KAM for Hamiltonian PDEs

Massimiliano Berti

Conference in honour of Ivar Ekeland
Paris Dauphine, 18-20- June 2014,

Two main frontiers of **KAM theory for PDEs**:

1. *PDEs in higher space dimension*

(NLW) Hamiltonian nonlinear wave equation

$$u_{tt} - \Delta u + V(x)u = \partial_u F(x,u)\,, \quad x \in \mathbb{T}^d\,, \quad u \in \mathbb{R}\,,$$

(NLS) Hamiltonian nonlinear Schrödinger equation

$$\mathrm{i}u_t - \Delta u + V(x)u = \partial_{\bar u} F(x,u)\,, \quad x \in \mathbb{T}^d\,, \; u \in \mathbb{C}\,,$$

- KP, etc. . .

2. *1-d PDEs with derivatives, Quasi-linear, fully-nonlinear PDEs*

(KdV) Quasi-linear Hamiltonian KdV

$$u_t + u_{xxx} + \partial_x u^2 + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0\,, \quad x \in \mathbb{T}$$

- Elasticity, Klein Gordon, . . .
- Water Waves . . .

KAM for PDEs

- **Nonlinear wave equation** (NLW), $d \geq 1$, *Philippe Bolle*
- **Nonlinear Schrödinger equation** (NLS), $d \geq 1$
  1. any space dimension $x \in \mathbb{T}^d$, $d \geq 1$
  2. Hamiltonian PDEs, semi-linear nonlinearities $f(x, u)$
  3. existence of quasi-periodic solutions,
  4. no-reducibility results, no informations on Lyapunov exponents/stability

- 1-**d derivative wave eq.,** *Luca Biasco, Michela Procesi*
  Quasi-linear KdV, *Pietro Baldi, Riccardo Montalto*
  1. 1-space dimension $x \in \mathbb{T}^1$
  2. other algebraic structures: reversibililty, ...
  3. quasi-linear/ fully-nonlinear
  4. reducibility results, informations on Lyapunov exponents/stability, ...

Techniques:

- Nash-Moser implicit function theorems
- KAM (Kolmogorov-Arnold-Moser) theory

- Key: new perturbative spectral analysis for the linearized PDE on approximate solutions

KdV

$$\partial_t u + u_{xxx} - 3\partial_x u^2 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0\,, \quad x \in \mathbb{T}$$

Quasi-linear Hamiltonian perturbation

$$\mathcal{N}_4 := -\partial_x\{(\partial_u f)(x, u, u_x)\} + \partial_{xx}\{(\partial_{u_x} f)(x, u, u_x)\}$$

$$\mathcal{N}_4 = a_0(x, u, u_x, u_{xx}) + a_1(x, u, u_x, u_{xx})u_{xxx}$$

$$\mathcal{N}_4(x, \varepsilon u, \varepsilon u_x, \varepsilon u_{xx}, \varepsilon u_{xxx}) = O(\varepsilon^4)\,, \quad \varepsilon \to 0$$

$$f(x, u, u_x) = O(|u|^5 + |u_x|^5),\ f \in C^q(\mathbb{T} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$$

Physically relevant for perturbative derivation from water-waves

- Control the effect of $\mathcal{N}_4 = O(\varepsilon^4 \partial_{xxx})$ over INFINITE times...

Hamiltonian PDE

$$u_t = X_H(u)\,, \quad X_H(u) := \partial_x \nabla_{L^2} H(u)$$

Hamiltonian KdV

$$H = \int_{\mathbb{T}} \frac{u_x^2}{2} + u^3 + f(x, u, u_x) dx$$

Phase space

$$H_0^1(\mathbb{T}) := \left\{ u(x) \in H^1(\mathbb{T}, \mathbb{R}) \ : \ \int_{\mathbb{T}} u(x) dx = 0 \right\}$$

Non-degenerate symplectic form:

$$\Omega(u, v) := \int_{\mathbb{T}} (\partial_x^{-1} u)\, v\, dx$$

Goal: look for small amplitude quasi-periodic solutions

Definition: quasi-periodic solution with n frequencies

$$u(t,x) = U(\omega t, x) \text{ where } U(\varphi, x) : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R},$$

$$\omega \in \mathbb{R}^n (= \textit{frequency vector}) \text{ is irrational } \omega \cdot k \neq 0\,,\ \forall k \in \mathbb{Z}^n \setminus \{0\}$$

$$\Longrightarrow \text{the linear flow } \{\omega t\}_{t \in \mathbb{R}} \text{ is DENSE on } \mathbb{T}^n$$

The torus-manifold

$$\mathbb{T}^n \ni \varphi \mapsto U(\varphi, x) \in \text{phase space}$$

is invariant under the flow evolution of the PDE:

$$\Phi_H^t \circ U = U \circ \Psi_\omega^t$$

$$\text{"}\textit{linear rotation}\text{"} : \ \Psi_\omega^t : \mathbb{T}^n \ni \varphi \to \varphi + \omega t \in \mathbb{T}^n$$

Linear Airy eq.

$$u_t + u_{xxx} = 0, \qquad x \in \mathbb{T}$$

Solutions: (superposition principle)

$$u(t,x) = \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j e^{\mathrm{i} j^3 t} e^{\mathrm{i} j x}$$

Eigenvalues j^3 = "NORMAL FREQUENCIES"
Eigenfunctions: $e^{\mathrm{i} j x}$ = "NORMAL MODES"

All solutions are 2π- periodic in time: COMPLETELY RESONANT

$\Rightarrow$ **Quasi-periodic solutions are a completely nonlinear phenomenon**

KdV is completely integrable

$$u_t + u_{xxx} - 3\partial_x u^2 = 0$$

$\exists$ infinitely many prime integrals (Lax). "Action-angle" variables:

Birkhoff-coordinates, Kappeler, analytic symplectic diffeo

$$\Psi : u(x) \mapsto (u_j)_{j\in\mathbb{Z}}\,, \quad \sum_j du_j \wedge d\bar{u}_j$$

New Hamiltonian system:

$$(H \circ \Psi)(I_1, I_2, \ldots)\,, \quad I_j := \tfrac{1}{2}|u_j|^2 = \mathrm{actions}$$
$$\dot{I}_j = 0\,, \quad \dot{\varphi}_j = W_j(I)\,, \quad \varphi_j := \arg u_j$$

$I_j(t) =$ prime integrals; frequencies $W_j(I)$ depends on the actions I

All solutions are periodic, quasi-periodic, almost periodic

Perturbed KdV

WHAT HAPPENS ADDING A SMALL PERTURBATION ?

$\Big($*Poincare': general problem of dynamics* $H(I) + \varepsilon P(\varphi, I)\Big)$

1. **KAM theory: most of these quasi-periodic solutions persists?**
2. **Arnold Diffusion: are there solutions whose Sobolev norm increases as $t \to +\infty$?**
3. **Birkhoff normal forms/Nekhoroshev theory: are there upper bounds for the growth of the Sobolev norms?**

KAM theory

Kuksin '98, Kappeler-Pöschel '03: KAM for KdV

$$u_t + u_{xxx} + uu_x + \varepsilon \partial_x f(x, u) = 0$$

1. SEMILINEAR PERTURBATION $\partial_x f(x, u)$
2. Also true for Hamiltonian perturbations

$$u_t + u_{xxx} + uu_x + \varepsilon \partial_x |\partial_x|^{1/2} f(x, |\partial_x|^{1/2} u) = 0$$

   of order 2

$|j^3 - i^3| \geq i^2 + j^2,\ i \neq j \implies$ KdV gains up to 2 spatial derivatives

3. for QUASI-LINEAR KdV? OPEN PROBLEM

Literature: KAM for "unbounded" perturbations

Liu-Yuan '10 for Hamiltonian DNLS (and Benjamin-Ono)

$$\mathrm{i} u_t - u_{xx} + M_\sigma u + \mathrm{i}\varepsilon f(u, \bar{u}) u_x = 0$$

Zhang-Gao-Yuan '11 Reversible DNLS

$$\mathrm{i} u_t + u_{xx} = |u_x|^2 u$$

Less dispersive $\Longrightarrow$ more difficult

Extending the Lyapunov-Schmidt approach of Craig-Wayne:

Bourgain '96, Derivative NLW

$$y_{tt} - y_{xx} + my + y_t^2 = 0 \,, \qquad m \neq 0,$$

Existence and stability of quasi-periodic solutions:

Berti-Biasco-Procesi, Ann. Sci. ENS '13, Arch. Rat. Mech. '14

$$y_{tt} - y_{xx} + my = g(x, y, y_x, y_t), \quad x \in \mathbb{T}$$

Reversibility in time-space

$$g(x, y, y_x, -v) = g(x, y, y_x, v), \; g(-x, y, -y_x, v) = g(x, y, y_x, v)$$

It rules out the nonlinearities y_t^3, y_x^3. The DNLW equations

$$y_{tt} - y_{xx} + my = y_t^3, \quad y_{tt} - y_{xx} + my = y_x^3,$$

do **not** possess periodic, quasi-periodic solutions

For quasi-linear nonlinearities? Formation of singularities?
Lax '64, Klainermann-Majda '82, for quasi-linear wave eq.
Periodic solutions:

Rabinowitz '71: periodic solutions of

$$y_{tt} - y_{xx} + \alpha y_t = \varepsilon F(x, t, y, y_t, y_x, y_{tx}, y_{xx}, y_{tt})$$

The small dissipation αy_t allows the existence of periodic solutions!

Iooss-Plotinikov-Toland: '01-'10. Periodic solutions of

Gravity Water Waves with Finite or Infinite depth

New ideas for conjugation of linearized operator

Main result:

- Existence and stability of quasi-periodic solutions of KdV eq. under QUASI-LINEAR HAMILTONIAN perturbations

$$\partial_t u + u_{xxx} - 3\partial_x u^2 + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0$$

General method to develop KAM theory for 1-d quasi-linear PDEs

Theorem ('14, P. Baldi, M. Berti, R. Montalto)

Let $f \in C^q$ (with $q := q(n)$ large enough), $f = O(|(u, u_x)|^5)$. Then, for "generic" choices of the "TANGENTIAL SITES"

$$S := \{-\bar{\jmath}_n, \dots, -\bar{\jmath}_1, \bar{\jmath}_1, \dots, \bar{\jmath}_n\} \subset \mathbb{Z} \setminus \{0\},$$

the Hamiltonian KdV equation

$$\partial_t u + u_{xxx} - 3\partial_x u^2 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0, \quad x \in \mathbb{T},$$

possesses small amplitude quasi-periodic solutions with Sobolev regularity H^s, $s \leq q$, of the form

$$u = \sum_{j \in S} \sqrt{\xi_j}\, e^{\mathrm{i}\omega_j^\infty(\xi)\, t} e^{\mathrm{i}jx} + o(\sqrt{\xi}), \ \omega_j^\infty(\xi) = j^3 + O(|\xi|)$$

for a "Cantor-like" set of "initial conditions" $\xi \in \mathbb{R}^n$ with density 1 at $\xi = 0$. The linearized equations at these quasi-periodic solutions are reduced to constant coefficients and are linearly **stable**.

Remarks: a similar result holds for

cubic perturbations: $a \in \mathbb{R}$

$$\partial_t u + u_{xxx} + \partial_x u^2 + a u^3 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0$$

mKdV: focusing/defocusing

$$\partial_t u + u_{xxx} \pm \partial_x u^3 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0$$

gKdV, generalized KdV (not integrable)

$$\partial_t u + u_{xxx} \pm \partial_x u^p + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0$$

by Birkhoff normal form techniques of Procesi-Procesi

1. The restriction of $\mathcal{C}_\varepsilon$ is not technical! Outside: "Chaos", "homoclinc/heteroclinics solutions", "Arnold Diffusion", ...
"Growth of Sobolev norms in 2-d cubic NLS"

$$\mathrm{i} u_t - \Delta u = |u|^2 u \,, \quad x \in \mathbb{T}^2$$

Colliander-Keel-Staffilani-Takaoka-Tao, Invent. Math. 2010

2. For Differentiable nonlinearities $f \in C^q$ the "chaotic effects" are stronger... and KAM theory more difficult

Linear stability

(L): linearized equation $\partial_t h = \partial_x \partial_u \nabla H(u(\omega t, x))h$

$$h_t + a_3(\omega t, x)h_{xxx} + a_2(\omega t, x)h_{xx} + a_1(\omega t, x)h_x + a_0(\omega t, x)h = 0$$

There exists a quasi-periodic (Floquet) change of variable

$$h = \Phi(\omega t)(\psi, \eta, v)\,, \quad \psi \in \mathbb{T}^\nu\,, \eta \in \mathbb{R}^\nu\,, \, v \in H^s_x \cap L^2_{S^\perp}$$

which transforms (L) into the **constant coefficients** system

$$\begin{cases} \dot\psi = b\eta \\ \dot\eta = 0 \\ \dot v_j = \mathrm{i}\mu_j v_j\,, \quad j \notin S\,, \ \mu_j \in \mathbb{R} \end{cases}$$

$\Longrightarrow \eta(t) = \eta_0, v_j(t) = v_j(0)e^{\mathrm{i}\mu_j t} \Longrightarrow \|v(t)\|_s = \|v(0)\|_s$: stability

Forced quasi-linear perturbations of Airy

Use $\omega = \lambda\vec{\omega} \in \mathbb{R}^n$ as 1-dim. parameter

Theorem (Baldi, Berti, Montalto, Math. Annalen 2014)

Let $\vec{\omega} \in \mathbb{R}^n$ diophantine. For every quasi-linear Hamiltonian nonlinearity f the perturbed Airy equation

$$\partial_t u + \partial_{xxx} u + \varepsilon f(\lambda\vec{\omega}t, x, u, u_x, u_{xx}, u_{xxx}) = 0$$

has a small quasi-periodic solution u with frequency $\omega = \lambda\vec{\omega}$ for all

$$\lambda \in \mathcal{C}_\varepsilon \subset [1/2, 3/2], \quad \lim_{\varepsilon \to 0} |\mathcal{C}_\varepsilon| = 1$$

Bifurcation problem: Let $\mathcal{F} : [0, \varepsilon_0) \times H^s \to H^{s-3}$ be

$$\mathcal{F}(\varepsilon, u) := \omega \cdot \partial_\varphi u + \partial_{xxx} u + \varepsilon f(\varphi, x, u, u_x, u_{xx}, u_{xxx})$$

Look for $u(\varphi, x)$ zeros $\mathcal{F}(\varepsilon, u) = 0$.

Small amplitude solutions:

$$\mathcal{F}(0,0) = 0, \qquad D_u \mathcal{F}(0,0) = \omega \cdot \partial_\varphi + \partial_{xxx}$$

eigenvectors: $e^{\mathrm{i}\ell\cdot\varphi} e^{\mathrm{i}jx}$ eigenvalues: $\mathrm{i}(-\omega \cdot \ell + j^3)$

Assumption: non-resonant case: small divisors

$$|\omega \cdot \ell - j^3| \geq \frac{\gamma}{1 + |\ell|^\tau}, \quad \forall (\ell, j) \in \mathbb{Z}^n \times \mathbb{Z}, \ j \neq 0, \ \tau > 0$$

$\Longrightarrow D_u \mathcal{F}(0,0)$ is invertible, but the inverse is **unbounded**:

$(\omega \cdot \partial_\varphi + \partial_{xxx})^{-1} : H^s \to H^{s-\tau}, \quad \tau :=$ "*LOSS OF DERIVATIVES*"

Nash-Moser Implicit Function Theorem

Newton tangent method for zeros of $\mathcal{F}(u) = 0$ + "smoothing":

$$u_{n+1} := u_n - S_n (D_u \mathcal{F})^{-1}(u_n)\mathcal{F}(u_n)$$

where S_n are regularizing operators (= "mollifiers")

- Advantage: QUADRATIC scheme

$$\|u_{n+1} - u_n\|_s \leq C(n)\|u_n - u_{n-1}\|_s^2$$

  $\Longrightarrow$ convergent also if $C(n) \to +\infty$

- Difficulty: invert $(D_u \mathcal{F})(u)$ in a whole neighborhood of the expected solution with good *tame* estimates of the inverse

For KdV: linearized equation on an **approximate** solution

$$h \to (D_u \mathcal{F})(u, \varepsilon)[h] :=$$
$$\omega \cdot \partial_\varphi + \partial_{xxx} + \varepsilon\big(a_3(\varphi, x)\partial_{xxx} + a_2(\varphi, x)\partial_{xx} + a_1(\varphi, x)\partial_x + a_0(\varphi, x)\big)$$

- Linear differential operator with non-constant coefficients
- not diagonal in Fourier basis
- "singular" perturbation problem: $L_\omega^{-1} T$ is unbounded

$$L_\omega := \omega \cdot \partial_\varphi - \partial_{xxx}$$

$$T := a_3(\varphi, x)\partial_{xxx} + a_2(\varphi, x)\partial_{xx} + a_1(\varphi, x)\partial_x + a_0(\varphi, x)$$

Key: spectral analysis of quasi-periodic operator

$$\mathcal{L} = \omega \cdot \partial_\varphi + \partial_{xxx} + a_3(\varphi, x)\partial_{xxx} + a_2(\varphi, x)\partial_{xx} + a_1(\varphi, x)\partial_x + a_0(\varphi, x)$$

$a_i = O(\varepsilon)$, $i = 0, 1, 2, 3$

Main problem: the non constant coefficients term $a_3(\varphi, x)\partial_{xxx}$!

- Usual KAM iterative scheme to **diagonalize** $\mathcal{L}$ is unbounded!

Idea to conjugate $\mathcal{L}$ to a diagonal operator

1. **"REDUCTION IN DECREASING SYMBOLS"**

$$\mathcal{L}_1 := \Phi^{-1}\mathcal{L}\Phi = \omega \cdot \partial_\varphi + m_3 \partial_{xxx} + m_1 \partial_x + R_0$$

- $R_0(\varphi, x)$ pseudo-differential operator of **order** 0, $R_0 = O(\varepsilon)$,
- $m_3 = 1 + O(\varepsilon)$, $m_1 = O(\varepsilon)$, $m_1, m_3 \in \mathbb{R}$, **constants**

Use Egorov type theorem!

2. **"REDUCTION OF THE SIZE of R_0"**

$$\mathcal{L}_n = \omega \cdot \partial_\varphi + m_3 \partial_{xxx} + m_1 \partial_x + r^{(n)} + \mathcal{R}_n$$

- *KAM quadratic scheme:* $\mathcal{R}_n = O(\varepsilon^{2^n})$, $r^{(n)} = \mathrm{diag}_{j \in \mathbb{Z}}(r_j^{(n)})$, Higher order term

$$\mathcal{L} := \omega \cdot \partial_\varphi + \partial_{xxx} + \varepsilon a_3(x)\partial_{xxx}$$

STEP 1: Under the **symplectic** change of variables

$$\Phi u := (1 + \beta_x(x))u(x + \beta(x))$$

we get

$$\begin{aligned} \mathcal{L}_1 := \Phi^{-1}\mathcal{L}\Phi &= \omega \cdot \partial_\varphi + (\Phi^{-1}(1 + \varepsilon a_3)(1 + \beta_x)^3)\partial_{xxx} + O(\partial_{xx}) \\ &= \omega \cdot \partial_\varphi + m_3\partial_{xxx} + O(\partial_{xx}) \end{aligned}$$

imposing

$$(1 + \varepsilon a_3)(1 + \beta_x)^3 = m_3 \,,$$

There exist solution $\beta = O(\varepsilon)$, $m_3 \approx 1$,

$\mathcal{L}_1$ has the leading order with CONSTANT COEFFICIENTS

A general approach for quasi-linear PDEs:

The family of symplectic transformations

$$u(x) \mapsto (1+\beta_x(x))u(x+\tau\beta(x)), \quad \tau \in [0,1],$$

are the flow of the time dependent Hamiltonian "transport eq."

$$\partial_\tau u = \partial_x (b(\tau,x)u), \quad b(\tau,x) := \frac{\beta(x)}{1+\tau\beta_x(x)} \qquad (1)$$

Question:

How a pseudo-differential operator, here

$$P_0 = (1+\varepsilon a_3(x))\partial_{xxx}, \quad p_0(x,\xi) = \mathrm{i}(1+\varepsilon a_3(x))\xi^3,$$

transforms under the flow $\Phi_{\tau_0}^{\tau} : H_x^s \to H_x^s$ of (1) ?

Egorov Theorem:

The transformed operator

$$P(\tau) := \Phi_0^\tau P_0 (\Phi_0^\tau)^{-1}$$

is a pseudo-differential operator of the same order of P_0, here 3, whose principle symbol $p(\tau, x, \xi)$ is obtained by the principal symbol $p_0(x, \xi) = \mathrm{i}(1 + \varepsilon a_3(x))\xi^3$ of P_0, following the Hamiltonian flow $\Psi_A^\tau : \mathbb{T} \times \mathbb{R} \mapsto \mathbb{T} \times \mathbb{R}$ of the classical Hamiltonian $A := b(\tau, x)\xi$ (associated to $\partial_\tau u = b(\tau, x)\partial_x u + \ldots$), namely

$$P(\tau) = \mathrm{Op}(p(\tau, x, \mathrm{i}\partial_x)) + \ldots , \quad p(\tau, x, \xi) = p_0 \circ \Psi_A^{\tau,0}(x, \xi)$$

PDEs in dimension $d \geq 2$

Main difficulties:

- 1) the eigenvalues of $-\Delta + V(x)$ appear in clusters of increasing size

For example $-\Delta e^{\mathrm{i}j\cdot x} = |j|^2 e^{\mathrm{i}j\cdot x}$ then $|j|^2 = |j_0|^2$, $j \in \mathbb{Z}^d$

- 2) The eigenfunctions of $-\Delta + V(x)$ may be "NOT localized with respect to exponentials"! (Feldman- Knörrer-Trubowitz)

$\Longrightarrow$ often used pseudo-PDE with Fourier multipliers

$$\mathrm{i}u_t - \Delta u + M_\sigma u = \varepsilon f \ , \quad M_\sigma e^{\mathrm{i}j\cdot x} = m_\sigma e^{\mathrm{i}j\cdot x}$$

and m_σ are used as parameters

Literature: $d \geq 2$: quasi-periodic solutions

- Newton method, 1^{th} order Melnikov
  - Bourgain, Annals '98, '05,
    NLS and NLW with Fourier multipliers
  - Wang, '11 completely resonant NLS-NLW,
  - Berti-Bolle, '10-'12, forced NLS-NLW, finite regularity, $V(x)$ multiplicative potential
- KAM theory: 2^{th} order Melnikov
  - Kuksin-Eliasson, Annals '10, NLS with Fourier multipliers
  - Procesi-Procesi '11, completely resonant NLS

Forced NLS and NLW

We look for quasi-periodic solutions of Hamiltonian

$$(NLS) \quad \mathrm{i}u_t - \Delta u + V(x)u = \varepsilon f(\omega t, x, u)$$

$$\omega = \lambda \bar{\omega}\,, \qquad \lambda \approx 1$$

in a FIXED diophantine direction

$$|\bar{\omega} \cdot \ell| \geq \frac{\gamma_0}{|\ell|^{\tau_0}}\,, \; \forall \ell \in \mathbb{Z}^\nu \setminus \{0\}\,,$$

In FINITE DIMENSION Eliasson '89 and Bourgain '94

Theorem (M.Berti, Philippe Bolle, JEMS '11)

Existence: $\exists s := s(d,\nu)$, $k := k(d,\nu) \in \mathbb{N}$, *such that:*
$\forall V, f \in C^k$, *there exist* $\varepsilon_0 > 0$, *such that* $\forall 0 < \varepsilon < \varepsilon_0$, *there is*

$$u(\varepsilon,\cdot) \in C^1([1/2,3/2]; H^s) \quad \textit{with} \quad \sup_{\lambda\in[1/2,3/2]} \|u(\varepsilon,\lambda)\|_s \stackrel{\varepsilon\to 0}{\longrightarrow} 0\,,$$

and a Cantor like set

$$\mathcal{C}_\varepsilon \subset [1/2,3/2] \qquad \textit{with} \qquad \lim_{\varepsilon\to 0} |\mathcal{C}_\varepsilon| = 1,$$

such that, $\forall \lambda \in \mathcal{C}_\varepsilon$, $u(\varepsilon,\lambda)$ *is a solution of NLS with* $\omega = \lambda\bar{\omega}$.
Regularity: *If* $V, f \in C^\infty$ *then* $u \in C^\infty$ *in space and time.*

- A similar result holds for NLW

$$(\omega\cdot\partial_\varphi)^2 u - \Delta u + V(x)u = \varepsilon f(\varphi,x,u)$$

About the Proof

KEY STEP: For "most" parameters $\lambda \in [1/2, 3/2]$ the linearized operator

$$\mathcal{L}_\varepsilon(\lambda) := (\lambda \bar{\omega} \cdot \partial_\varphi)^2 - \Delta + V(x) + \varepsilon (\partial_u f)(\varphi, x, u(\varphi, x))$$

is invertible and TAME estimate in HIGHER Sobolev norms, i.e.

$$\|\mathcal{L}_\varepsilon^{-1}(\lambda) h\|_s \leq \|h\|_{s+\tau} \|u\|_{s_0} + \|h\|_{s_0} \|u\|_s \,, \ \forall s_0 \leq s \leq k$$

- **Step 1)** L^2-estimates: lower bounds for the eigenvalues of the **self adjoint** operator $\mathcal{L}_\varepsilon(\lambda)$: eigenvalues are smooth in $\lambda \in [1/2, 3/2]$
- **Step 2)** Tame-estimates in high norm
  KEY OBSERVATION: many eigenvalues are NOT small !

Separation properties of singular sites

Singular sites : $(\ell, j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d$ such that

$$\textbf{NLW)} \quad |-(\omega \cdot \ell)^2 + |j|^2 + m| < \rho$$

$$\textbf{NLS)} \quad |-\omega \cdot \ell + |j|^2 + m| < \rho$$

must be more and more "rare" as $\rho \to 0$

- **(NLW)** Integer points near a "cone"
- **(NLS)** Integer points near a "paraboloid"

GROUP THE SINGULAR SITES INTO LARGE CLUSTERS

Next step:

KAM for autonomous NLW with multiplicative potential:

$$u_{tt} - \Delta u + V(x)u = a(x)u^3 + O(u^4)$$

in preparation with Philippe Bolle

Further difficulties:

- bifurcation analysis
- the tangential and the normal variables are coupled

Happy Birthday Ivar!!