property.
- The viscosity solutions are variational.
- $G = V$.

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- "The" variational resolution G satisfies the semi-group property.
- The viscosity solutions are variational.
- $G = V$.

These properties are not true for all Hamiltonians. In general:

Theorem (Wei)

$$V^t = lim_{n \longrightarrow \infty} (G^{t/n})^n$$

The convex case

In the case where H is convex in p (with $\partial^2_{pp}H > 0$) the equality $V = G$ holds.

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In this case, there is an underlying optimisation problem and the Lax-Oleinik formula holds:

$$V^t u_0(q) = G^t u_0(q) = \min_Q \left(u_0(Q(0)) + \mathcal{A}_0^t(Q, P_Q) \right)$$

on curves Q such that $Q(t) = q$. Here

$$P_Q(s) = argmax\left(p \cdot \dot{Q}(s) - H(Q(s), p) \right).$$

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$$P_Q(s) = argmax \big(p \cdot \dot{Q}(s) - H(Q(s), p) \big).$$

In particular, $G^t u_0(q)$ is the smallest critical value of the functional $u_0 + \mathcal{A}$.

The Hopf formula

Theorem

If $H(q,p) = h(p)$*, and* u_0 *is convex, then*

$$G^t u_0(q) = V^t u_0(q) = \sup_p \big(p \cdot q - u_0^*(p) - th(p)\big)$$

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$$u_0(q) = \sup_p pq - u_0^*(p) = \sup_p f_p(q)$$

and, for each p,

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and, for each p,

$$G^t(f_p) = V^t(f_p) = pq - u_0^*(p) - th(p).$$

In this case, $G^t u_0(q)$ is the largest critical value of $u_0 + \mathcal{A}$.

Generalized Hopf setting

$H(q,p)$ is arbitrary and u_0 is semi-concave.
It means that there exists $K > 0$ such that $q \longmapsto u_0(q) - K\|q\|^2$ is concave.

Theorem

There exists $T > 0$ (which depends only on A and K) such that, for $t \in [0,T]$,

1. *$G^t u_0(q)$ is the smallest critical value of the functional*
$$(Q,P) \longmapsto u_0(Q(0)) + \mathcal{A}_0^t(Q,P)$$
with endpoint $Q(t) = q$.
2. $V^t u_0 \leqslant G^t u_0$

Generalized Hopf setting

A function is semi-concave if and only if it can be written as a minimum of C^2 functions.
More precisely, there exists a family $\mathcal{F}_0 \subset C^2$ such that

$$u_0 = \min_{f_0 \in \mathcal{F}_0} f_0$$

and $\|d^2 f_0(q)\| \leqslant K$ for each $q \in \mathbb{R}^d$, $f_0 \in \mathcal{F}_0$.

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There exists $T > 0$ (which depends only on A and K) such that the Cauchy problem for (HJ) with initial condition f_0 has a C^2 solution f on $[0, T] \times \mathbb{R}^d$, for each $f \in \mathcal{F}_0$. We call $\mathcal{F} \in C^2([0, T] \times \mathbb{R}^d, \mathbb{R})$ the set of these solutions.

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In the Hopf setting (functions f_0 linear, H independant from q), they are equal.
We will see that it's not true in general.
For each $f_0 \in \mathcal{F}_0$, we have $f_0 \geqslant u_0$ hence

$$f_t = G^t f_0 \geqslant G^t u_0 \quad , \quad f_t = V^t f_0 \geqslant G^t u_0$$

$$\min f \geqslant g \quad , \quad \min f \geqslant v$$

The main Lemma

Lemma

Assume that the set $\underline{\mathcal{F}}_0 \subset C^2$ is sufficiently large for the following property to hold:
For each $q \in \mathbb{R}^d$, and each $p \in \partial u_0(q)$, there exists $f_0 \in \underline{\mathcal{F}}_0$ such that

$$f_0(q) = u_0(q) \quad , \quad df_0(q) = p.$$

Then for each $(t, q), t \leqslant T$, each critical value of $u_0 + \mathcal{A}$ under the constraint $Q(t) = q$ is of the form $f(t, q)$ for some $f \in \underline{\mathcal{F}}$.

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- $\Rightarrow g \geqslant \min_{f \in \underline{\mathcal{F}}} f \Rightarrow g = \min_{f \in \underline{\mathcal{F}}} f$.
- Such a $\underline{\mathcal{F}}_0$ exists.
- $g = \min_{f \in \underline{\mathcal{F}}} f \geqslant v$.

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Assume that the set $\underline{\mathcal{F}}_0 \subset C^2$ *is sufficiently large for the following property to hold:*
For each $q \in \mathbb{R}^d$*, and each* $p \in \partial u_0(q)$*, there exists* $f_0 \in \underline{\mathcal{F}}_0$ *such that*

$$f_0(q) = u_0(q) \quad , \quad df_0(q) = p.$$

Then for each $(t, q), t \leqslant T$*, each critical value of* $u_0 + \mathcal{A}$ *under the constraint* $Q(t) = q$ *is of the form* $f(t, q)$ *for some* $f \in \underline{\mathcal{F}}$*.*

- $\Rightarrow g \geqslant \min_{f \in \underline{\mathcal{F}}} f \Rightarrow g = \min_{f \in \underline{\mathcal{F}}} f$.
- Such a $\underline{\mathcal{F}}_0$ exists.
- $g = \min_{f \in \underline{\mathcal{F}}} f \geqslant v$.
- Each critical point of $u_0 + \mathcal{A}$ is larger than $\min_{f \in \underline{\mathcal{F}}} f = g$.

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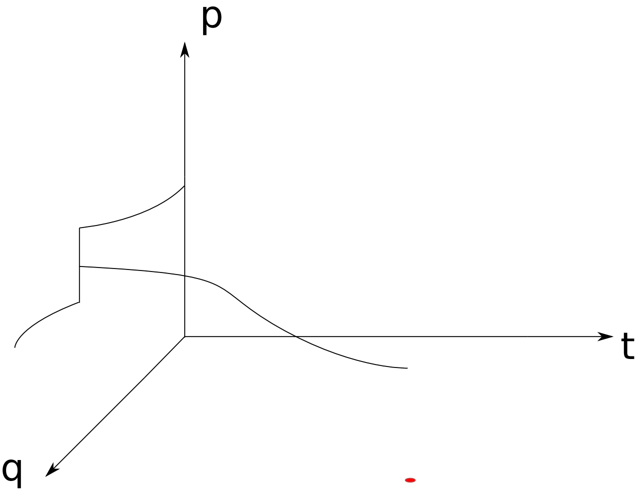
- $\Rightarrow g \geqslant \min_{f \in \underline{\mathcal{F}}} f \Rightarrow g = \min_{f \in \underline{\mathcal{F}}} f$.
- Such a $\underline{\mathcal{F}}_0$ exists.
- $g = \min_{f \in \underline{\mathcal{F}}} f \geqslant v$.
- Each critical point of $u_0 + \mathcal{A}$ is larger than $\min_{f \in \underline{\mathcal{F}}} f = g$.
- Theorem is proved.

Proof of the Lemma



$$(t, q) \quad , \quad C = u_0(Q(0)) + \mathcal{A}_0^t(Q, P)$$

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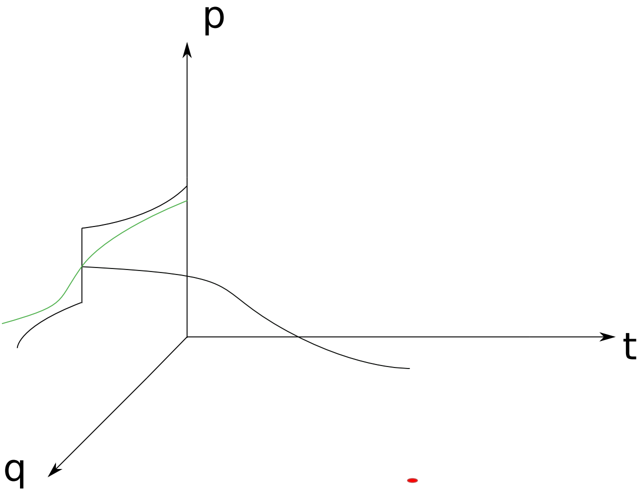
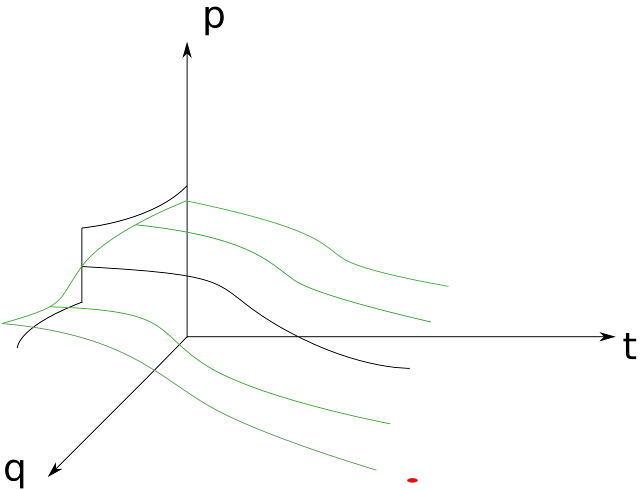


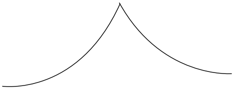
$f_0 \in \underline{\mathcal{F}}_0$ such that $f_0(Q(0)) = u_0(Q(0))$ and $df_0(Q(0)) = P(0)$.

Proof of the Lemma

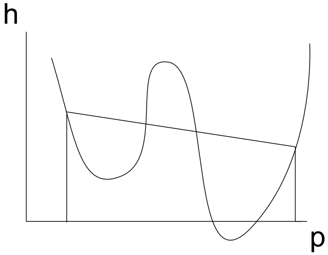


$$f(t,q) = f_0(Q(0)) + \mathcal{A}_0^t(Q,P)$$
$$= u_0(Q(0)) + \mathcal{A}_0^t(Q,P) = C$$

Example

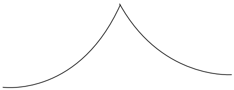


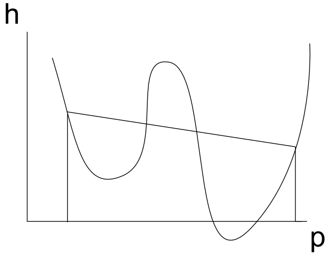
u_0



$H(q,p) = h(p)$

Example

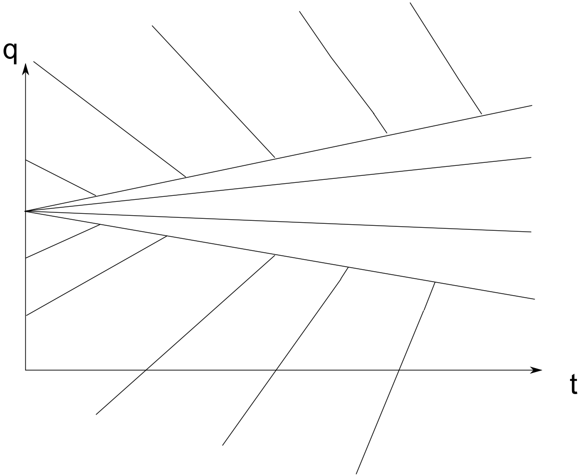




u_0 $\quad$ $H(q,p) = h(p)$

Valentine Roos (extrapolating on Qiaoling Wei) proved that $v < g$:

Example



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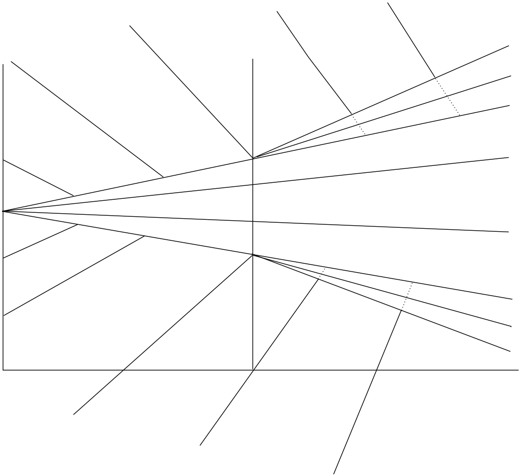

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