The Schrödinger problem on networks and an application to the identification of hedonic models

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1 Introduction

This talk deals with the estimation of hedonic models.

- Early papers of hedonic models:

Hedonic regression: Waugh (1928), Court (1939)

Hedonic equilibrium models: Tinbergen (1956), Rosen (1975)

- Criticism of Rosen's methodology: Brown and Rosen (1982), Epple (1987), Bartik (1987)

- Identification of hedonic models: Ekeland, Heckman, Nesheim (2002, 2004), Heckman, Matzkin, Nesheim (2003, 2010).

- Existence and qualitative properties: Ekeland (2010). Chiappori, McCann, and Nesheim (2010) show equivalence with a matching problem. Queyranne (2011) suggested the problem is equivalent to a max-flow problem.

- Applications:

– Price indices: Griliches (1971), Boskin et al. (1996), Lancaster (1966)

– Valuation of amenities: Triplett (1969), Berndt et al. (1995), Berndt and Rappaport (2001), Pakes (2003)

– Value of Statistical Life: Thaler and Rosen (1976), Schelling (1987), Viscusi and Aldy's (2003) and Viscusi (2008)

– Wine: Golan and Shalit (1993), Oczkowski (1994), Nerlove (1995), Combris et al. (1997)

2 The hedonic equilibrium

Discrete characteristics of good: $z \in \mathcal{Z}$ (e.g. quality parameter of wine, observable–e.g. ratings by different experts, measurable characteristics).

Producer's type $x \in \mathcal{X}$ (eg. producer's sub-region; production process characteristics; ownership; year of production).

Consumer's type $y \in \mathcal{Y}$ (e.g. country; distribution channel used...)

Price of quality $z \in \mathcal{Z}$ is p_z. It is possible for a producer or a consumer to stay out of the market (e.g. by consuming non-rated wines or alternative bewerages) and receive no transfer. Such option is denoted $z = 0$. We denote

$$\mathcal{Z}^0 = \mathcal{Z} \cup \{0\}$$

the full set of consumption options including alternatives.

Classical hedonic model:

- producing and selling quality z yields profit $\alpha_{xz} + p_z$ to producer. Think of $\alpha_{xz} = -c_{xz}$ where c_{xz} is the cost to x of producing quality z.

- Buying and consuming quality z yields utility $\gamma_{yz} - p_z$ to consumer.

Consumer can only buy 0 or 1 (indivisible) unit.

Note that consumers' preferences do not depend on x. Consumer only value quality z, have no preferences over producers' identity.

A producer x has the possibility to remain out of the market, in which case her utility is $\alpha_{x0} = 0$ and she gets no transfer. The same applies for a consumer y which gets $\gamma_{y0} = 0$ if staying out of the market.

There are n_x producers of type x and m_y consumers of type y (exogenously given). The number of units produced and sold at quality z is endogenously determined.

Literature on hedonic model (Tinbergen, 1956, Ekeland et al., 2004) classically assumes continuous characteristics z and differentiability. Result: There exists an equilibrium price p_z such that supply, given by the maximization of producer profit

$$\max_z (\alpha_{xz} + p_z)$$

equates demand, given by the maximization of consumer surplus

$$\max_z \left(\gamma_{yz} - p_z\right).$$

We will focus on discrete hedonic models: z is discrete and consumption units are indivisible, and n_x and m_y

are integers. The first result is that an equilibrium exists, despite indivisibilities.

Supply and demand. Let μ_{xz} be the number of producers of type x supplying quality z, and μ_{yz} be the number of consumers of type y demanding quality z.

Counting equations. Counting of types of producers and consumers imposes

$$\sum_{z\in\mathcal{Z}} \mu_{xz} \leq n_x \text{ for all } x \in \mathcal{X} \quad (1)$$

$$\sum_{z\in\mathcal{Z}} \mu_{yz} \leq m_y \text{ for all } y \in \mathcal{Y} \quad (2)$$

Balance equations. Balancing supply and demand for each quality of good requires

$$\sum_{x\in\mathcal{X}} \mu_{xz} = \sum_{y\in\mathcal{Y}} \mu_{yz} \text{ for all } z \in \mathcal{Z}. \quad (3)$$

We denote

$$\mu \in \mathcal{M}$$

the set of $\mu \geq 0$ satisfying the type counting equations (1), (2) and the balance equations (3). The hedonic model with indivisibilities requires further that μ_{xz} and μ_{yz} be integral, which we denote

$$\mu \in \mathcal{M}_{int}.$$

An equilibrium is given by the data of $\left(\mu_{xz}, \mu_{yz}, p_z\right)$ where $\mu_{xz} \geq 0$ is the supply of quality z arising from producer of type x, $\mu_{yz} \geq 0$ is the demand of quality z arising from consumers of type y. Equilibrium requires that, given prices, no individual (producer or consumer) choosing a certain quality would get strictly more utility with another quality, that is

$$\begin{aligned} \mu_{xz} &> 0 \text{ implies } \alpha_{xz} + p_z \geq \alpha_{xz'} + p_{z'} \text{ for all } z' \\ \mu_{yz} &> 0 \text{ implies } \gamma_{yz} - p_z \geq \gamma_{yz'} - p_{z'} \text{ for all } z'. \end{aligned}$$

Theorem 1. *There is a nonempty set of equilibria in the hedonic model with indivisibilities.*

This result is proven by an appeal to the concept of network flows.

3 Queyranne's reformulation

The network. Define a set of *nodes* by $\mathcal{N} = \mathcal{X} \cup \mathcal{Z} \cup \mathcal{Y}$, and a set of *arcs* $\mathcal{A}$ which is a subset of $\mathcal{N} \times \mathcal{N}$ and is such that if $ww' \in \mathcal{A}$, then $w'w \notin \mathcal{A}$. Here, the set of arcs is $\mathcal{A} = (\mathcal{X} \times \mathcal{Z}) \cup (\mathcal{Y} \times \mathcal{Z})$.

A *vector* is defined as an element of $\mathbb{R}^{\mathcal{A}}$. Here, we introduce the following *direct surplus vector*

$$\phi_{ww'} \; : \; = \alpha_{xz} \text{ if } w = x \text{ and } w' = z \tag{4a}$$
$$\phi_{ww'} \; : \; = \gamma_{yz} \text{ if } w = z \text{ and } w' = y. \tag{4b}$$

For two nodes w and w', a *path* from w to w' is a chain

$$(w_0 w_1), (w_1 w_2), ..., (w_{T-2} w_{T-1}), (w_{T-1} w_T)$$

such that $w_i w_{i+1} \in \mathcal{A}$ for each i. T is the *length* of the path. Here, the only nontrivial paths are of length 2 and are of the form $(xz), (zy)$ where $x \in \mathcal{X}$, $z \in \mathcal{Z}$ and $y \in \mathcal{Y}$.

For two nodes w and w', we define the *reduced surplus*, or *indirect surplus* as the surplus associated to the optimal path from w to w'. Here, for $x \in \mathcal{X}$, $y \in \mathcal{Y}$, the indirect suplus Φ_{xy} of producer x and consumer y is

$$\Phi_{xy} := \max_{z \in \mathcal{Z}} \left(\alpha_{xz} + \gamma_{yz} \right). \tag{5}$$

For $w \in \mathcal{N}$, we let N_w be the algebraic quantity of mass leaving the network at w. Hence N_w is the flow of mass being consumed ($N_w > 0$) or produced ($N_w < 0$) at w. The nodes such that $N_w < 0$ (resp. $N_w = 0$ and $N_w > 0$) are called the source nodes, whose set is denoted $\mathcal{S}$ (resp. intermediate nodes $\mathcal{I}$ and target nodes $\mathcal{T}$). Here, for $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $z \in \mathcal{Z}$, we set

$$N_x := -n_x \ , \ N_y := m_y \ , \ N_z := 0 \tag{6}$$

so that the set of source nodes is $\mathcal{S} := \mathcal{X}$, the set of intermediate nodes is $\mathcal{I} := \mathcal{Z}$, and the set of target nodes is $\mathcal{T} := \mathcal{Y}$.

Gradient, flows. We define a *potential* as an element of $\mathbb{R}^{\mathcal{N}}$. We define the *gradient matrix* as the matrix of general term ∇_{aw}, $a \in \mathcal{A}$, $w \in \mathcal{N}$ such that

$$\begin{aligned} \nabla_{aw} &= -1 \text{ if } a \text{ is out of } w,\ \nabla_{aw} = 1 \text{ if } a \text{ is into } w, \\ \nabla_{aw} &= 0 \text{ else,} \end{aligned}$$

so that, for a potential $f \in \mathbb{R}^{\mathcal{N}}$, ∇f is the vector such that for $a = ww' \in \mathcal{A}$, one has $(\nabla f)_{ww'} = f_{w'} - f_w$. Here, set the potential of surpluses U as

$$U_x := -u_x \ , \ U_z := -p_z \ , \ U_y := v_y, \tag{7}$$

and

$$(\nabla U)_{xz} = u_x - p_z \text{ and } (\nabla U)_{zy} = v_y + p_z. \tag{8}$$

We define the *divergence* matrix ∇^* (sometimes also called *node-edge*, or *incidence matrix*) as the transpose of the gradient matrix: $\nabla^*_{xa} := \nabla_{ax}$. As a result, for a vector v,

$$(\nabla^* v)_{ww'} = \sum_z v_{zw'} - \sum_z v_{wz}.$$

A *flow* is a nonnegative vector $\mu \in \mathbb{R}_+^{\mathcal{A}}$ that satisfies the *balance of mass equation*, that is

$$
\begin{aligned}
(N - \nabla^* \mu)_w &\geq 0, \; w \in \mathcal{S} &(9)\\
(N - \nabla^* \mu)_w &= 0, \; w \in \mathcal{I} &(10)\\
(N - \nabla^* \mu)_w &\leq 0, \; w \in \mathcal{T} &(11)
\end{aligned}
$$

Here, $\mu : \left(\mu_{xz}, \mu_{zy}\right)$ is a flow if and only if μ_{xz} and μ_{zy} satisfy the people counting and market clearing equations, that is

$$\sum_z \mu_{xz} \leq n_x \,, \; \sum_z \mu_{zy} \leq m_y \text{ and } \sum_{x \in \mathcal{X}} \mu_{xz} = \sum_{y \in \mathcal{Y}} \mu_{zy}.$$

Maximum surplus flow. We now consider the *maximum surplus flow problem*, that is

$$
\begin{aligned}
&\max_{\mu \in \mathbb{R}_+^{\mathcal{A}}} \sum_{a \in A} \mu_a \phi_a &(12)\\
s.t. \qquad &\mu \text{ satisfies (9), (10), (11),}
\end{aligned}
$$

whose value coincides with the value of its dual version, that is

$$
\begin{aligned}
& \min_{U \in \mathbb{R}^{\mathcal{N}}} \sum_{w \in \mathcal{N}} U_w N_w && (13) \\
s.t. \quad & U_w \leq 0, \ \forall w \in \mathcal{S} \\
& U_w \geq 0, \ \forall w \in \mathcal{T} \\
& \nabla U \geq \phi,
\end{aligned}
$$

and by complementary slackness, for $w \in \mathcal{S} \cup \mathcal{T}$, $U_w \neq 0$ implies $N_w = (\nabla^* \mu)_w$. A standard result is that if N has only integral entries, then (12) has an integral solution μ.

Here the solution U of (13) is related to the solution to the hedonic model by Equations (7), that is $u_x = -U_x$, $p_z = -U_z$, $v_y = U_y$. Using (8) and (4), $\nabla U \geq \phi$ implies $u_x - p_z = U_z - U_x \geq \phi_{xz} = \alpha_{xz}$ and $v_y + p_z = U_y - U_z \geq \phi_{zy} = \gamma_{zy}$, thus, using complementary slackness one recovers

$$u_x = \max_z \left(\alpha_{xz} + p_z\right)^+ \text{ and } v_y = \max_z \left(\gamma_{zy} - p_z\right)^+ .$$

Further, if n and m have only integral entries, then there is an integral solution μ to (12). Therefore:

Theorem (Queyranne). The hedonic equilibrium problem can be reformulated as a matching flow problem as described above.

This reformulation has several advantages. First, it establishes the existence of a hedonic equilibrium, and its integrality.

Theorem. Consider a market given by n_x producers of type x, m_y consumers of type y, and where productivity of producer x is given by α_{xz}, and utility of consumer y is γ_{yz}.Then:

(i) There exists a hedonic equilibrium $\left(p_z, \mu_{xz}, \mu_{yz}\right)$;

(ii) $\left(\mu_{xz}, \mu_{yz}\right)$ are solution to the primal problem of the expression of the social welfare

$$\max_{\mu_{xz},\mu_{yz}\geq 0} \sum_{xz} \mu_{xz}\alpha_{xz} + \sum_{yz} \mu_{yz}\gamma_{yz} \tag{14}$$

$$\sum_z \mu_{xz} \leq n_x \text{ and } \sum_z \mu_{yz} \leq m_y \text{ and } \sum_x \mu_{xz} = \sum_y \mu_{yz},$$

while (p_z) is obtained from the solution of the dual expression of the social welfare

$$\min_{u_x,v_y\geq 0;p_z} \sum_x n_x u_x + \sum_y m_y v_y \tag{15}$$

$$u_x \geq \alpha_{xz} + p_z \text{ and } v_y \geq \gamma_{yz} - p_z.$$

(iii) If n_x and m_y are integral for each x and y, then μ_{xz} and μ_{yz} can be taken integral.

In particular, the equilibrium prices (p_z) as well as the quantities μ_{xz}, μ_{yz} supplied at equilibrium can be computed efficiently using one of the many maximum flows algorithms.

4 Introducing heterogeneities

Consider an individual producer i and an individual consumer j, let x_i and y_j be their observed characteristics. We assume that there are unobserved heterogeneities in productivities and tastes, following Galichon and Salanié (2013).

Individual producer i (of observable characteristics x_i) has profitability from producing z

$$\alpha_{x_i z} + \varepsilon_{iz} + p_z$$

Individual consumer j (of observable characteristics y_j) has utility from consumption of z

$$\gamma_{y_j z} + \eta_{jz} - p_z.$$

We make two important assumptions.

Assumption L (Large market). *The number of individuals on the market* $N = \sum_{x \in \mathcal{X}} n_x + \sum_{y \in \mathcal{Y}} m_y$ *goes to infinity; and the ratios* (n_x/N) *and* (m_y/N) *converge.*

Assumption D (Distribution of Unobserved Variation in Surplus).

a) For any producer i of observable characteristics x, *the* ε_{iz} *are drawn from a* $(|\mathcal{Z}| + 1)$*-dimensional distribution* P_x

b) For any consumer j of observable characteristics y, *the* η_{jz} *are drawn from a* $(|\mathcal{Z}| + 1)$*-dimensional distribution* Q_y.

Example 1. *The utility shocks* ε *and* η *have the iid Type-I Extreme Value (Gumbel) distribution, namely their C.D.F. is*

$$F(x) = \exp\left(-\exp\left(-\frac{x+\gamma}{\beta}\right)\right).$$

While the previous example has the flavour of discrete choice theory, the next one is very much in line with the tradition of hedonic models.

Example 2. *As in Ekeland et al. (2004), consider* $\alpha(z,x,\varepsilon) = \varphi(z) + z\,(g(x)+\varepsilon)$. *This is a particular case of a model where*

$$\varepsilon_{iz} = -\sigma_{x_i z}.\varepsilon_i \tag{16}$$

where the utility shock ε_i *is iid across individuals.*

Producers' choice. Assume that, at equilibrium, producers of group x get average utility U_{xz} from quality z. Then sum of the surpluses of the producers of all groups is

$$G(U) = \sum_{x\in\mathcal{X}} n_x \mathbb{E}_{\mathbf{P}_x}\left[\max_{z\in\mathcal{Z}^0}(U_{xz}+\varepsilon_{iz})\right] \tag{17}$$

where the expectation is taken over a random vector of utility shocks $(\varepsilon_{i0},\ldots,\varepsilon_{i|\mathcal{Z}|}) \sim \mathbf{P}_x$. By the Envelope theorem, the number of producers choosing quality z is given by

$$\begin{aligned}\mu_{xz} &= n_x \mathbf{P}_x\left(x \text{ choose s } z\right) \\ &= \frac{\partial G(U)}{\partial U_{xz}}. \qquad (18)\end{aligned}$$

Note, however, that – as econometricians – one would like to solve exacty the inverse problem, i.e. determine whenever possible U_{xz} as a function of μ_{xz}. Hence one would like to inverse Eq. (18).

To do this, introduce the *generalized entropy* G^* as the convex conjugate (Legendre-Fenchel transform) of G as

$$G^*(\mu) = \max_{U_{xz}} \left(\sum_{x,y} \mu_{xz} U_{xz} - G(U) \right) \qquad (19)$$

where μ_{xz} is the number of producers choosing quality z. By the envelope theorem, one has

$$U_{xz} = \frac{\partial G^*(\mu)}{\partial \mu_{xz}}. \qquad (20)$$

and note that the first order conditions in (17) are exactly the envelope theorem in (19), hence equations (18) and (20) hold simultaneously, which means that (20) is the condition we are looking for in order to solve our inverse problem.

Consumers' choice. Similarly, the sum of the expected utilities of the consumers of all groups is

$$H(V) = \sum_{y \in \mathcal{Y}} m_y \mathbb{E}_{\mathbf{Q}_y} \left[\max_{z \in \mathcal{Z}^0} (V_{yz} + \eta_{yz}) \right]$$

and the associated generalized entropy is denoted

$$H^*(\mu).$$

Example 1 continued. When the utility shocks ε and η have the iid Type-I Extreme Value (Gumbel) distribution, one has the logit structure

$$\begin{aligned} G(U) &= \sum_{x \in \mathcal{X}} n_x \log \left(1 + \sum_{z \in \mathcal{Z}^0} e^{U_{xz}} \right) \\ G^*(\mu) &= \sum_{x \in \mathcal{X}} \sum_{z \in \mathcal{Z}^0} \mu_{xz} \log \frac{\mu_{xz}}{n_x}, \end{aligned}$$

where μ_{x0} is implicitely defined by $\mu_{x0} = n_x - \sum_{y\in\mathcal{Y}} \mu_{xy}$. Similar formulas hold for H_y and H_y^*. Hence G^* is a usual entropy function.

Example 2 continued. Recall specification (16)

$$\varepsilon_{iz} = -\sigma_{x_i z}.\varepsilon_i.$$

Then one has

$$G(U) = \sum_{x\in\mathcal{X}} n_x G_x(U)$$

where

$$G_x(U) = \mathbb{E}_{\mathbf{P}_x}\left[\max_{z\in\mathcal{Z}^0}(U_{xz} - \sigma_{xz}.\varepsilon)\right]$$

thus

$$G_x^*(\mu_{.|x}) = \max_{U_{xz}}\left(\sum_{z\in\mathcal{Z}^0}\mu_{z|x}U_{xz} - \mathbb{E}_{\mathbf{P}}\left[\max_{z\in\mathcal{Z}^0}(U_z - \sigma_{xz}.\varepsilon)\right]\right)$$

Letting $\bar{U}_\varepsilon = -\max_{z\in\mathcal{Z}^0}(U_z - z.\varepsilon)$, one has

$$-\bar{U}_\varepsilon \geq U_z + \sigma_{xz}.\varepsilon$$

thus

$$-\sigma_{xz}.\varepsilon \geq U_z + \bar{U}_\varepsilon$$

hence

$$\begin{aligned} G_x^*(\mu_{.|x}) &= \max_{U_z + \bar{U}_\varepsilon \leq -\sigma_{xz}.\varepsilon} \sum_{z \in \mathcal{Z}^0} \frac{\mu_{xz}}{n_x} U_z + \mathbb{E}_{\mathbf{P}}\left[\bar{U}_\varepsilon\right] \\ &= \min_{Z \sim \frac{\mu_{x.}}{n_x}, \varepsilon \sim \mathbf{P}} \mathbb{E}\left[-\sigma_{xZ}.\varepsilon\right] \\ &= -\max_{Z \sim \frac{\mu_{x.}}{n_x}, \varepsilon \sim \mathbf{P}} \mathbb{E}\left[\sigma_{xZ}.\varepsilon\right] \end{aligned}$$

In the one-dimensional case as in Ekeland et al., the solution to the previous problem can be given explicitely, and in the case where $\varepsilon \sim \mathcal{U}([0,1])$, G^* is quadratic with respect to μ_{xz}.

We can compute the social welfare in this economy.

Theorem 2. *(i) The optimal social welfare in this economy is given by*

$$\mathcal{W} = \max_{p_z} G(\alpha + p_.) + H(\gamma - p_.) \qquad (21)$$

alternatively, it can be expressed as

$$\mathcal{W} = \max_{\mu \in \mathcal{M}} \sum_{x \in \mathcal{X}, z \in \mathcal{Z}} \mu_{xz} \alpha_{xz} + \sum_{y \in \mathcal{Y}, z \in \mathcal{Z}} \mu_{yz} \gamma_{yz} - \mathcal{E}(\mu) \tag{22}$$

where $\mathcal{E}(\mu)$ is a generalized entropy function, defined by

$$\mathcal{E}(\mu) = G^* (\mu) + H^* (\mu) .$$

(ii) Further the equilibrium $\left(p_z, \mu_{xz}, \mu_{yz}\right)$ is unique and is such that (p_z) is optimal for (21) and $\left(\mu_{xz}, \mu_{yz}\right)$ is optimal for (22).

(iii) Finally, at equilibrium, surplus of producer i is

$$\alpha_{x_i z} + p_z + \varepsilon_{iz}$$

and surplus of consumer j is

$$\gamma_{y_j z} - p_z + \eta_{jz}.$$

In other words, producers and consumers keep their surplus shock at equilibrium.

5 Identification

As a result of the first order conditions in the previous theorem, the model is exactly identified from $\left(p_z, \mu_{xz}, \mu_{yz}\right)$.

Theorem 3. *The producers and consumers systematic surpluses at equilibrium are identified from μ_{xz} and μ_{yz} and p_z by*

$$\begin{aligned}
\alpha_{xz} &= \frac{\partial G^*(\mu)}{\partial \mu_{xz}} - p_z \\
\gamma_{yz} &= \frac{\partial H^*(\mu)}{\partial \mu_{yz}} + p_z
\end{aligned}$$

Let us see what this identification formula becomes on our Logit example.

Example 1 continued. When ε and η are iid Gumbel, one has

$$
\begin{aligned}
\alpha_{xz} &= \log\left(\frac{\mu_{xz}}{\mu_{x0}}\right) - p_z \\
\gamma_{yz} &= \log\left(\frac{\mu_{yz}}{\mu_{y0}}\right) + p_z.
\end{aligned}
$$

6 Computation

Nonparametric identification did not require to solve for the equilibrium. However, parametric identification does. The primitives of the model are the functions α_{xz}^{λ} and γ_{yz}^{λ}; and the unknowns are supply μ_{xz}, demand μ_{yz} and the price p_z.

6.1 Logit case

In the setting of Example 1, recall that one has

$$\begin{aligned} \mu_{xz} &= \mu_{x0} e^{\alpha_{xz}+p_z} \\ \mu_{yz} &= \mu_{0y} e^{\gamma_{yz}-p_z}. \end{aligned}$$

The balance equation is

$$\sum_{x\in\mathcal{X}} \mu_{x0} e^{\alpha_{xz}+p_z} = \sum_{y\in\mathcal{Y}} \mu_{y0} e^{\gamma_{yz}-p_z} \ \forall z \in \mathcal{Z}$$

yielding

$$e^{2p_z} = \frac{\sum_{y\in\mathcal{Y}} \mu_{y0} e^{\gamma_{yz}}}{\sum_{x\in\mathcal{X}} \mu_{x0} e^{\alpha_{xz}}}$$

The type counting equations become

$$\begin{aligned} \mu_{x0} + \mu_{x0} \sum_{z\in\mathcal{Z}} e^{\alpha_{xz}+p_z} &= n_x \ \forall x \in \mathcal{X} \\ \mu_{y0} + \mu_{y0} \sum_{z\in\mathcal{Z}} e^{\gamma_{yz}-p_z} &= m_y \ \forall y \in \mathcal{Y} \end{aligned}$$

This suggests an iterative algorithm for the computation of μ_{x0}, μ_{y0} and p_z.

1. Initialize: $\mu_{x0} = \frac{1}{\sum_x n_x}$, $\mu_{y0} = \frac{1}{\sum_y m_y}$, $p_z = 0$.

2. At step k, assume P^k, μ_{xz}^k, and μ_{yz}^k have been computed

2a. Compute new market clearing price

$$p_z^{k+1} = \frac{1}{2} \log \left(\frac{\sum_{y \in \mathcal{Y}} \mu_{y0}^k e^{\gamma_{yz}}}{\sum_{x \in \mathcal{X}} \mu_{x0}^k e^{\alpha_{xz}}} \right)$$

2b. Compute supply under new price

$$\mu_{x0}^{k+1} = \frac{n_x}{\sum_{z \in \mathcal{Z}^0} e^{\alpha_{xz} + p_z^{k+1}}}$$

Compute demand under new price

$$\mu_{y0}^{k+1} = \frac{m_y}{\sum_{z \in \mathcal{Z}^0} e^{\gamma_{yz} - p_z^{k+1}}}$$

Iterate until nearing a fixed point.

6.2 General case

In the general case, we solve for the prices p_z using the fact that the latter is optimal for

$$\max_{P_z} G(\alpha + p_\cdot) + H(\gamma - p_\cdot).$$

This suggests a tâtonnement algorithm.

1. Guess an intial value of p_z^0. E.g. $p_z^0 = 0$.

2. Assuming P_z^k is known, compute

$$\mu_{xz}^{k+1} = \frac{\partial G(\alpha + p_\cdot^k)}{\partial U_{xz}}$$
$$\mu_{yz}^{k+1} = \frac{\partial H(\gamma - p_\cdot^k)}{\partial V_{yz}}$$

Update (p_z) in proportion to the excess demand

$$p_z^{k+1} = p_z^k - \delta^k \left(\sum_{x \in \mathcal{X}} \mu_{xz}^{k+1} - \sum_{y \in \mathcal{Y}} \mu_{yz}^{k+1} \right).$$

3. Stop when $\left(p_z^{k+1}\right)$ is close enough to $\left(p_z^k\right)$.

The algorithm converges to the equilibrium prices and supply and demand.

7 And the Schrödinger problem?

Recall the network flow problem (here with forced participation)

$$\begin{aligned} &\max_{\mu \in \mathbb{R}_+^{\mathcal{A}}} \sum_{a \in A} \mu_a \phi_a \\ s.t. \quad &\nabla^* \mu = N \;\; [U] \end{aligned}$$

and its dual

$$\begin{aligned} &\min_{U \in \mathbb{R}^{\mathcal{N}}} \sum_{w \in \mathcal{N}} U_w N_w \\ s.t. \quad &\nabla U \geq \phi \;\; [\mu]\,. \end{aligned}$$

Consider the same problem, but with an entropic penalization

$$\begin{aligned} &\max \sum_a \mu_a \phi_a - \sigma \mathcal{E}(\mu) \\ &\text{s.t. } \nabla^* \mu = N \;\; [U] \end{aligned}$$

which expresses as

$$\min_U \langle N, U \rangle + \max_\mu \langle \mu, \phi - \nabla U \rangle - \sigma \mathcal{E}(\mu)\,.$$

By FOC with respect to μ,

$$\frac{\phi_a - (\nabla U)_a}{\sigma} = \frac{\partial \mathcal{E}(\mu)}{\partial \mu_a}$$

thus

$$\frac{\phi_{ww'} - (U_{w'} - U_w)}{\sigma} = \frac{\partial \mathcal{E}(\mu)}{\partial \mu_{ww'}}$$

and in particular, when $\mathcal{E}(\mu) = \sum_a \mu_a \ln \mu_a$,

$$\begin{aligned} \frac{\phi_{ww'} - (U_{w'} - U_w) - \sigma}{\sigma} &= \ln \mu_{ww'} \\ \nabla^* \mu &= N \end{aligned}$$

which, in the case of a bipartite network, boils down to the two-margin Bernstein-Schrödinger problem

$$\frac{\phi_{xy} - (V_y - U_x) - \sigma}{\sigma} = \ln \mu_{xy}$$
$$\sum_y \mu_{xy} = n_x, \ \sum_x \mu_{xy} = m_y$$

A problem studied under various forms by a number of authors: Czisar; Sinkhorn; Yuille, Rüschendorf in the static case; Schrödinger; Bernstein; Föllmer; Nelson; Léonard; Zambrini in the dynamic case.

7.1 Challenges

- Provide a consistent estimation of the distribution of the unobservable heterogeneity.

- Relax assumption that utility is quasi-linear with respect to money; convex analytic formulation no longer applies; no variational principles. Approach related to matching with imperfectly transferable utility. Rich structure regardless (lattice structure, monotonicity, submodularity).

- Relax one-to-one assumption: one consumer chooses a bundle of goods. Need to model complementarty and substitutability between goods; trade-off between quality and quantity.Cf. Kelso-Crawford.

THANK YOU!

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