From Ekeland's Hopf-Rinow theorem to optimal incompressible transport theory

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MINIMIZING GEODESICS AND CLOSEST POINTS

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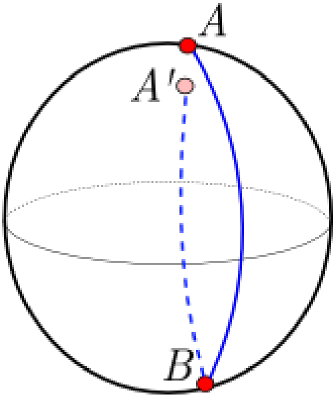
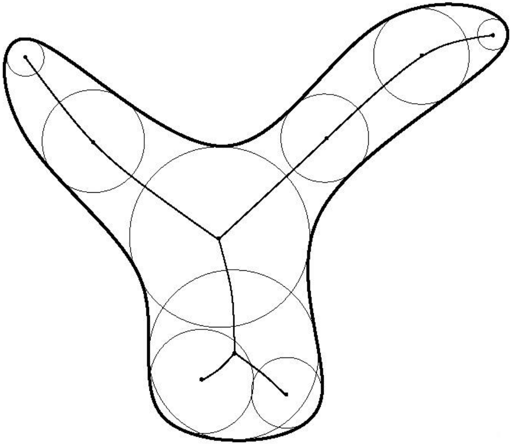
This implies that, for each k, M_k must minimize on S its distance to the mid-point $(M_{k-1} + M_{k+1})/2$.

According to Ekeland's Hopf-Rinow theorem (J. Diff. Geo. 1979), under suitable assumptions, minimizing geodesics (that may not exist) are generically unique.



Edelstein's theorem: in a Hilbert space, generically a point has a unique projection (closest point) on a given closed bounded subset.



AN INFINITE DIMENSIONAL EXAMPLE : THE (SEMI-) GROUP OF VOLUME-PRESERVING MAPS

Consider a bounded domain D in R^d (this could be generalized to a Riemanniann manifold) and the Hilbert space $H = L^2(D, R^d)$. Let $VPM(D)$ be the semi-group of all volume-preserving maps

$$VPM(D) = \{M \in H, \ \int_D q(M(x))dx = \int_D q(x)dx, \ \forall q \in C(R^d)\}$$

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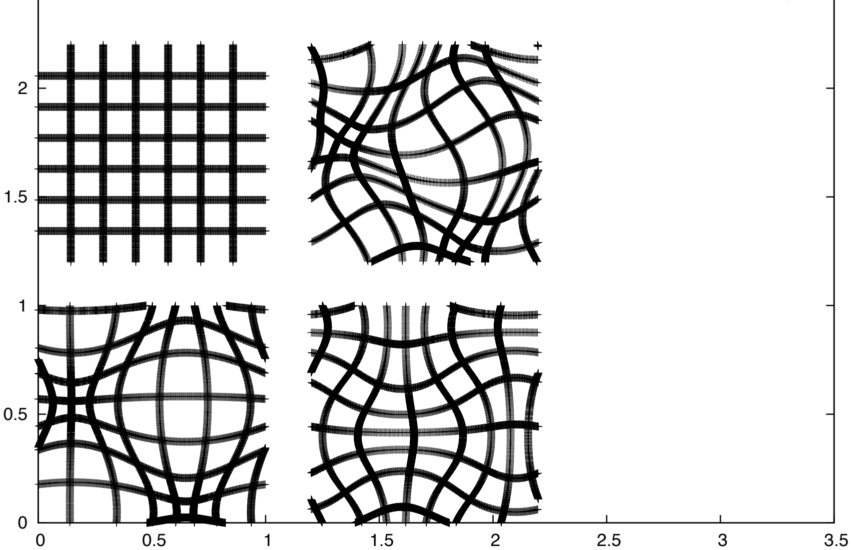
N.B. This semi-group contains, as a dense subset, the group of orientation and volume preserving diffeomorphisms $SDiff(D)$ of D, provided $d \geq 2$.



Three maps of the (periodized) square: guess which one is volume-preserving!

A discontinuous volume-preserving map

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namely the rigid permutation of 16 sub-cells.

THE CLOSEST POINT PROBLEM

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$$= \sup_p \int_D (p^c(T(x)) + p(x))dx, \quad p^c(y) = \inf_{m \in D} \frac{1}{2}|m - z|^2 - p(m)$$

A MONGE-AMPERE-KANTOROVICH SOLUTION

THEOREM **Let μ and ν respectively be the Lebesgue measure on D and its image ν by the given map T. Assume ν to be absolutely continuous w.r.t. the Lebesgue measure.**

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Arnold's geometric interpretation of Euler's theory of incompressible fluids (1755).

VLADIMIR ARNOLD

Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits

Annales de l'institut Fourier, tome 16, n° 1 (1966), p. 319-361.

<http://www.numdam.org/item?id=AIF_1966__16_1_319_0>

XXI. Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélerations actuelles que nous venons de trouver, & nous obtiendrons les trois équations suivantes :

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right)$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$

$$R - \frac{1}{q}\left(\frac{dp}{dz}\right) = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right)$$

Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la considération de la continuité du fluide :

$$\left(\frac{dq}{dt}\right)+\left(\frac{d.qu}{dx}\right)+\left(\frac{d\,qv}{dy}\right)+\left(\frac{d.qw}{dz}\right)=o.$$

Si le fluide n'étoit pas compreſſible, la denſité q ſeroit la même en Z, & en Z′, & pour ce cas on auroit cette équation :

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=o.$$

qui eſt auſſi celle ſur laquelle j'ai établi mon Mémoire latin allégué ci-deſſus.

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(*) If we assume the domain D and the (modified) pressure field $x \to \lambda\frac{|x|^2}{2} - p_t(x)$ to be both convex, for some $\lambda \in R$, it is sufficient that $(t_1 - t_0)^2\lambda < \pi^2$.

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The (constant speed) minimizing geodesic problem can be written as a saddle point problem, just by using a time-dependent Lagrange multiplier to relax the constraint for M_t to belong to $VPM(D)$

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This is trivially bounded from below by

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which naturally leads to a dual least action principle

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and reads (after a short calculation)

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with $J_p(y,z) = \inf \int_{t_0}^{t_1} (\frac{1}{2}|\frac{d\xi_t}{dt}|^2 - p_t(\xi_t))\, dt$ **where the infimum is taken over all curves** $\xi_t \in D$ **such that** $\xi_{t_0} = y \in D$, $\xi_{t_1} = z \in D$.

THIS IS A GENERALIZATION OF KANTOROVICH 1942 OPTIMAL TRANSPORT THEORY, ALSO SIMILAR TO WEAK KAM THEORY

Jellyfish! a more and more frequent (almost) incompressible optimal transport problem...

APPROXIMATE MINIMIZING GEODESICS

DEFINITION Let us assume D to be convex, fix $t_0 = 0, \ t_1 = 1$ and consider two maps $M_0, M_1 \in VPM(D)$. We say that $(M_t^\epsilon) \in SDiff(D)$ is an ϵ-minimizing geodesic if

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where $\frac{1}{2} d(M_0, M_1)^2$ denotes the maximal dual action. The existence of such approximations is in no way trivial and is a consequence of a key density result due to A. Shnirelman (GAFA 1994) for Y.B. "generalized incompressible flows" (JAMS 1991).

THE MAIN RESULT ON MINIMIZING GEODESICS: EXISTENCE OF A UNIQUE ACCELERATION

MAIN THEOREM Let us assume D to be convex, with $d \geq 3$, fix $t_0 = 0, \ t_1 = 1$ and consider two maps $M_0, M_1 \in VPM(D)$. Then, there is a UNIQUE pressure-gradient Dp_t such that for all (M_t^ϵ) ϵ-minimizing geodesics, we have in the sense of distributions

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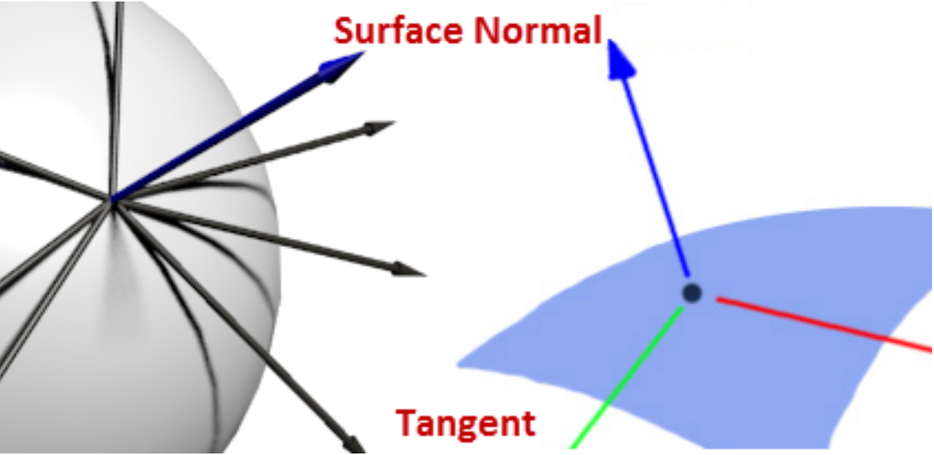
This result essentially goes back to YB CPAM 1999, with substantial improvements in Ambrosio-Figalli ARMA 2008. It is a combination of solving the dual least action problem and using Shnirelman's density result for "generalized flows", GAFA 1994.



Uniqueness of the acceleration for minimizing geodesics along the (infinite dimensional) semi-group of volume-preserving maps, in the case of incompressible fluids.

There is no similar results for the minimizing geodesics along the (finite dimensional) group of orthogonal transforms, in the case of rigid solids.

Toward modern art!

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 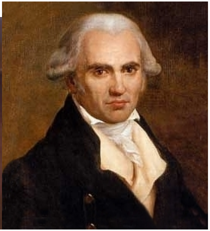

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 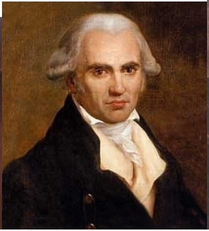 

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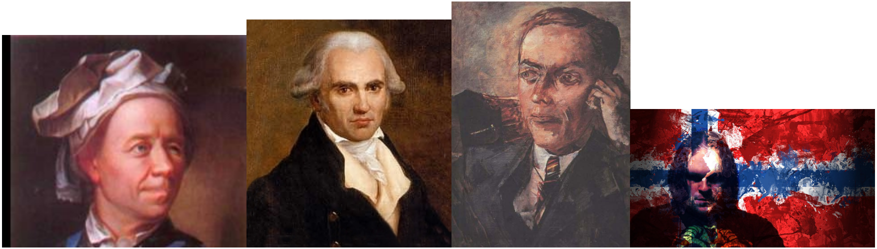
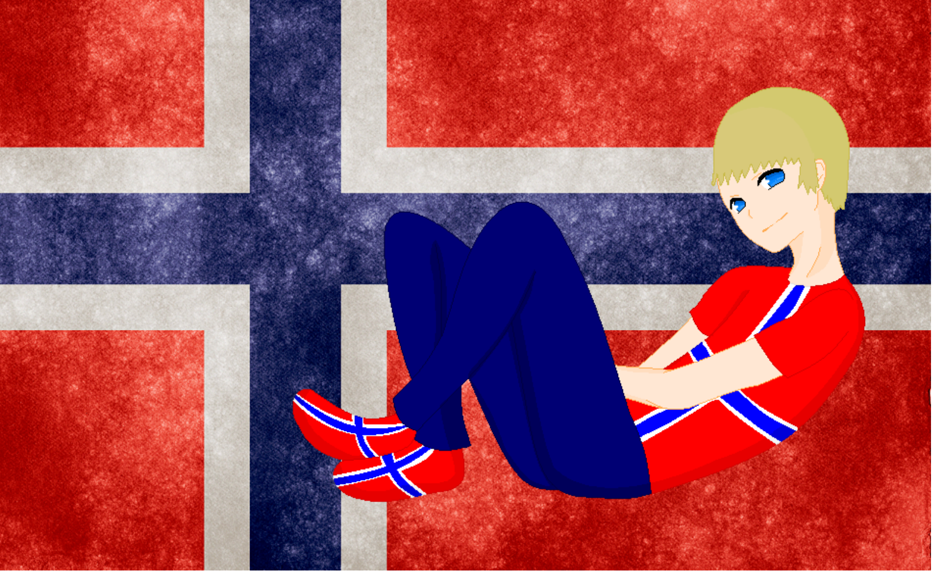


Leonard, Gaspard, Leonid and ....CédrIvar?



Ivar younger than ever! Happy birthday!

(more conventional art)

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MINIMIZING GEODESICS AND CONVEXITY

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1) UNIQUENESS OF THE ACCELERATION This remarkable feature comes from the convexity of the problem in infinite dimension. There is no equivalent result for finite dimensional configuration spaces such as $SO(3)$, on which geodesic curves correspond to rigid body motions in classical mechanics.
2) LIMITED REGULARITY OF THE PRESSURE GRADIENT The pressure gradient was proven first (YB CPAM 1999) to be a locally bounded measure.

MINIMIZING GEODESICS AND CONVEXITY

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2) LIMITED REGULARITY OF THE PRESSURE GRADIENT The pressure gradient was proven first (YB CPAM 1999) to be a locally bounded measure. Local L^2 integrability in time (with measure values in space) was shown by Ambrosio and Figalli in 2008.

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OTHER GEOMETRIC ANALYSIS ISSUES

GEODESIC COMPLETENESS This amounts to globally solving the initial value problem for the Euler equations. This is an outstanding problem for nonlinear evolution PDEs, which has not been discussed in this lecture. (You are welcome to ask questions after the talk!)

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MINIMIZING GEODESICS Shnirelman has proven (Math USSR Sb 1986) that existence of minimizing geodesics along SDiff(D) may fail when d $\geq$ 3. Remarkably enough, as already seen, the case d $\geq$ 3 turns out to be "easy", with a crucial use of the convex structure of the dual problem. However the Hopf-Rinow theorem has not been proven in this framework. The case d $=$ 2 is clearly linked to symplectic geometry and seems extremely difficult: a fascinating strategy has been developed by Shnirelman, by adding braid constraints to the minimization problem, which certainly deserves further investigations.