Decoupling DeGiorgi systems via multi-marginal mass transport

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On the occasion of Ivar's 70th birthday

Paris,
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De Giorgi's original conjecture

Suppose that u is an entire solution of the Allen-Cahn equation

$$\Delta u = H'(u) = u^3 - u \quad \text{on} \quad \mathbb{R}^N \tag{1}$$

satisfying

$$|u(\mathbf{x})| \leq 1, \ \frac{\partial u}{\partial x_N}(\mathbf{x}) > 0 \text{ for } \mathbf{x} = (\mathbf{x}', x_N) \in \mathbb{R}^N.$$

Then, at least for $N \leq 8$ the level sets of u must be hyperplanes.

The story is now essentially settled and here are the main milestones:

- For $N = 2$ by (Ghoussoub-Gui 1997)
- For $N = 3$ by (Ambrosio-Cabré 2000)
- For $N = 4, 5$, if u is anti-symmetric, by (Ghoussoub-Gui 2003)
- For $N \leq 8$, if u satisfies the additional natural assumption (Savin 2003)

$$\lim_{x_N \to \pm\infty} u(\mathbf{x}', x_N) \to \pm 1.$$

- Counterexample for $N \geq 9$, by (Del Pino-Kowalczyk-Wei 2008)

In low dimensions, the nonlinearity can be more general than $H'(u) = u^3 - u$.

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FIGURE 2. Don't try this in seven dimensions.

From the following article:
Mathematics: How to melt if you must
Ivar Ekeland
Nature **392**, 654-657(16 April 1998)
doi:10.1038/33541

DeGiorgi type results for systems-Motivation

Berestycki, Lin, Wei and Zhao considered a system of $m = 2$ equations which appears as a limiting elliptic system arising in phase separation for multiple states Bose-Einstein condensates.

$$\begin{cases} \Delta u = uv^2 & \text{in } \mathbb{R}^N, \\ \Delta v = vu^2 & \text{in } \mathbb{R}^N, \end{cases} \tag{2}$$

- They show that any positive solution (u, v) of the system (2) such that $\partial_N u > 0$, $\partial_N v < 0$, and which satisfies

$$\int_{B_{2R} \setminus B_R} u^2 + v^2 \leq CR^4,$$

  is necessarily one-dimensional, i.e., there exists $\mathbf{a} \in \mathbb{R}^N$, $|\mathbf{a}| = 1$ such that

$$((u(x), v(x)) = (U(\mathbf{a} \cdot x), V(\mathbf{a} \cdot x)),$$

  where (U, V) is a solution of the corresponding one-dimensional system.
- In particular, any such "monotone" solution (u, v) which satisfies $u(x), v(x) = O(|x|^k)$ is necessarily one-dimensional provided $N \leq 4 - 2k$.
- Berestycki,Terracini, Wang and Wei: Same result for stable solutions. But polynomial growth not enough.

DeGiorgi conjecture for systems

Consider the gradient system

$$\Delta u = \nabla H(u) \ \text{ in } \ \mathbb{R}^N, \tag{3}$$

where $u : \mathbb{R}^N \to \mathbb{R}^m$, $H \in C^2(\mathbb{R}^m)$ and $\nabla H(u) = \frac{\partial H}{\partial u_i}(u_1, u_2, ...u_m))_i$.
Consider solutions whose components $(u_1, u_2, ...u_m)$ are strictly monotone in x_N. They need not be all increasing (or decreasing).

1. Say that the level set of the component u_i is a hyperplane if $u_i(x', x_N) = g_i(\mathbf{a}_i \cdot x' - x_N)$ for some $\mathbf{a}_i \in \mathbf{S}^{N-1}$.

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4. Say that the system is decoupled at $u = (u_i)_{i=1}^m$, if there exist $V_i, i = 1, ..., m$ such that

$$\Delta u_i = V_i'(u_i(x)) \quad i = 1, ..., m. \tag{4}$$

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Conjecture

Suppose $u = (u_i)_{i=1}^m$ is a monotone bounded entire solutions of the system (3). Under what conditions on H, one can show that at least in low dimensions

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Which of the main ideas extend to systems?

For $H \in C^2(\mathbb{R}^2)$, consider the system

$$\begin{cases} \Delta u = H_u(u,v) & \text{in } \mathbb{R}^N, \\ \Delta v = H_v(u,v) & \text{in } \mathbb{R}^N, \end{cases}$$

where the energy of a solution (u, v) on a ball B_R is defined as

$$E_R(u,v) = \int_{B_R} \frac{1}{2}|\nabla u|^2 d\mathbf{x} + \int_{B_R} \frac{1}{2}|\nabla v|^2 d\mathbf{x} + \int_{B_R} H(u,v) d\mathbf{x}$$

Natural questions and open problems:

1. Suitable linear Liouville theorems for systems? YES
2. Suitable monotonicity of solutions? YES

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Consider any solution u of $\Delta u = H'(u)$ such that $\phi := \frac{\partial u}{\partial x_N} > 0$. Take any other directional derivative $\psi := \nabla u \cdot \nu$. Then $\sigma := \frac{\psi}{\phi}$ satisfies the linear

$$-\mathrm{div}(\phi^2(x)\nabla\sigma) = 0 \qquad x \in \mathbb{R}^n, \tag{5}$$

So it suffices to establish a Liouville theorem of the type:

If $\mathrm{div}(\phi^2(x)\nabla\sigma) = 0$ in $\mathbb{R}^n$, then σ is constant.

This has been verified under the following conditions:

- If $\phi(x) \geq \delta > 0$ and σ is bounded below. (Liouville and Moser)
- If $\phi\sigma$ bounded (Berestycki-Caffarelli-Nirenberg, Ghoussoub-Gui)
- If $\int_{B_{2R}\setminus B_R} \phi^2\sigma^2 \leq CR^2$(Ambrosio-Cabre).

Let's use the same linearization trick on

$$\begin{cases} \Delta u = H_u(u,v) \text{ in } \mathbb{R}^N, \\ \Delta v = H_v(u,v) \text{ in } \mathbb{R}^N. \end{cases} \tag{6}$$

Let $\phi := \partial_N u > 0$ and $\psi := \nabla u \cdot \eta$ for any fixed $\eta = (\eta', 0) \in \mathbb{R}^{N-1} \times \{0\}$.
Let $\tilde{\phi} := \partial_N v < 0$ and $\tilde{\psi} := \nabla v \cdot \eta$ for the given $\eta = (\eta', 0) \in \mathbb{R}^{N-1} \times \{0\}$.
Then $(\phi, \tilde{\phi})$ and $(\psi, \tilde{\psi})$ satisfy the following systems

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and

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$\sigma := \frac{\psi}{\phi}$ and $\tau := \frac{\tilde{\psi}}{\tilde{\phi}}$ are then solutions of the linear system

$$\begin{cases} \mathrm{div}(\gamma(x)\nabla\sigma) = \lambda(x)(\sigma - \tau) \text{ in } \mathbb{R}^N, \\ \mathrm{div}(\tilde{\gamma}(x)\nabla\tau) = -\lambda(x)(\sigma - \tau) \text{ in } \mathbb{R}^N, \end{cases} \tag{9}$$

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Lemma (Linear Liouville theorems for systems) Let (σ, τ) be a solution of

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such that

$$\lambda(x) = H_{uv}(u, v)\phi(x)\tilde{\phi}(x) \leq 0$$

and

$$\int_{B_{2R} \setminus B_R} \gamma\sigma^2 + \tilde{\gamma}\tau^2 \leq CR^2.$$

Then, σ and τ are constants.

- This is a direct extension of the single equation case initiated by Berestycki-Caffarelli-Nirenberg and used by Ghoussoub-Gui in dimension 2, and by Ambrosio-Cabre in dimension 3.

Say that a solution $u = (u_k)_{k=1}^m$ is *H-monotone* if the following hold:

1. For every $i \in \{1, \ldots, m\}$, u_i is strictly monotone in the x_N-variable (i.e., $\partial_N u_i \neq 0$).
2. For $i < j$, we have

$$H_{u_i u_j} \partial_N u_i(x) \partial_N u_j(x) \leq 0 \text{ for all } x \in \mathbb{R}^N. \tag{11}$$

The existence of an H-monotone solution implies that there exists $(\theta_i)_i$ that do not change sign (in this case $\theta_i = \partial_N u_i$)

$$H_{u_i u_j} \theta_i \theta_j \leq 0 \text{ for all } x \in \mathbb{R}^N. \tag{12}$$

This will be our definition of orientability of H around u.

Stable solutions

Definition: A solution u of the system (3) on a domain Ω is said to be

(i) *stable*, if the second variation of the corresponding energy functional is nonnegative, i.e., if for every $\zeta_k \in C_c^1(\Omega)$, $k = 1, ..., m$,

$$\sum_i \int_\Omega |\nabla \zeta_i|^2 + \sum_{i,j} \int_\Omega H_{u_i u_j} \zeta_i \zeta_j \geq 0, \tag{13}$$

(ii) *pointwise stable*, if there exist $(\phi_i)_{i=1}^m$ in $C^1(\Omega)$ that do not change sign and $\lambda \geq 0$ such that

$$\Delta \phi_i = \sum_j H_{u_i,u_j} \phi_j - \lambda \phi_i \text{ in } \Omega \text{ for all } i = 1, ..., m, \tag{14}$$

and

$$H_{u_i,u_j} \phi_j \phi_i \leq 0 \text{ for } 1 \leq i < j \leq m. \tag{15}$$

- H-monotone solution $\Longrightarrow$ pointwise stable solution $\Longrightarrow$ stable solution.
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Orientable systems

We say that the system

$$\Delta u = \nabla H(u) \text{ in } \mathbb{R}^N,$$

(or the non-linearity H) is **orientable around** u, if there exist constants $(\theta_k)_{k=1}^m$ such that for all i, j with $1 \leq i < j \leq m$, we have $H_{u_i u_j} \theta_i \theta_j \leq 0$.

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Assume that $m, N \geq 1$ and $\Omega \subset \mathbb{R}^N$ is an open set. Let u be a stable solution $u \in C^2(\Omega)$ of (3). Then, for any $\eta = (\eta_k)_{k=1}^m \in C_c^1(\Omega)$, the following holds:

$$\sum_i \int_\Omega |\nabla u_i|^2 |\nabla \eta_i|^2 \geq \sum_i \int_{|\nabla u_i| \neq 0} \left(|\nabla u_i|^2 \mathcal{A}_i^2 + |\nabla_T |\nabla u_i||^2 \right) \eta_i^2$$

$$+ \sum_{i \neq j} \int_\Omega \left(\nabla u_i \cdot \nabla u_j \eta_i^2 - |\nabla u_i||\nabla u_j| \eta_i \eta_j \right) H_{u_i u_j},$$

where ∇_T stands for the tangential gradient along a given level set of u_i and $\mathcal{A}_i^2$ is the sum of the squares of the principal curvatures of such a level set. The case of a single equation ($m = 1$) is:

$$\int_\Omega |\nabla u|^2 |\nabla \eta|^2 \geq \int_{|\nabla u| \neq 0} \left(|\nabla u|^2 \mathcal{A}^2 + |\nabla_T |\nabla u|^2 \right) \eta^2$$

- Sternberg-Zumbrun to study semilinear phase transitions problems.
- Farina-Sciunzi-Valdinoci to reprove some results about the De Giorgi's conjecture.
- Cabré to prove the boundedness of extremal solutions of semilinear elliptic equations with Dirichlet boundary conditions on a convex domain up to dimension four.

De Giorgi type results

Consider the standard test function

$$\chi(x) := \begin{cases} \frac{1}{2}, & \text{if } |x| \leq \sqrt{R}, \\ \frac{\log \frac{R}{|x|}}{\log R}, & \text{if } \sqrt{R} < |x| < R, \\ 0, & \text{if } |x| \geq R. \end{cases}$$

Since the system (3) is *orientable*, there exist nonzero functions $\theta_k \in C^1(\mathbf{R}^N)$, $k = 1, \cdots, m$, which do not change sign such that

$$H_{u_i u_j} \theta_i \theta_j \leq 0, \quad \text{for all} \quad i, j \in \{1, \cdots, m\} \text{ and } \quad i < j. \tag{16}$$

Consider $\eta_k := sgn(\theta_k)\chi$ for $1 \leq k \leq m$, where $sgn(x)$ is the Sign function. The geometric Poincaré inequality yields

$$\int_{B_R \setminus B_{\sqrt{R}}} \sum_i |\nabla u_i|^2 |\nabla \chi|^2 \geq \sum_i \int_{|\nabla u_i| \neq 0} \left(|\nabla u_i|^2 \kappa_i^2 + |\nabla_T |\nabla u_i||^2 \right) \chi^2$$

$$+ \sum_{i \neq j} \int_{\mathbf{R}^N} \left(\nabla u_i \cdot \nabla u_j - sgn(\theta_i) sgn(\theta_j) |\nabla u_i||\nabla u_j| \right) H_{u_i u_j} \chi^2 = I_1 + I_2.$$

I_1 is clearly nonnegative. Moreover, $H_{u_i u_j} sgn(\theta_i) sgn(\theta_j) \leq 0$ for all $i < j$, and therefore, I_2 can be written as

$$I_2 = \sum_{i \neq j} \int_{\mathbf{R}^N} \left(sgn(H_{u_i u_j}) \nabla u_i \cdot \nabla u_j + |\nabla u_i||\nabla u_j| \right) H_{u_i u_j} sgn(H_{u_i u_j}) \chi^2 \geq 0.$$

Since

$$\int_{B_R\setminus B_{\sqrt{R}}} \sum_i |\nabla u_i|^2 |\nabla \chi|^2 \le C \begin{cases} \frac{1}{\log R}, & \text{if } N = 2, \\ \frac{R^{N-2}+R^{(N-2)/2}}{|N-2||\log R|^2}, & \text{if } N \neq 2, \end{cases}$$

So in dimension $N = 2$, the left hand side of (17) goes to zero as $R \to \infty$.
Since $I_1 = 0$, one concludes that all u_i for $i = 1, \cdots, m$ are one-dimensional.
Since $I_2 = 0$ and provided $H_{u_i u_j}$ is not identically zero, we have for all $x \in \mathbb{R}^2$,

$$-sgn(H_{u_i u_j}) \nabla u_i \cdot \nabla u_j = |\nabla u_i||\nabla u_j|.$$

Hence

Right framework

The concept of "orientable system" seems to be the right framework for dealing with systems of three or more equations. For an orientable system

- Liouville Theorem then holds for linearization.
- The notions of variational stability and spectral (or pointwise) stability coincide.

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The results

Theorem

(Fazly-Ghoussoub) If the dimension $N = 2$, then any bounded stable solution $u = (u_i)_{i=1}^m$ of a system $\Delta u = \nabla H(u)$ on $\mathbb{R}^N$, where H is orientable is necessarily one-dimensional.
Moreover, for $i \neq j$, $\nabla u_i = C_{i,j} \nabla u_j$ for all $x \in \mathbb{R}^2$, where $C_{i,j}$ are constants whose sign is opposite to the one of $H_{u_i u_j}$.

Theorem

(Fazly-Ghoussoub) If $N \leq 3$ and $u = (u_i)_{i=1}^m$ is an H-monotone bounded solution of of a system $\Delta u = \nabla H(u)$ on $\mathbb{R}^N$, then all the compoents of u are one-dimensional.

Theorem

(Ghoussoub-Pass) If $N \leq 3$ and u is an H-monotone solution, then

1. *The components $(u_i)_{i=1}^m$ have common level sets, which are also hyperplanes.*
2. *The system is decoupled at $(u_i)_{i=1}^m$ into m separate ODEs.*

Monge-Kantorovich problems

Given probability measures $\mu_i, i = 1, ..., m$ on $\Omega_i \subset \mathbb{R}$, the optimal transport (or Monge-Kantorovich) problem we consider consists of minimizing

$$\inf \left\{ \int_{\Omega_1 \times \ldots \times \Omega_m} H(p_1, p_2, ..., p_m) d\gamma(p_1, p_2, ..., p_m) \right\} \tag{17}$$

among probability measures γ on $\Omega_1 \times ... \times \Omega_m$ whose i-marginals are μ_i. In this setting, H is called the *cost function*.
(*) If H is bounded below on $\Omega_1 \times ... \times \Omega_m$, then there exists a solution $\bar{\gamma}$ to the Kantorovich problem (17)

Open problem: For which costs H, the solution is unique and is "supported on a graph", that is $\bar{\gamma}$ is of the form $\bar{\gamma} = (I, T_2, T_3, ..., T_{m-1})_{\#}\mu_1$ for a suitable family of point transformations (T_i) such that $T_{i\#}\mu_1 = \mu_i$.

Case $m = 2$ (Original Monge problem) is by now well understood.
Case $m \geq 3$ Mostly open.

Duality

If H is bounded below on $\Omega_1 \times ... \times \Omega_m$, then there exists an m-tuple of functions $(\bar{V}_1, \bar{V}_2, ..., \bar{V}_m)$, which maximizes the following dual problem

$$\sup \left\{ \sum_{i=1}^{m} \int_{\Omega_i} V_i(p_i) d\mu_i ; \ (V_1, V_2, ..., V_m) \right\} \tag{18}$$

among all m-tuples $(V_1, V_2, ..., V_m)$ of functions $V_i \in L^1(\mu_i)$ for which

$$\sum_{i=1}^{m} V_i(p_i) \leq H(p_1, ..., p_m) \text{ for all } (p_1, ..., p_m) \in \Omega_1 \times ... \times \Omega_m.$$

They satisfy for all $i = 1, ..., m$,

$$\bar{V}_i(p_i) = \inf_{\substack{p_j \in \mathbb{R} \\ j \neq i}} \left(H(p_1, p_2, ..., p_m) - \sum_{j \neq i} \bar{V}_j(p_j) \right),$$

$$\inf \left\{ \int_{\Pi_{i=1}^{m} \Omega_i} H(p_1, p_2, ..., p_m) d\gamma ; \gamma \right\} = \sup \left\{ \sum_{i=1}^{m} \int_{\Omega_i} V_i(p_i) d\mu_i ; \ (V_1, V_2, ..., V_m) \right\}$$

Theorem (Carlier): If $\mu_1, ..., \mu_m$ are continuous probability measures on $\mathbb{R}$, and $\frac{\partial^2 H}{\partial q_i \partial q_j} < 0$ on the product of their support. Then, there is a unique solution to the optimal transportation problem (17), given by $\gamma = (I, f_2, f_3, ... f_m)_{\#}\mu_1$, where $f_i : \mathbb{R} \to \mathbb{R}$ is the unique increasing map pushing forward μ_1 to μ_i.

- How to make the condition invariant of changes of variables $p_i \mapsto q_i$, that is for all $i \neq j$

$$\frac{\partial^2 H}{\partial q_i \partial q_j} < 0 \text{ for all } q = (q_1, q_2, ..., q_m) \in \mathbb{R}^m.$$

- Say that $H : \mathbb{R}^m \to \mathbb{R}$ is said to be compatible if for all distinct i, j, k, we have

$$\frac{\partial^2 H}{\partial p_i \partial p_j} \left(\frac{\partial^2 H}{\partial p_k \partial p_j} \right)^{-1} \frac{\partial^2 H}{\partial p_k \partial p_i} < 0 \text{ for all } p = (p_1, p_2, ..., p_m) \in \mathbb{R}^m. \qquad (19)$$

Corollary: If H is compatible, then there is a unique solution to (17) of the form $\gamma = (I, g_1, g_2, ..., g_m)_{\#}\mu_1$, where $g_i : \mathbb{R} \to \mathbb{R}$ is *increasing* if $\frac{\partial^2 H}{\partial p_1 \partial p_i} < 0$ and *decreasing* if $\frac{\partial^2 H}{\partial p_1 \partial p_i} > 0$, and is the unique such map pushing μ_1 to μ_i.

Classical notion of submodular functions

The function $H : A \subset \mathbb{R}^m \to \mathbb{R}$ is submodular (or 2-increasing in Economics) if

$$H(\mathbf{x} + h\mathbf{e}_i + k\mathbf{e}_j) + H(\mathbf{x}) \leq H(\mathbf{x} + h\mathbf{e}_i) + H(\mathbf{x} + k\mathbf{e}_j) \qquad (i \neq j, \quad h, k > 0),$$

where $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{e}_i$ denotes the i-th standard basis vector in $\mathbb{R}^m$.

Theorem (Lorentz, ..., Burchard-Hajaiej): H is submodular on $\mathbb{R}^m_+$ if and only if the following extended Hardy-Littlewood inequality holds:
For all choices of real-valued non-negative measurable functions $(u_1, ..., u_m)$ that vanish at infinity, we have

$$\int_{\mathbb{R}^N} H(u_1^*(x), ..., u_m^*(x))dx \leq \int_{\mathbb{R}^N} H(u_1(x), ..., u_m(x)dx, \qquad (20)$$

where u_i^* is the symmetric decreasing rearrangement of u_i for $i = 1, ..., m$.

Link to mass transport

Lemma: The following are equivalent for a function $H \in C^2(\mathbb{R}^m, \mathbb{R})$.

1. H is orientable.
2. H is compatible.
3. After a change of variables, $\frac{\partial^2 H}{\partial q_i \partial q_j} < 0$.
4. After a change of variables, H is submodular.

Connection to mass transport.

Proposition: Let $u = (u_1, u_2, ..., u_m)$ be a function on $\mathbb{R}^N$, whose components are monotone in the last variable. Let μ be a probability measure on $\mathbb{R}$, absolutely continuous with respect to Lebesgue measure. For each $x' \in \mathbb{R}^{N-1}$, let $\mu_i^{x'}$ be the pushforward of μ by the map $u_i^{x'} : x_N \mapsto u_i(x', x_N)$ and set $\gamma^{x'} := (u_1^{x'}, u_2^{x'}, ..., u_m^{x'})_{\#}\mu$. Then the following are equivalent:

1. u is H-monotone.
2. For each $x' \in \mathbb{R}^{N-1}$, the measure $\gamma^{x'}$ is optimal for the Monge-Kantorovich problem (17), when the marginals are given by $\mu_i^{x'}$.

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Theorem (Ghoussoub-Pass): Let $u = (u_1, ..., u_m)$ be a bounded solution to $\Delta u = \nabla H(u)$.

1. If $u = (u_1, ..., u_m)$ is monotone, then there exist functions $V_i(x', p_i)$ such that u solves the system of decoupled equations:

$$\Delta u_i = \frac{\partial V_i}{\partial p_i}(x', u_i(x)) \quad \text{for } i = 1, ..., m. \tag{21}$$

   Furthermore, along the solution, we have

$$\sum_{i=1}^m V_i(x', u_i(x)) = H(u_1(x), u_2(x), ..., u_m(x)) \text{ for } x \in \mathbb{R}^N. \tag{22}$$

2. If $u = (u_1, ..., u_m)$ is H-monotone, then for all $p = (p_1, p_2, ..., p_m) \in \mathbb{R}^m$,

$$\sum_{i=1}^m V_i(x', p_i) \leq H(p_1, p_2, ..., p_m). \tag{23}$$

3. If the u_i have common level sets, then the V_i can be chosen to be independent of x', that is $V_i(x', p_i) = V_i(p_i)$.

Conjecture: Any H-monotone solution to the system has common level sets.

Proof

1. Fix $x' \in \mathbb{R}^{N-1}$, and define $V_i(\cdot, x')$ on the range of $x_N \mapsto u_i(x', x_N)$ as follows. For p_i in this range, monotonicity ensures the existence of a unique $x_N = x_N(p_i)$ such that $p_i = u_i(x', x_N)$. We can therefore set
$$\frac{\partial V_i}{\partial p_i}(p_i, x') = \frac{\partial H}{\partial p_i}(u(x', x_N)).$$
Actually one can write an explicit expression for the V_i's:
$$V_i(\tilde{p}_i, x') = \int_0^{\tilde{p}_i} \frac{\partial H}{\partial p_i}(u(x', x_N(p_i)))dp_i + H(u(x', 0)). \tag{24}$$
2. If u is H-monotone, then for each $x' \in \mathbb{R}^{n-1}$ the V_i's play the role of Kantorovich potentials, and so we have
$$\sum_{i=1}^{m} V_i(p_i, x') \leq H(p_1, p_2, ..., p_m).$$
3. If the u_i have common level sets, then the image of $(u_1^{x'}, u_2^{x'}, ..., u_m^{x'})$ is independent of x' and the measures $\gamma^{x'}$ are then all supported on the same set, and so we can choose the $V_i(p_i, x') = V_i(p_i)$ to be independent of x'. The image of $(u_1, u_2, ..., u_m)$ is equal to the image of $(u_1^{x'}, u_2^{x'}, ..., u_m^{x'})$ for any fixed x', and any measure supported on this set is optimal for its marginals.

Decoupling

Corollary

- *If* $N \leq 3$ *and* $u = (u_1, \ldots, u_m)$ *is H-monotone, then there exist* $V_i, i = 1, \ldots, m$ *such that*

$$\Delta u_i = V_i'(u_i(x)) \text{ on } \mathbb{R}^N \text{ for } i = 1, \ldots, m. \tag{25}$$

- *For some constant* C *independent of the solution,*

$$\sum_{i=1}^m |\nabla u_i(x)|^2 \leq 2H(u_1(x), \ldots, u_m(x)) + C \text{ for all } x \in \mathbb{R}^N. \tag{26}$$

- *If* $N = 2$*, then* $C = 0$*, hence an extension of Modica's inequality.*

Allen-Cahn potentials with quadratic interaction

Consider $H(u) = \sum_{i \neq j} |u_i - u_j|^2 + \sum_{i=1}^{m} W(u_i)$, where $W(u_i) = \frac{1}{4}(u_i^2 - 1)^2$ is the Allen-Cahn potential. Note that $\frac{\partial^2 H}{\partial u_i \partial u_j} = -2 < 0$, hence orientable. The decoupled system is

$$\Delta u_i = V_i'(u_i) = W'(u_i) = u_i(u_i^2 - 1).$$

and the one dimensional solutions are of the form $u_i(x) = \tanh(\frac{a \cdot x - b}{\sqrt{2}})$, for constants $b \in \mathbb{R}$, $a \in \mathbb{R}^n$, with $|a| = 1$.
These functions, with a common a, b are a one dimensional solution, with common level sets, to the original coupled system

$$\Delta u_i = H'(u_i) = 2(m-1)u_i - 2\sum_{j \neq i}^{m} u_j + u_i(u_i^2 - 1).$$

We do not know whether there are other H-monotone solutions to this equation.

Products of Allen-Cahn potentials

Consider $H(u_1, u_2, ...u_m) = m\log[\frac{1}{m}\sum_{i=1}^m e^{\frac{1}{4}(u_i^2-1)^2}]$. Assuming $|u_i| \leq 1$, the decoupled system becomes

$$\Delta u_i = V_i'(u_i) = W'(u_i) = u_i(u_i^2 - 1). \tag{27}$$

Note that the mixed second order partial derivatives change signs, as

$$\frac{\partial^2 H}{\partial u_i \partial u_j} = \frac{-mu_i u_j(u_i^2-1)(u_j^2-1)e^{\frac{1}{4}(u_i^2-1)^2}e^{\frac{1}{4}(u_j^2-1)^2}}{[\sum_{k=1}^m e^{\frac{1}{4}(u_k^2-1)^2}]^2}.$$

However, H is orientable in any region where $u_i \neq 0$ and $|u_i| \leq 1$. Again, solutions to the decoupled problem

$$u_i(x) = \pm\tanh(\frac{a\cdot x - b}{\sqrt{2}}).$$

One can check directly that these functions solve the original system.

Coupled quadratic system

For $H(u_1, u_2) = \frac{1}{2}u_1^2 u_2^2$, finding solutions with common level sets to the system can be reformulated as looking for a concave function F_1, with conjugate F_2, such that both F_1 and F_2 are increasing on $[0, \infty)$, as well as non-negative functions u_1, u_2, solving the decoupled system

$$\Delta u_1 = 2F_1(u_1^2)$$
$$\Delta u_2 = 2F_2(u_2^2).$$

Any such solution (u_1, u_2) is an H-monotone solution to the original system, with common level sets.
For $N \leq 3$, they are one dimensional. In this case Berestycki et al. found a solution with $u_1(-\infty) = 0$, $u_1(\infty) = \infty$ and $u_2(-\infty) = \infty$, $u_2(-\infty) = 0$.
We can use this solution to build decoupling potentials as follows:

$$V_1(t) = \int_0^t s u_2^2(u_1^{-1}(s))\, ds \text{ and } V_2(t) = \int_0^t s u_1^2(u_2^{-1}(s))\, ds.$$

$F_i(q_i) = 2V_i(\sqrt{q_i})$ are then concave conjugates. while u_1 and u_2 solve the decoupled system

$$\Delta u_i = V_i'(u_i) = u_i F'(u_i^2).$$

We suspect that in low dimensions, any H-monotone solution with appropriate limits must satisfy this decoupled system.

Energy minimizing solutions on Finite domains

Let $\Omega \subset \mathbb{R}^{N-1}$ be a bounded domain. For $u = (u_1, ...u_m) : \Omega \times [0,1] \to \mathbb{R}^m$, set

$$E(u) = \int_{\Omega \times [0,1]} \frac{1}{2} \sum_{i=1}^{m} |\nabla u_i|^2 + H(u) dx$$

Under certain conditions, minimizers of the energy E on bounded domains have common level sets and are 1-dimensional. Heuristically,

- The H-term in the energy that forces the u_i to have the same level sets, so the image of $(u_1, ...u_m)$ is optimal for its marginals.
- The Dirichlet term then forces these level sets to be hyperplanes.
- For a single equation, one dimensional rearrangements reduce the Dirichlet energy. On the other hand, the term $\int H(u_1)dx$ is unchanged by rearrangement, as $u_1 \# dx$ remains the same.
- For systems, $\int H(u_1, u_2, ..., u_m)dx$ will be lowered if the appropriate rearrangements $\bar{u}_i$ can be made so that $(\bar{u}_1, ..., \bar{u}_m)\# dx$ solves the optimal transport problem (17) with marginals $u_i \# dx$. For an orientable H, this is possible. For a non-orientable H, the solution to the optimal transport problem may concentrate on a higher dimensional set.

Gibbons type results

Theorem

Suppose H is orientable and that the set $\{a_i, b_i\}_{i=1}^m$ of constants is consistent with the orientability condition, that is

$$H_{i,j}(u)(b_i - a_i)(b_j - a_j) < 0 \text{ for all } i \neq j. \tag{28}$$

Assume u minimizes the energy E in the class of functions $v = (v_1, ...v_m) : \Omega \times [0,1] \to \mathbb{R}^m$ satisfying

1. $v_i(x', 0) = a_i, \; v_i(x', 1) = b_i,$
2. *v is H-monotone with respect to x_N.*

Then, $u(x', x_N) = u(x_N)$ is one dimensional and the components u_i's have common level sets.

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FIGURE 2. Don't try this in seven dimensions.

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