KAM for Hamiltonian PDEs

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Two main frontiers of KAM theory for PDEs:

1. **PDEs in higher space dimension**

   **(NLW) Hamiltonian nonlinear wave equation**
   \[ u_{tt} - \Delta u + V(x)u = \partial_u F(x, u), \quad x \in \mathbb{T}^d, \quad u \in \mathbb{R}, \]

   **(NLS) Hamiltonian nonlinear Schrödinger equation**
   \[ iu_t - \Delta u + V(x)u = \partial_{\bar{u}} F(x, u), \quad x \in \mathbb{T}^d, \quad u \in \mathbb{C}, \]

   - KP, etc...

2. **1-d PDEs with derivatives, Quasi-linear, fully-nonlinear PDEs**

   **(KdV) Quasi-linear Hamiltonian KdV**
   \[ u_t + u_{xxx} + \partial_x u^2 + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0, \quad x \in \mathbb{T} \]

   - Elasticity, Klein Gordon, ...
   - Water Waves ...
Nonlinear wave equation (NLW), $d \geq 1$, Philippe Bolle

Nonlinear Schrödinger equation (NLS), $d \geq 1$

1. any space dimension $x \in \mathbb{T}^d$, $d \geq 1$
2. Hamiltonian PDEs, semi-linear nonlinearities $f(x, u)$
3. existence of quasi-periodic solutions,
4. no-reducibility results, no informations on Lyapunov exponents/stability

1-d derivative wave eq., Luca Biasco, Michela Procesi

Quasi-linear KdV, Pietro Baldi, Riccardo Montalto

1. 1-space dimension $x \in \mathbb{T}^1$
2. other algebraic structures: reversibililty, ...
3. quasi-linear/ fully-nonlinear
4. reducibility results, informations on Lyapunov exponents/stability, ...
Techniques:

- **Nash-Moser implicit function theorems**
- **KAM (Kolmogorov-Arnold-Moser) theory**
- **Key**: new perturbative spectral analysis for the linearized PDE on approximate solutions
KdV

\[ \partial_t u + u_{xxx} - 3\partial_x u^2 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0, \quad x \in \mathbb{T} \]

Quasi-linear Hamiltonian perturbation

\[ \mathcal{N}_4 := -\partial_x \{(\partial_u f)(x, u, u_x)\} + \partial_{xx} \{(\partial_{ux} f)(x, u, u_x)\} \]

\[ \mathcal{N}_4 = a_0(x, u, u_x, u_{xx}) + a_1(x, u, u_x, u_{xx})u_{xxx} \]

\[ \mathcal{N}_4(x, \varepsilon u, \varepsilon u_x, \varepsilon u_{xx}, \varepsilon u_{xxx}) = O(\varepsilon^4), \quad \varepsilon \to 0 \]

\[ f(x, u, u_x) = O(|u|^5 + |u_x|^5), \quad f \in C^q(\mathbb{T} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \]

Physically relevant for perturbative derivation from water-waves

- Control the effect of \( \mathcal{N}_4 = O(\varepsilon^4 \partial_{xxx}) \) over INFINITE times...
Hamiltonian PDE

\[ u_t = X_H(u), \quad X_H(u) := \partial_x \nabla_{L^2} H(u) \]

Hamiltonian KdV

\[ H = \int_T \frac{u_x^2}{2} + u^3 + f(x, u, u_x) \, dx \]

Phase space

\[ H^1_0(T) := \left\{ u(x) \in H^1(T, \mathbb{R}) : \int_T u(x) \, dx = 0 \right\} \]

Non-degenerate symplectic form:

\[ \Omega(u, v) := \int_T (\partial_x^{-1} u) v \, dx \]
Goal: look for small amplitude quasi-periodic solutions

Definition: quasi-periodic solution with $n$ frequencies

$$u(t, x) = U(\omega t, x)$$ where $U(\varphi, x) : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$,
$$\omega \in \mathbb{R}^n (= \text{frequency vector})$$ is irrational $\omega \cdot k \neq 0, \forall k \in \mathbb{Z}^n \setminus \{0\}$

$\implies$ the linear flow $\{\omega t\}_{t \in \mathbb{R}}$ is dense on $\mathbb{T}^n$

The torus-manifold

$$\mathbb{T}^n \ni \varphi \mapsto U(\varphi, x) \in \text{phase space}$$

is invariant under the flow evolution of the PDE:

$$\Phi^t_H \circ U = U \circ \Psi^t_\omega$$

"linear rotation" : $\Psi^t_\omega : \mathbb{T}^n \ni \varphi \to \varphi + \omega t \in \mathbb{T}^n$
Linear Airy eq.

\[ u_t + u_{xxx} = 0, \quad x \in \mathbb{T} \]

Solutions: (superposition principle)

\[ u(t, x) = \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j e^{ij^3 t} e^{ijx} \]

Eigenvalues \( j^3 \) = "NORMAL FREQUENCIES"

Eigenfunctions: \( e^{ijx} \) = "NORMAL MODES"

All solutions are 2π-periodic in time: COMPLETELY RESONANT

⇒ Quasi-periodic solutions are a completely nonlinear phenomenon
KdV is completely integrable

\[ u_t + u_{xxx} - 3 \partial_x u^2 = 0 \]

∃ infinitely many prime integrals (Lax). "Action-angle" variables:

**Birkhoff-coordinates**, Kappeler, analytic symplectic diffeo

\[ \Psi : u(x) \mapsto (u_j)_{j \in \mathbb{Z}}, \quad \sum_j du_j \wedge d\tilde{u}_j \]

**New Hamiltonian system:**

\[ (H \circ \Psi)(l_1, l_2, \ldots), \quad l_j := \frac{1}{2} |u_j|^2 = \text{actions} \]

\[ \dot{l}_j = 0, \quad \dot{\varphi}_j = W_j(l), \quad \varphi_j := \arg u_j \]

\[ l_j(t) = \text{prime integrals; frequencies } W_j(l) \text{ depends on the actions } l \]

**All solutions are periodic, quasi-periodic, almost periodic**
What happens adding a small perturbation?

\((\text{Poincare’}: \text{general problem of dynamics } H(I) + \varepsilon P(\varphi, I))\)

1. **KAM theory**: most of these quasi-periodic solutions persists?

2. **Arnold Diffusion**: are there solutions whose Sobolev norm increases as \(t \to +\infty\)?

3. **Birkhoff normal forms/Nekhoroshev theory**: are there upper bounds for the growth of the Sobolev norms?
KAM theory

Kuksin ’98, Kappeler-Pöschel ’03: KAM for KdV

\[ u_t + u_{xxx} + uu_x + \varepsilon \partial_x f(x, u) = 0 \]

1. **SEMILINEAR PERTURBATION** \( \partial_x f(x, u) \)
2. Also true for Hamiltonian perturbations

\[ u_t + u_{xxx} + uu_x + \varepsilon \partial_x |\partial_x|^{1/2} f(x, |\partial_x|^{1/2} u) = 0 \]

of order 2

\[ |j^3 - i^3| \geq i^2 + j^2, \quad i \neq j \implies \text{KdV gains up to 2 spatial derivatives} \]

3. **for QUASI-LINEAR KdV? OPEN PROBLEM**
### Literature: KAM for "unbounded" perturbations

#### Liu-Yuan '10 for Hamiltonian DNLS (and Benjamin-Ono)

\[ iu_t - u_{xx} + M_\sigma u + i\varepsilon f(u, \bar{u})u_x = 0 \]

#### Zhang-Gao-Yuan '11 Reversible DNLS

\[ iu_t + u_{xx} = |u_x|^2 u \]

Less dispersive \(\implies\) more difficult

Extending the Lyapunov-Schmidt approach of Craig-Wayne:

#### Bourgain '96, Derivative NLW

\[ y_{tt} - y_{xx} + my + y_t^2 = 0, \quad m \neq 0, \]
Existence and stability of quasi-periodic solutions:


\[ y_{tt} - y_{xx} + my = g(x, y, y_x, y_t), \quad x \in \mathbb{T} \]

Reversibility in time-space

\[ g(x, y, y_x, -v) = g(x, y, y_x, v), \quad g(-x, y, -y_x, v) = g(x, y, y_x, v) \]

It rules out the nonlinearities \( y_t^3, y_x^3 \). The DNLW equations

\[ y_{tt} - y_{xx} + my = y_t^3, \quad y_{tt} - y_{xx} + my = y_x^3, \]

do not possess periodic, quasi-periodic solutions
For quasi-linear nonlinearities? Formation of singularities? Lax ’64, Klainermann-Majda ’82, for quasi-linear wave eq. Periodic solutions:

Rabinowitz ’71: periodic solutions of

\[ y_{tt} - y_{xx} + \alpha y_t = \varepsilon F(x, t, y, y_t, y_x, y_{tx}, y_{xx}, y_{tt}) \]

The small dissipation \( \alpha y_t \) allows the existence of periodic solutions!

Iooss-Plotinikov-Toland: ’01-’10. Periodic solutions of Gravity Water Waves with Finite or Infinite depth

New ideas for conjugation of linearized operator
Main result:

- Existence and stability of quasi-periodic solutions of KdV eq. under quasi-linear Hamiltonian perturbations

$$\partial_t u + u_{xxx} - 3\partial_x u^2 + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0$$

General method to develop KAM theory for 1-d quasi-linear PDEs
Theorem ('14, P. Baldi, M. Berti, R. Montalto)

Let $f \in C^q$ (with $q := q(n)$ large enough), $f = O(||u, u_x||^5)$. Then, for "generic" choices of the "TANGENTIAL SITES"

$$S := \{-j_n, \ldots, -j_1, j_1, \ldots, j_n\} \subset \mathbb{Z} \setminus \{0\},$$

the Hamiltonian KdV equation

$$\partial_t u + u_{xxx} - 3\partial_x u^2 + N_4(x, u, u_x, u_{xx}, u_{xxx}) = 0, \quad x \in \mathbb{T},$$

possesses small amplitude quasi-periodic solutions with Sobolev regularity $H^s$, $s \leq q$, of the form

$$u = \sum_{j \in S} \sqrt{|\xi_j|} e^{i\omega_j^\infty(\xi)} t e^{ijx} + o(\sqrt{|\xi|}), \quad \omega_j^\infty(\xi) = j^3 + O(|\xi|)$$

for a "Cantor-like" set of "initial conditions" $\xi \in \mathbb{R}^n$ with density 1 at $\xi = 0$. The linearized equations at these quasi-periodic solutions are reduced to constant coefficients and are linearly stable.
Remarks: a similar result holds for

**cubic perturbations:** \( a \in \mathbb{R} \)

\[
\partial_t u + u_{xxx} + \partial_x u^2 + au^3 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0
\]

**mKdV: focusing/defocusing**

\[
\partial_t u + u_{xxx} \pm \partial_x u^3 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0
\]

**gKdV, generalized KdV (not integrable)**

\[
\partial_t u + u_{xxx} \pm \partial_x u^p + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0
\]

by Birkhoff normal form techniques of Procesi-Procesi
The restriction of $C_\varepsilon$ is not technical! Outside: "Chaos", "homoclinic/heteroclinics solutions", "Arnold Diffusion", .... "Growth of Sobolev norms in 2-d cubic NLS"

$$iu_t - \Delta u = |u|^2 u, \quad x \in \mathbb{T}^2$$

Colliander-Keel-Staffilani-Takaoka-Tao, Invent. Math. 2010

For Differentiable nonlinearities $f \in C^q$ the "chaotic effects" are stronger... and KAM theory more difficult
Linear stability

(L): linearized equation $\frac{\partial h}{\partial t} = \partial_x \partial_u \nabla H(u(\omega t, x))h$

\[ h_t + a_3(\omega t, x)h_{xxx} + a_2(\omega t, x)h_{xx} + a_1(\omega t, x)h_x + a_0(\omega t, x)h = 0 \]

There exists a quasi-periodic (Floquet) change of variable

\[ h = \Phi(\omega t)(\psi, \eta, v), \quad \psi \in \mathbb{T}^\nu, \eta \in \mathbb{R}^\nu, v \in H^s_x \cap L^2_s \]

which transforms (L) into the constant coefficients system

\[
\begin{align*}
\dot{\psi} &= b\eta \\
\dot{\eta} &= 0 \\
\dot{v}_j &= i\mu_j v_j, \quad j \notin S, \ \mu_j \in \mathbb{R}
\end{align*}
\]

\[ \eta(t) = \eta_0, v_j(t) = v_j(0)e^{i\mu_j t} \implies \|v(t)\|_s = \|v(0)\|_s : \text{stability} \]
Forced quasi-linear perturbations of Airy

Use \( \omega = \lambda \vec{\omega} \in \mathbb{R}^n \) as 1-dim. parameter

**Theorem (Baldi, Berti, Montalto, Math. Annalen 2014)**

Let \( \vec{\omega} \in \mathbb{R}^n \) diophantine. For every *quasi-linear Hamiltonian* nonlinearity \( f \) the perturbed Airy equation

\[
\partial_t u + \partial_{xxx} u + \varepsilon f(\lambda \vec{\omega} t, x, u, u_x, u_{xx}, u_{xxx}) = 0
\]

has a small quasi-periodic solution \( u \) with frequency \( \omega = \lambda \vec{\omega} \) for all

\[
\lambda \in C_\varepsilon \subset [1/2, 3/2], \quad \lim_{\varepsilon \to 0} |C_\varepsilon| = 1
\]
Bifurcation problem: Let $\mathcal{F} : [0, \varepsilon_0) \times H^s \to H^{s-3}$ be

$$
\mathcal{F}(\varepsilon, u) := \omega \cdot \partial \varphi u + \partial_{xxx} u + \varepsilon f(\varphi, x, u, u_x, u_{xx}, u_{xxx})
$$

Look for $u(\varphi, x)$ zeros $\mathcal{F}(\varepsilon, u) = 0$.

Small amplitude solutions:

$$
\mathcal{F}(0, 0) = 0, \quad D_u \mathcal{F}(0, 0) = \omega \cdot \partial \varphi + \partial_{xxx}
$$

eigenvectors: $e^{il \cdot \varphi} e^{ijx}$

eigenvalues: $i(-\omega \cdot l + j^3)$

Assumption: non-resonant case: small divisors

$$
|\omega \cdot l - j^3| \geq \frac{\gamma}{1 + |l|^\tau}, \quad \forall (l, j) \in \mathbb{Z}^n \times \mathbb{Z}, j \neq 0, \tau > 0
$$

$\implies$ $D_u \mathcal{F}(0, 0)$ is invertible, but the inverse is unbounded:

$$(\omega \cdot \partial \varphi + \partial_{xxx})^{-1} : H^s \to H^{s-\tau}, \quad \tau := "LOSS \ OF \ DERIVATIVES"$$
Nash-Moser Implicit Function Theorem

Newton tangent method for zeros of $\mathcal{F}(u) = 0$ + "smoothing":

$$u_{n+1} := u_n - S_n(D_u\mathcal{F})^{-1}(u_n)\mathcal{F}(u_n)$$

where $S_n$ are regularizing operators (= "mollifiers")

- **Advantage:** QUADRATIC scheme

$$\|u_{n+1} - u_n\|_s \leq C(n)\|u_n - u_{n-1}\|_s^2$$

$\implies$ convergent also if $C(n) \to +\infty$

- **Difficulty:** invert $(D_u\mathcal{F})(u)$ in a whole neighborhood of the expected solution with good *tame* estimates of the inverse
For KdV: linearized equation on an approximate solution

\[ h \rightarrow (D_u \mathcal{F})(u, \varepsilon)[h] := \]
\[ \omega \cdot \partial \varphi + \partial_{xxx} + \varepsilon (a_3(\varphi, x) \partial_{xxx} + a_2(\varphi, x) \partial_{xx} + a_1(\varphi, x) \partial_x + a_0(\varphi, x)) \]

- Linear differential operator with non-constant coefficients
- not diagonal in Fourier basis
- "singular" perturbation problem: \( L_\omega^{-1} T \) is unbounded

\[ L_\omega := \omega \cdot \partial \varphi - \partial_{xxx} \]
\[ T := a_3(\varphi, x) \partial_{xxx} + a_2(\varphi, x) \partial_{xx} + a_1(\varphi, x) \partial_x + a_0(\varphi, x) \]
Key: spectral analysis of quasi-periodic operator

\[ \mathcal{L} = \omega \cdot \partial \varphi + \partial_{xxx} + a_3(\varphi, x) \partial_{xxx} + a_2(\varphi, x) \partial_{xx} + a_1(\varphi, x) \partial_x + a_0(\varphi, x) \]

\[ a_i = O(\varepsilon), \quad i = 0, 1, 2, 3 \]

Main problem: the non constant coefficients term \( a_3(\varphi, x) \partial_{xxx} \)!

- Usual KAM iterative scheme to diagonalize \( \mathcal{L} \) is unbounded!
Idea to conjugate $\mathcal{L}$ to a diagonal operator

1. "REDUCTION IN DECREASING SYMBOLS"

   $\mathcal{L}_1 := \Phi^{-1} \mathcal{L} \Phi = \omega \cdot \partial \varphi + m_3 \partial_{xxx} + m_1 \partial_x + R_0$

   - $R_0(\varphi, x)$ pseudo-differential operator of order 0, $R_0 = O(\varepsilon)$,
   - $m_3 = 1 + O(\varepsilon)$, $m_1 = O(\varepsilon)$, $m_1, m_3 \in \mathbb{R}$, constants

   Use Egorov type theorem!

2. "REDUCTION OF THE SIZE of $R_0$"

   $\mathcal{L}_n = \omega \cdot \partial \varphi + m_3 \partial_{xxx} + m_1 \partial_x + r^{(n)} + \mathcal{R}_n$

   - KAM quadratic scheme: $\mathcal{R}_n = O(\varepsilon^{2n})$, $r^{(n)} = \text{diag}_{j \in \mathbb{Z}}(r_j^{(n)})$, 
Higher order term

\[ \mathcal{L} := \omega \cdot \partial \varphi + \partial_{xxx} + \varepsilon a_3(x) \partial_{xxx} \]

STEP 1: Under the **symplectic** change of variables

\[ \Phi u := (1 + \beta_x(x))u(x + \beta(x)) \]

we get

\[ \mathcal{L}_1 := \Phi^{-1} \mathcal{L} \Phi = \omega \cdot \partial \varphi + (\Phi^{-1}(1 + \varepsilon a_3)(1 + \beta_x)^3) \partial_{xxx} + O(\partial_{xx}) \]
\[ = \omega \cdot \partial \varphi + m_3 \partial_{xxx} + O(\partial_{xx}) \]

imposing

\[ (1 + \varepsilon a_3)(1 + \beta_x)^3 = m_3, \]

There exist solution \( \beta = O(\varepsilon), m_3 \approx 1, \)
\( \mathcal{L}_1 \) has the leading order with **constant coefficients**
A general approach for quasi-linear PDEs:

The family of symplectic transformations

\[ u(x) \mapsto (1 + \beta_x(x))u(x + \tau \beta(x)), \quad \tau \in [0, 1], \]

are the flow of the time dependent Hamiltonian "transport eq."

\[ \partial_\tau u = \partial_x(b(\tau, x)u), \quad b(\tau, x) := \frac{\beta(x)}{1 + \tau \beta_x(x)} \tag{1} \]

**Question:**

How a pseudo-differential operator, here

\[ P_0 = (1 + \varepsilon a_3(x))\partial_{xxx}, \quad p_0(x, \xi) = i(1 + \varepsilon a_3(x))\xi^3, \]

transforms under the flow \( \Phi_{\tau_{\tau_0}} : H^s_x \to H^s_x \) of (1)?
Egorov Theorem:

The transformed operator

\[ P(\tau) := \Phi_0^\tau P_0(\Phi_0^\tau)^{-1} \]

is a pseudo-differential operator of the same order of \( P_0 \), here 3, whose principle symbol \( p(\tau, x, \xi) \) is obtained by the principal symbol \( p_0(x, \xi) = i(1 + \varepsilon a_3(x))\xi^3 \) of \( P_0 \), following the Hamiltonian flow \( \Psi_A^\tau : T \times \mathbb{R} \mapsto T \times \mathbb{R} \) of the classical Hamiltonian \( A := b(\tau, x)\xi \) (associated to \( \partial_\tau u = b(\tau, x)\partial_x u + \ldots \)), namely

\[ P(\tau) = \text{Op}(p(\tau, x, i\partial_x)) + \ldots, \quad p(\tau, x, \xi) = p_0 \circ \Psi_A^{\tau;0}(x, \xi) \]
PDEs in dimension $d \geq 2$

Main difficulties:

1) the eigenvalues of $-\Delta + V(x)$ appear in clusters of increasing size

For example $-\Delta e^{ij \cdot x} = |j|^2 e^{ij \cdot x}$ then $|j|^2 = |j_0|^2, j \in \mathbb{Z}^d$

2) The eigenfunctions of $-\Delta + V(x)$ may be "NOT localized with respect to exponentials"! (Feldman- Knörrer-Trubowitz)

$\implies$ often used pseudo-PDE with Fourier multipliers

$$iu_t - \Delta u + M_\sigma u = \varepsilon f , \quad M_\sigma e^{ij \cdot x} = m_\sigma e^{ij \cdot x}$$

and $m_\sigma$ are used as parameters
Literature: \( d \geq 2 \): quasi-periodic solutions

- Newton method, 1\(^{th}\) order Melnikov
  - Bourgain, Annals '98, '05, NLS and NLW with Fourier multipliers
  - Wang, '11 completely resonant NLS-NLW,
  - Berti-Bolle, '10-'12, forced NLS-NLW, finite regularity, \( V(x) \) multiplicative potential

- KAM theory: 2\(^{th}\) order Melnikov
  - Kuksin-Eliasson, Annals '10, NLS with Fourier multipliers
  - Procesi-Procesi '11, completely resonant NLS
Forced NLS and NLW

We look for quasi-periodic solutions of Hamiltonian

\[ \text{(NLS)} \quad iu_t - \Delta u + V(x)u = \varepsilon f(\omega t, x, u) \]

\[ \omega = \lambda \bar{\omega}, \quad \lambda \approx 1 \]

in a **FIXED** diophantine direction

\[ |\bar{\omega} \cdot \ell| \geq \frac{C_0}{|\ell|^{\gamma_0}}, \quad \forall \ell \in \mathbb{Z}^\nu \setminus \{0\}, \]

In **FINITE DIMENSION** Eliasson ’89 and Bourgain ’94
Theorem (M. Berti, Philippe Bolle, JEMS ’11)

**Existence:**  \( \exists s := s(d, \nu), k := k(d, \nu) \in \mathbb{N}, \text{ such that: } \)
\( \forall V, f \in C^k, \text{there exist } \varepsilon_0 > 0, \text{ such that } \forall 0 < \varepsilon < \varepsilon_0, \text{there is} \)
\( u(\varepsilon, \cdot) \in C^1([1/2, 3/2]; H^s) \text{ with } \sup_{\lambda \in [1/2, 3/2]} \| u(\varepsilon, \lambda) \|_s \xrightarrow{\varepsilon \to 0} 0, \)

and a Cantor like set
\( C_\varepsilon \subset [1/2, 3/2] \text{ with } \lim_{\varepsilon \to 0} |C_\varepsilon| = 1, \)

such that, \( \forall \lambda \in C_\varepsilon, u(\varepsilon, \lambda) \) is a solution of NLS with \( \omega = \lambda \bar{\omega} . \)

**Regularity:** If \( V, f \in C^\infty \) then \( u \in C^\infty \) in space and time.

- A similar result holds for NLW

\[ (\omega \cdot \partial_\varphi)^2 u - \Delta u + V(x)u = \varepsilon f(\varphi, x, u) \]
KEY STEP: For "most" parameters $\lambda \in [1/2, 3/2]$ the linearized operator

$$L_\varepsilon(\lambda) := (\lambda \bar{\omega} \cdot \partial \varphi)^2 - \Delta + V(x) + \varepsilon(\partial u f)(\varphi, x, u(\varphi, x))$$

is invertible and TAME estimate in HIGHER Sobolev norms, i.e.

$$\|L_\varepsilon^{-1}(\lambda) h\|_s \leq \|h\|_{s+\tau} \|u\|_{s_0} + \|h\|_{s_0} \|u\|_s, \quad \forall s_0 \leq s \leq k$$

- **Step 1)** $L^2$-estimates: lower bounds for the eigenvalues of the self adjoint operator $L_\varepsilon(\lambda)$: eigenvalues are smooth in $\lambda \in [1/2, 3/2]$

- **Step 2)** Tame-estimates in high norm

   **Key observation:** many eigenvalues are not small!
Separation properties of singular sites

**Singular sites**: \((\ell, j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d\) such that

- **NLW** \(|-(\omega \cdot \ell)^2 + |j|^2 + m| < \rho\)

- **NLS** \(|-\omega \cdot \ell + |j|^2 + m| < \rho\)

must be more and more "rare" as \(\rho \to 0\)

- **(NLW)** Integer points near a "cone"
- **(NLS)** Integer points near a "paraboloid"

**GROUP THE SINGULAR SITES INTO LARGE CLUSTERS**
Next step:

**KAM for autonomous NLW with multiplicative potential:**

\[ u_{tt} - \Delta u + V(x)u = a(x)u^3 + O(u^4) \]

in preparation with Philippe Bolle

*Further difficulties:*
- bifurcation analysis
- the tangential and the normal variables are coupled
Happy Birthday Ivar!!