Hamiltonian Dynamics and Symplectic Rigidity
Ivar-Fest, Paris, June 19th, 2014

June 19, 2014

One can make a convincing point that the (modern) fields of **Dynamical Systems** and **Symplectic Geometry** originate with Poincaré.

One can make a convincing point that the (modern) fields of **Dynamical Systems** and **Symplectic Geometry** originate with Poincaré.

Both fields intersect nontrivially in the area of **Hamiltonian Dynamics**.

Since Poincaré the developments of the fields of Dynamical Systems and Symplectic Geometry have been quite separate. However, it became recently very evident that there is a field best called

Symplectic Dynamics

which is combination of two developed fields with its own set of integrated ideas and having roots in

Dynamical Systems

Symplectic Geometry

A PRIMER IN SYMPLECTIC GEOMETRY

Let (M, ω) be a symplectic manifold.

Let (M, ω) be a symplectic manifold.

What does this actually mean?

Let (M, ω) be a symplectic manifold.

What does this actually mean?

How should we think about it?

Let (M, ω) be a symplectic manifold.

What does this actually mean?

How should we think about it?

Why should we care?

Let (M, ω) be a symplectic manifold.

What does this actually mean?

How should we think about it?

Why should we care?

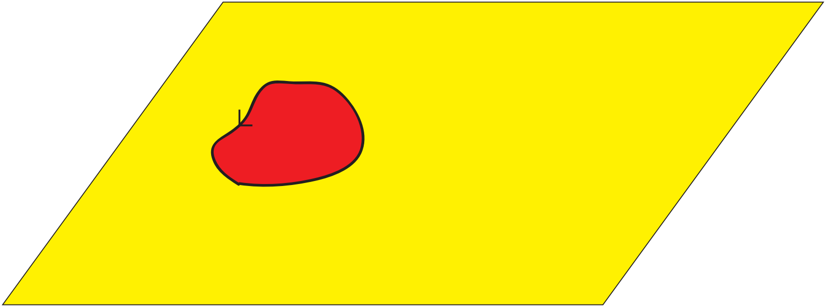
The basic example is $\mathbb{R}^{2n}$

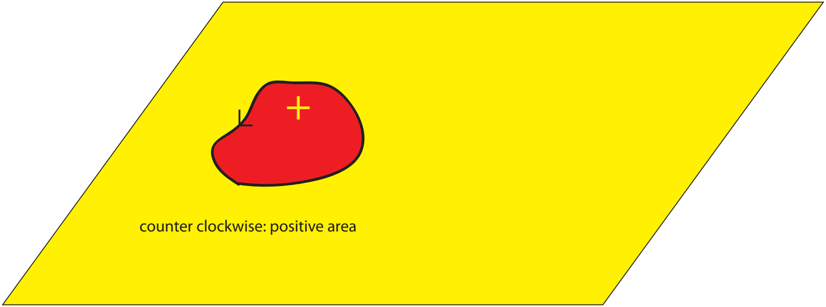
$$\mathbb{R}^{2n} = \mathbb{R}^2 \oplus \ldots \oplus \mathbb{R}^2, \quad z = (z_1, \ldots, z_n), \; z_j = (q_j, p_j).$$

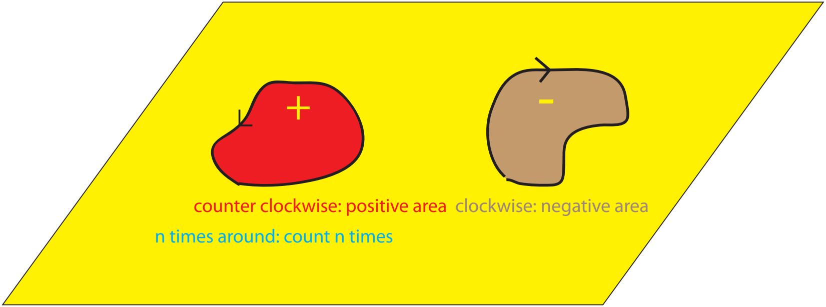
$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j.$$

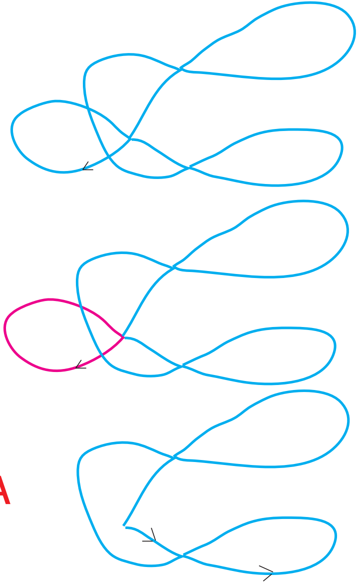
In dimension two a symplectic map preserves area and orientation.

This can be reformulated as follows using loops in the plane.



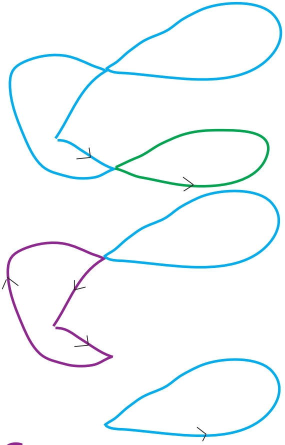

+
counter clockwise: positive area


+
-
counter clockwise: positive area
clockwise: negative area
n times around: count n times



-A

$-A$



$-A+B$

$-A+B-C$

$-A+B-C+D$

$$A = \int_{\text{Loop}} \lambda \quad \text{with} \quad d\lambda = \omega.$$

Action associated to a loop in the plane.

We shall also refer to it as

AREA (the algebraic area associated to a loop).

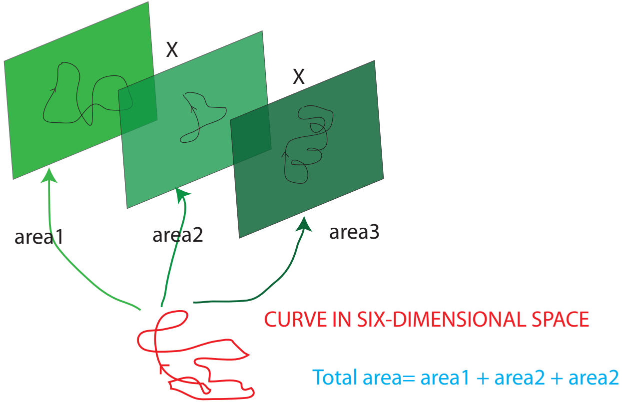
We can associate (signed) AREA to a closed curve in the plane $\mathbb{R}^2$!

We can associate (signed) AREA to a closed curve in the plane $\mathbb{R}^2$!

What do we in higher dimensions?

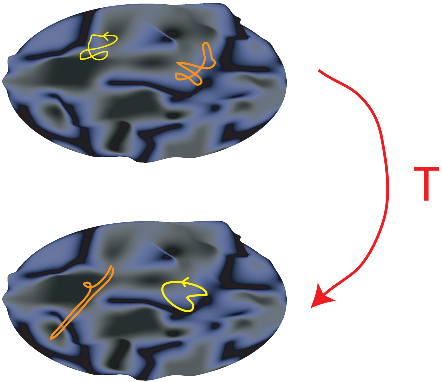
We can associate (signed) AREA to a closed curve in the plane $\mathbb{R}^2$!

What do we in higher dimensions?



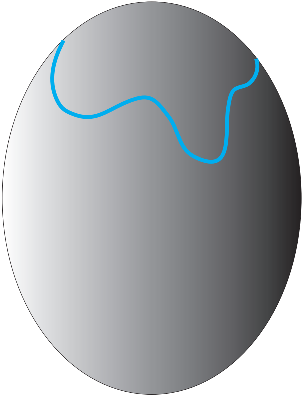
2n-dim space with the structure that we can talk about the area associated to a (small) loop

SYMPLECTIC SPACE



T maps loops to loops with the same area

SYMPLECTIC MAP



Ambiguity of AREA due to topology

Symplectic Geometry is the study of spaces on which it makes sense to talk about the 'algebraic area' associated to small loops and the maps between them preserving area.

Felix Klein: Geometry is the study of invariants under a group of transformations.

Euclidean Geometry
Riemannian Geometry
Metric Geometry

Symplectic Geometry

Points and a notion of distance

AREA of small loops

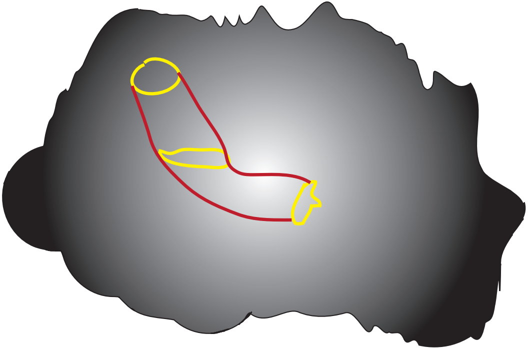
Transformations preserving distance

Transformations preserving AREA of loops

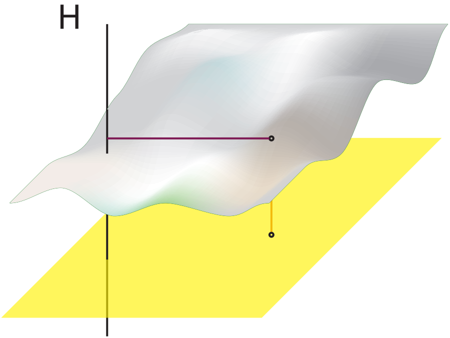
The importance of symplectic spaces and their transformations comes from the fact many physical systems with some preserved quantity can be modeled this way:

P phase space

$$T : P \to P \text{ or } T_t : P \to P$$



T_t



$$<\,-\,-\,-\,-\,-\,> \quad (T_t)$$

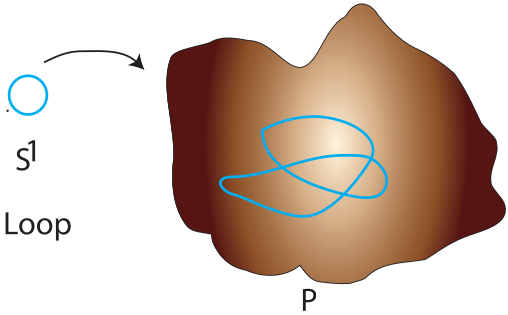
Dynamics is equivalent to giving a possibly time-dependent energy distribution, and the energy is preserved if not depending on time, or otherwise

$$H(t_1, x(t_1)) = H(t_0, x(t_0)) + \int_{t_0}^{t_1} \frac{\partial H}{\partial t}(t, x(t))\,dt$$

Dynamical systems of this kind: Hamiltonian Systems

Poincaré: The fact that the AREA of loops is preserved easily introduces chaos into a system.

Poincaré: Chaos in Celestial Mechanics.



The space of all loops in P is an infinite-dimensional manifold $\Lambda(P)$.
Given an Hamiltonian system, i.e. an energy $H : P \to \mathbb{R}$ we have a natural function

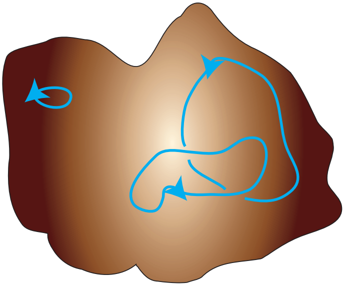
$$\Phi_H : \Lambda(P) \to \mathbb{R} : \Phi_H(x) = \textbf{area}(x) - \int_{S^1} H(x(t))dt.$$

Perhaps an relationship between the shape of $\Lambda(P)$ and the critical points of Φ_H?

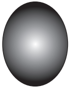
What are the critical points of Φ_H?

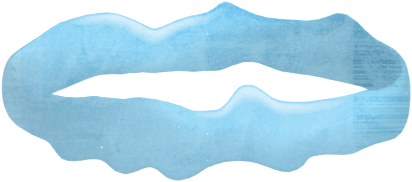
Energy H $< - - - >$ Dynamical system (T_t) on P

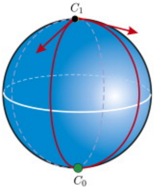
Critical points of Φ_H to a periodic movements, i.e. states moving on loops.

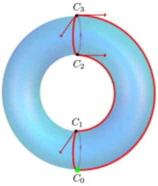


Morse theory is about the relationship between the topology of a smooth manifold and the critical points of a smooth function.









One can learn about the number of critical points from the shape of the space and vice versa.

WAIT A MINUTE!

WAIT A MINUTE!

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XXIX (1976)

Periodic Orbits near an Equilibrium and a Theorem by Alan Weinstein*

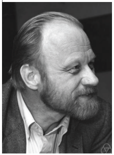
J. MOSER

One could try to base the solution of this boundary value problem on the well known variational principle which calls for the extrema of the functional

$$S=\int \sum y_k\,dx_k-\lambda(H-c)\,ds=\int_0^1\left(\left(y,\frac{dx}{ds}\right)-\lambda(H-c)\right)ds \tag{1.5}$$

over the class of periodic vector functions $z(s)=(x(s),y(s))$ of period 1 and the parameters λ. The Euler equations for this variational principle are given precisely by (1.2) together with $H=c$.

However, this variational principle is very degenerate, for example even the Legendre condition is violated, and is certainly not suitable for an existence proof. To overcome this difficulty we solve the above boundary



RABINOWITZ 78

EKELAND 78 --

AMANN-ZEHNDER 80
CONLEY-ZEHNDER 82 --

CONVEX HAMILTONIAN SYSTEMS

ARNOLD CONJECTURE FOR TORI

Ekeland-Lasry on n solutions

Ekeland's Morse theory

GROMOV 85

NON SQUEEZING

Geometry of the Action

Pseudoholomorphic curve theory

Ekeland-Hofer (1988-1989)
Symplectic Topology and Hamiltonian Dynamics I & II

- **Connection of Symplectic Rigidity with Hamiltonian Dynamics.**
- **Symplectic Capacities**

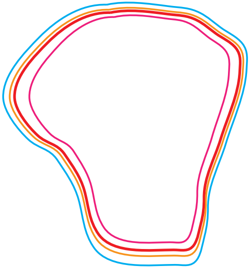
Ekeland-Hofer (1988-1989)
Symplectic Topology and Hamiltonian Dynamics I & II

- **Connection of Symplectic Rigidity with Hamiltonian Dynamics.**
- **Symplectic Capacities**

$$\mathbb{R}^{2n} \supset \Omega \to (c_1(\Omega), c_2(\Omega), ...,)$$

- $c_k(\omega) \in [0, \infty]$, $\lim_k c_k(\Omega) \to \infty$ for $\Omega \neq \varnothing$.
- $c_k(\{(x_1, y_1, .., x_n, y_n) \mid x_1^2 + y_1^2 < 1\}) < \infty$
- $\Psi(\Omega) \subset \Omega'$, then $c_k(\Omega) \leq c_k(\Omega')$.
- For precompact Ω with smooth boundary of restricted contact type $c_k(\Omega) = A(P_\Omega^k)$ for a periodic orbit on $\partial\Omega$.

Ekeland-Hofer (1988-1989)
Symplectic Topology and Hamiltonian Dynamics I & II



• Given a smooth connected compact hypersurface $\Sigma \subset \mathbb{R}^{2n}$ take a smooth foliation of its neighborhood

$$\Sigma_t, \ t \in [-1, 1]$$

with associated Ω_t with $\partial\Omega_t = \Sigma_t$. Then there exists a subset $T \subset [-1, 1]$ of measure 2 so that for every $t \in T$ the EH-capacities of Ω_t are represented by periodic orbits on the boundary.
Almost existence mechanism (Hofer-Zehnder 87)

Symplectic capacities are an important tool in symplectic geometry. There are relationships to Gromov-Witten theory, Floer Theory, Symplectic Field Theory.

A sizable number of questions in symplectic geometry can be phrased as a question about the existence of a symplectic capacity with suitable properties.

Symplectic capacities measure the size of sets and relate them to dynamics, symplectic isotopy etc.

A capacity associates to a symplectic manifold of dimension $2n$ a number $c(M,\omega) \in [0,\infty]$. It should satisfy

- $c(M, r\omega) = r \cdot c(M,\omega)$.
- $(M,\omega) \xrightarrow{symp} (M',\omega')$ then $c(M,\omega) \leq c(M',\omega')$.
- A capacity is said to be proper provided $c(M,\omega) > 0$ for $M \neq \varnothing$ and

$$c(\{(x_1, y_1, .., x_n, y_n) \mid x_1^2 + y_1^2 < 1\}) < \infty.$$

If we have a finite number number of capacities $c_1, .., c_k$ and a positively 1-homogeneous map f we can build a new one $c(U) = f(c_1(U), .., c_k(U))$. We can also take point-wise limits of capacities. Using these procedures a family $\mathcal{F}$ of capacity produces a set of capacities $\mathfrak{G}(\mathcal{F})$

Some results in dimension 4

Introduce the following notation.

$P(a,b) = \{(z_1,z_2) \mid \pi|z_1|^2 < a,\ \pi|z_2|^2 < b\}$

$E(a,b) = \{(z_1,z_2) \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} < 1\}$

$B(a) = E(a,a)$.

Some results in dimension 4

• The EH-capacities $(c_1 \leq c_2 \leq, \dots)$ distinguish ellipsoids $E(a,b)$.
• The capacity v, $v(U) = (\frac{1}{2}\int_U \omega \wedge \omega)^{\frac{1}{2}}$, on the space of contact type balls is not in $\mathfrak{G}(EH)$.
• There exists a sequence of proper capacities ECH $(d_0 \leq d_1 \leq, \dots.)$ so that

$$\lim_{k\to\infty} \frac{d_k(U)}{\sqrt{k}} = 2 \cdot v(U)$$

for contact type balls. (Cristofaro-Gardiner/ Gripp/ Hutchings)
• Neither $\mathfrak{G}(ECH)$ nor $\mathfrak{G}(EH)$ contains the other
For example

$$P(1,1) \to E(a,2a)$$

ECH says $a \geq 1$, EH says $a \geq \frac{3}{2}$, which is known to be optimal.
• For $P(1,2) \to E(c,c)$ ECH and EH say $c \geq 2$, but the optimal answer is $c = 3$ (Hind/Lisi)

What is known in higher dimensions?

• The embedding problem is not classified by EH.
• There is not even conjecture about when $E(a,b,c) \to E(a',b',c')$. In fact all obvious generalizations of the four-dimensional situation are wrong.
• No increasing sequence of proper capacities is known with $\mathfrak{G}(\mathcal{F})$ containing v.

In symplectic geometry most straight forward ideas or generalizations are wrong!

Brouwer: An area and orientation preserving homeomorphism of the open two-disk has a fixed point.

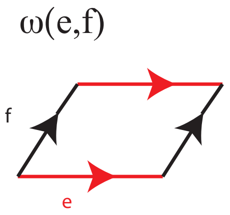
Does a symplectic diffeomorphism of the open four-ball symplectically isotopic to the identity have a fixed point: No! (Morrison)

Conjecture (Hofer): A symplectic diffeomorphism f of the (open) $E(a_1, .., a_n)$ symplectically isotopic to the identity has a fixed point provided the a_i are independent over the integers.

ACTION

and

PSEUDOHOLOMORPHIC CURVES



ω – **Area** : $\text{area}_\omega(e,f) := \omega(e,f).$

$$\omega(e,f) = -\omega(e,f)$$
$$\omega(e,\lambda f_1 + f_2) = \omega(e,f_1) + \lambda\omega(e,f_2)$$

Metric Area: $\text{area}^g_{met}(e,f) := \sqrt{|e|^2|f|^2 - g(e,f)^2}$

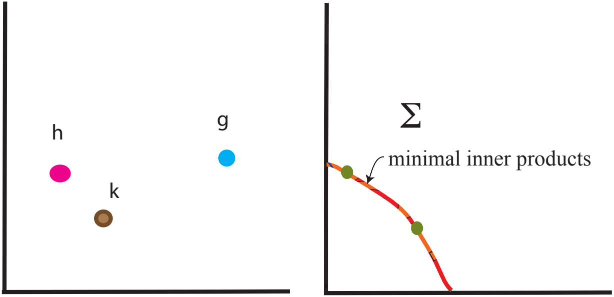
$$g(e,f) = g(f,e),\ \ g(e,e) > 0 \text{ for } e \neq 0$$
$$g(e,f_1 + \lambda f_2) = g(e,f_1) + \lambda g(e,f_2)$$
$$|f| = \sqrt{g(f,f)}$$

ω encapsulates the infinitesimal version of measuring signed area! After a choice of an "inner product g" on $\mathbb{R}^{2n}$ we can measure an associated honest area.

Consider all ways of measuring honest area so that it exceeds the signed area!

$$\Sigma = \{g \mid \text{area}_\omega(e, f) \le \text{area}^g_{met}(e, f) \text{ for all } e, f\}.$$

We give Σ a partial order by saying $g \le h$ if and only if $g(e, e) \le h(e, e)$ for all $e \in \mathbb{R}^{2n}$. Of interest in Σ are the minimal elements.



$\mathbb{R}^{2n}$ skew-symmetric non-degenerate bilinear form ω, which is the (infinitesimal) way to measure signed area.
FACT: $g \in \Sigma$ is minimal if and only if there exists a linear map (matrix) J satisfying

$$J^2 = -Id$$
$$\omega(e, J(f)) = g(e, f) \quad .$$

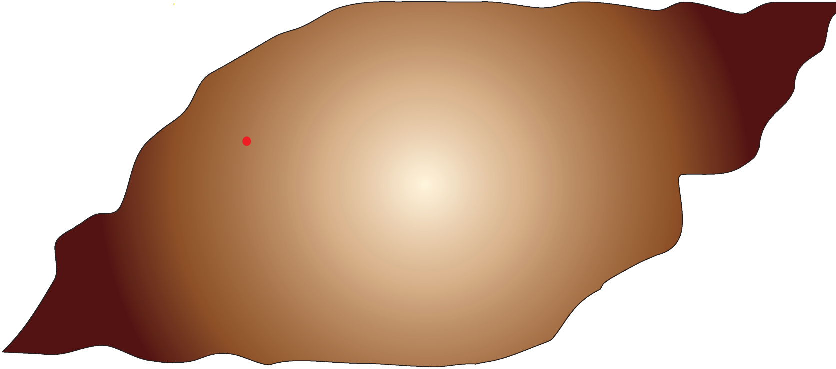
Given a minimal g there is an associated way to define the multiplication of a vector $\mathbb{R}^{2n}$ with a complex number.

$$(a + ib)v = a + bJ(v).$$

There are many choices of such g and J, respectively.

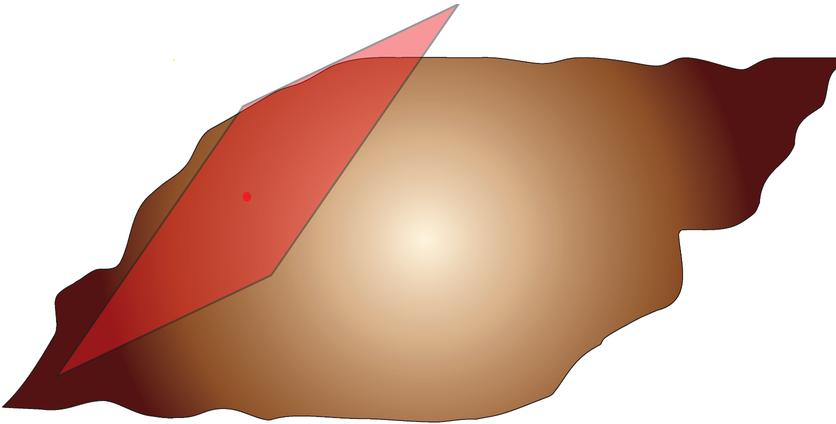
Gromov (1985):

There is a connection between Symplectic Geometry and particular Minimal Surfaces.



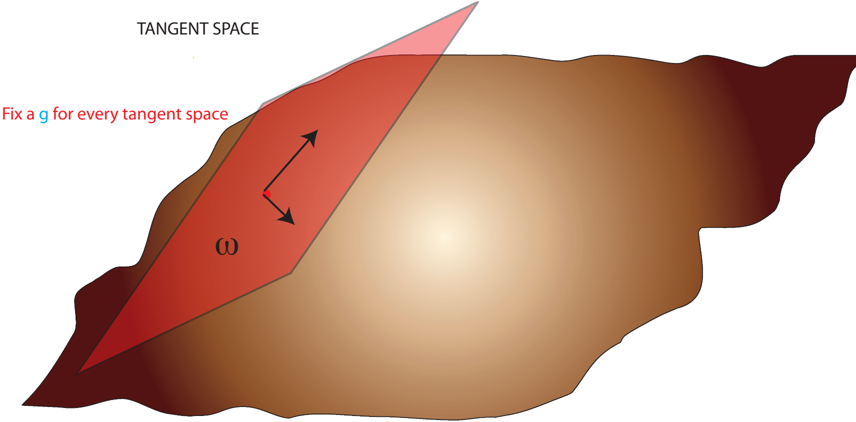
Gromov (1985):

There is a connection between Symplectic Geometry and particular Minimal Surfaces.



Gromov (1985):

There is a connection between Symplectic Geometry and particular Minimal Surfaces.



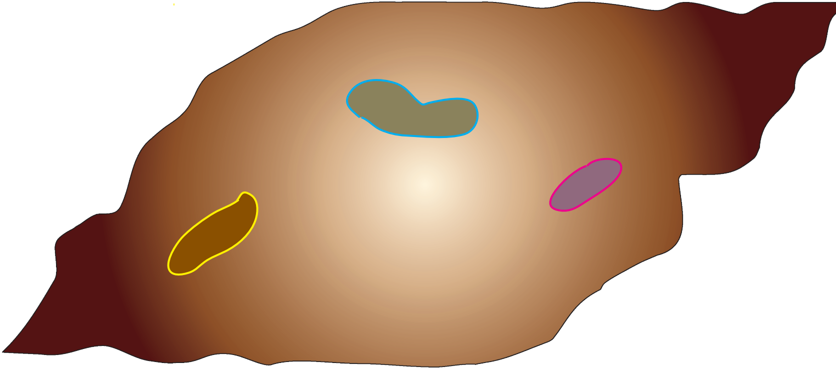
Gromov (1985):

There is a connection between Symplectic Geometry and particular Minimal Surfaces.

We have the notion of SIGNED AREA, which can be positive and negative.
Gromov introduces compatible auxiliary structures to measure some HONEST AREA (an auxiliary structure from the geometry in which length and distance play a role, Riemannian geometry)

Study 2-dimensional surfaces for which, in the small

$$\text{SIGNED AREA} = \text{HONEST AREA}$$



Particular two-dimensional surfaces in symplectic space (depend on the choice of the minimal g)

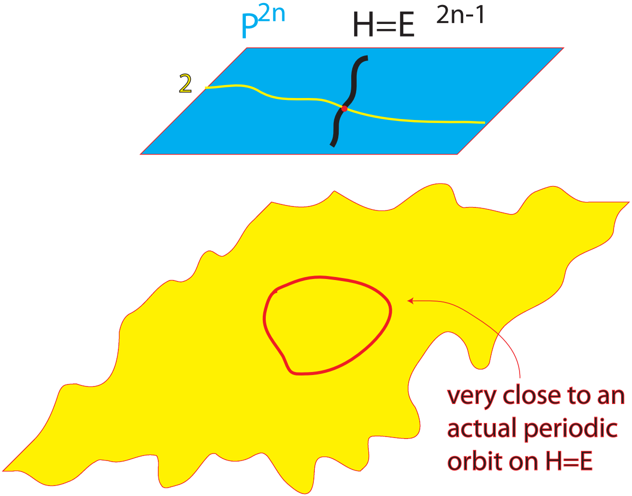
Significance in Dynamics

For the following we are working in a symplectic space P with and energy $H : P \to \mathbb{R}$ and are interested to understand some specifics about the dynamics on the energy surface $H = E$, let's say

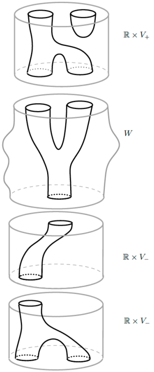
Periodic movements on $H = E$.

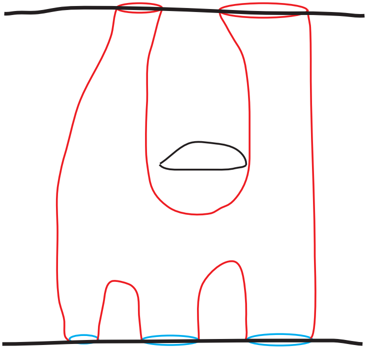
Recall that there are many different measurements of "Honest Area" compatible with the "Signed Area".

Under very general assumptions one can pick this auxiliary structure "Honest Area" very carefully so that one can produce a surface with the properties highlighted in the following picture.



P^{2n}
H=E 2n-1
2
very close to an
actual periodic
orbit on H=E





Periodic orbits on the boundary of domains carry obstructions and pseudoholomorphic curves between them relations.
There is a lot of algebraic structure which can be bundled as
Symplectic Field Theory
Eliashberg-Givental-Hofer 2000, Hofer-Wysocki-Zehnder 2003–

We conclude with some not so obvious applications of this kind of technology due to B. Bramham.

Consider an area-preserving diffeomorphism $T : D \to D$ and normalize area so that $\mu(D) = 1$.

For almost every point $x \in D$ and L^1-map $f : D \to \mathbb{R}$ the following limit exists

$$\hat{f}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$$

This limit depends in general on the point x. When does it not depend?

If $T(E) = E$ implies $\mu(E) = 0$ or 1. (T is ergodic)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int_D f d\mu \quad a.e.$$

A stronger notion is strong mixing:

$$\lim_{k\to\infty} \mu(A \cap T^k(B)) = \mu(A) \cdot \mu(B).$$

An intermediate notion is weak mixing:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} |\mu(A \cap T^k(B)) - \mu(A)\mu(B)| = 0$$

$$\text{strong mixing} \implies \text{weak mixing} \implies \text{ergodic.}$$

An irrational number α is a Liouville number provided for all integers $k \geq 0$ there exists $(p, q) \in \mathbb{Z} \times \mathbb{N}$ relatively prime so that

LIOUVILLE NUMBER

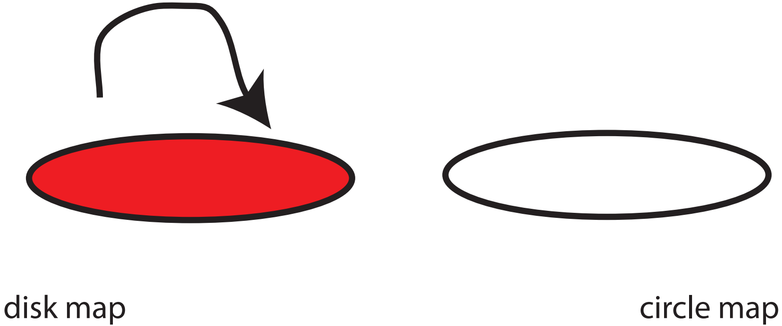
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^k}.$$

Otherwise it is called a

DIOPHANTINE NUMBER

Liouville: Fast approximation by fractions
Diophantine: Slow approximation

$T : D \to D$ is called irrational pseudo-rotation provided it has one periodic point and the rotation number $\rho(T) \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ is an irrational multiple of 2π (mod 2π).



$T : D \to D$ is called irrational pseudo-rotation provided it has one periodic point and the rotation number $\rho(T) \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ is an irrational multiple of 2π (mod 2π).

Anosov-Katok (1970) and Fayad-Saprykina (2005): For every Liouvillean rotation number there exists a pseudo rotation which is weak mixing in particular ergodic.

The question then was since the Anosov-Katok examples: Can there be strong mixing pseudo-rotations?

Herman Conjecture (1998): An irrational pseudo-rotation with diophantine rotation number is smoothly conjugated to a real rotation. He also showed in unpublished work that in the diophantine case they can never be mixing in any sense.

Theorem (Bramham)

Pseudo rotations with "super-Liouvillean" rotation numbers cannot be strongly mixing.

For all $k \geq 1$ there exists a relatively prime $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with

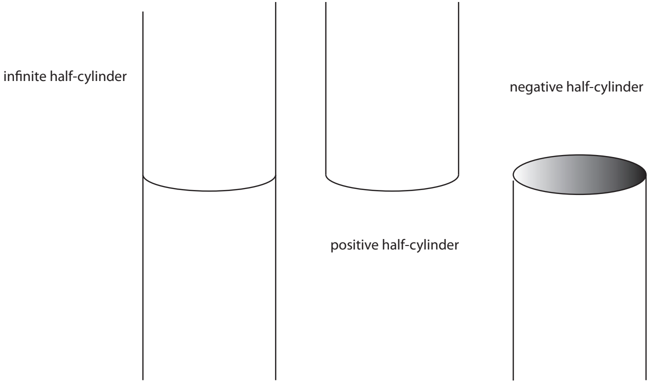
$$|\alpha - \frac{p}{q}| < \frac{1}{e^{kq}}$$

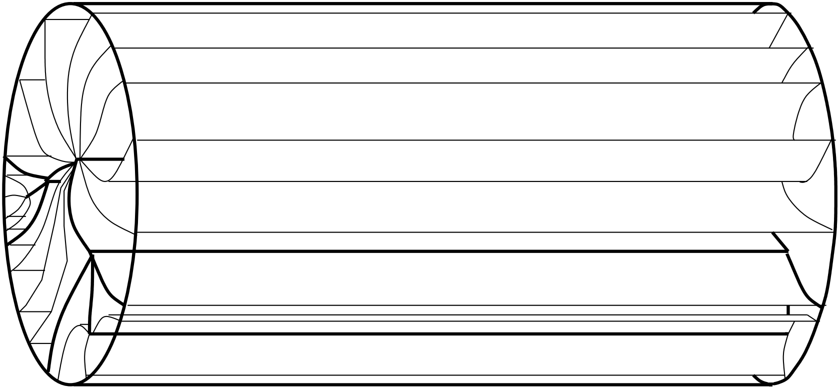
Theorem (Bramham)

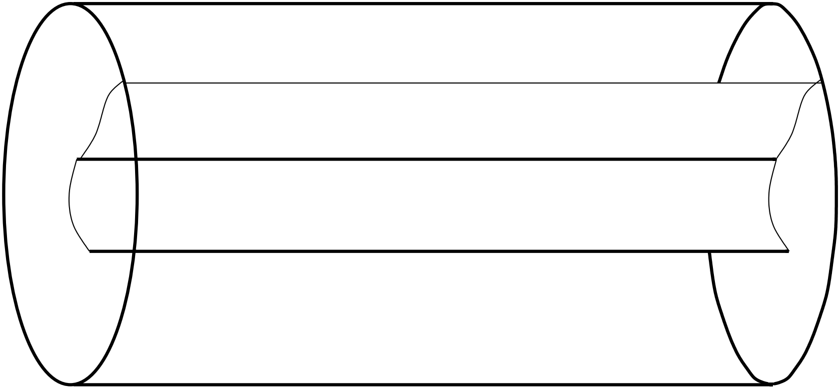
Every smooth, area preserving diffeomorphism of the closed 2-disk having not more than one periodic point is the uniform limit of periodic smooth diffeomorphisms. In particular every smooth irrational pseudo-rotation can be C^0-approximated by integrable systems.

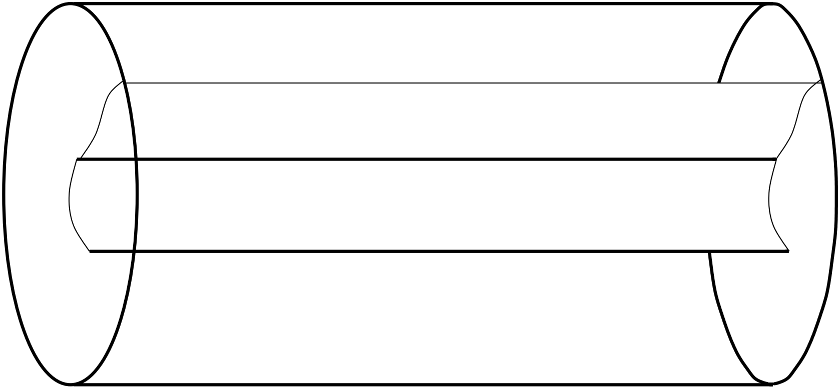
This partially answers a long standing question of A. Katok regarding zero entropy Hamiltonian systems in low dimensions. These are special cases of Bramham's more general theory of using holomorphic curves to study area-preserving disk maps.

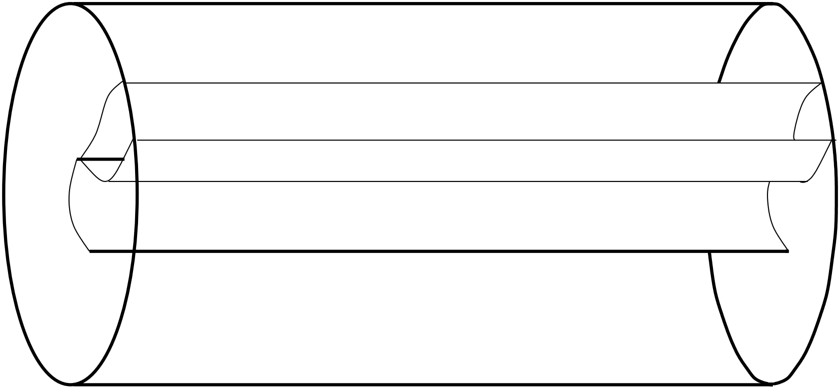
Some pictures illustrating Bramham's results.

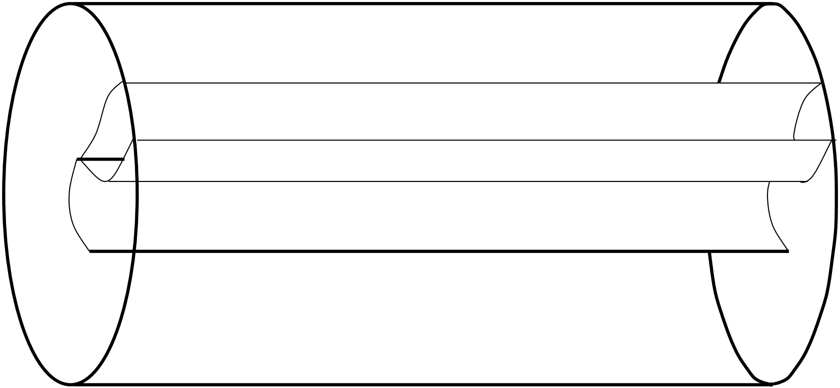


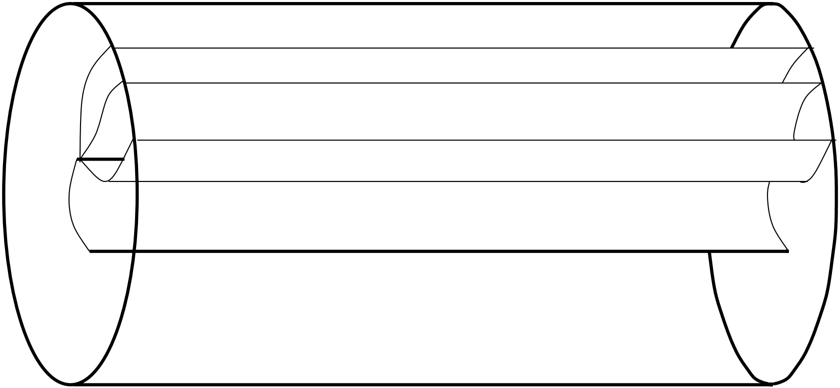


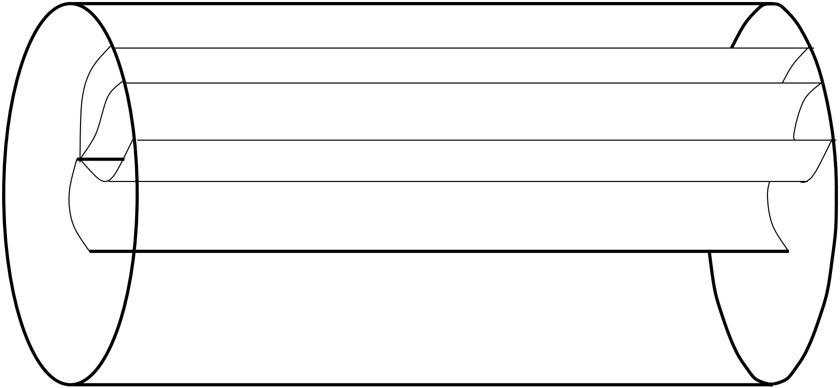


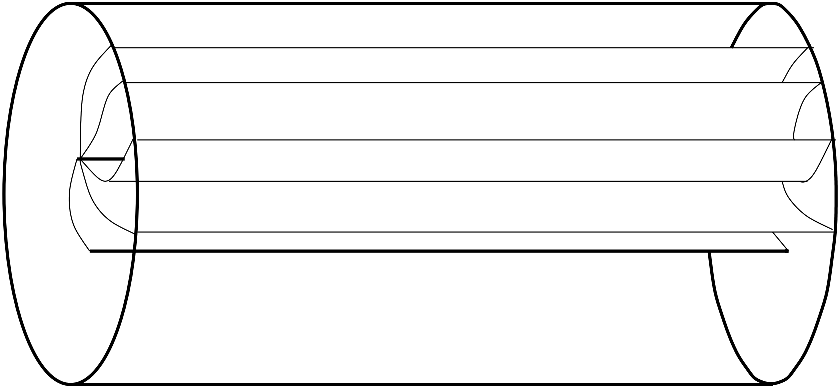


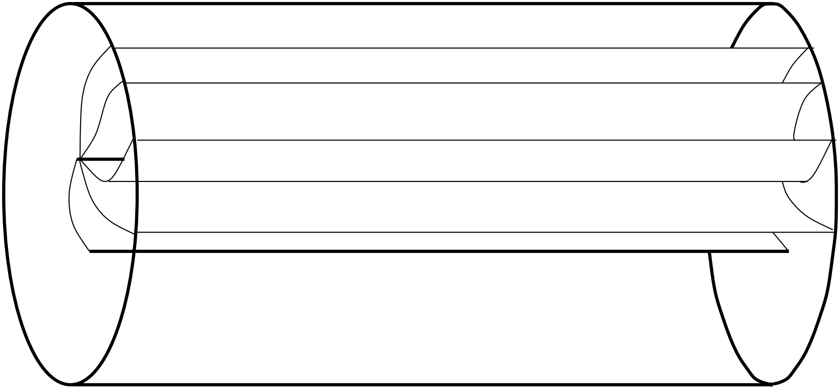


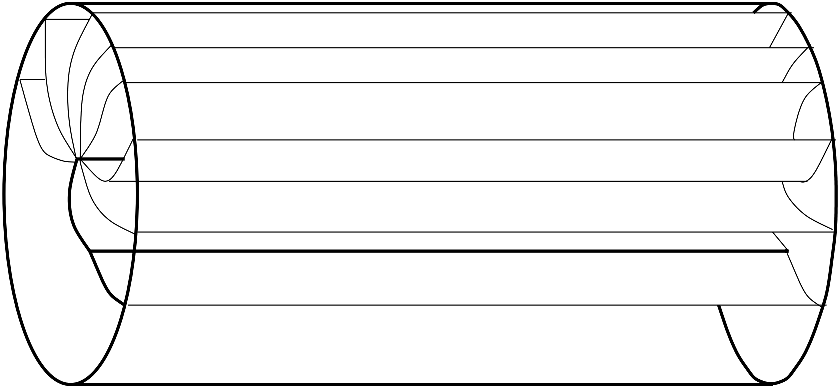


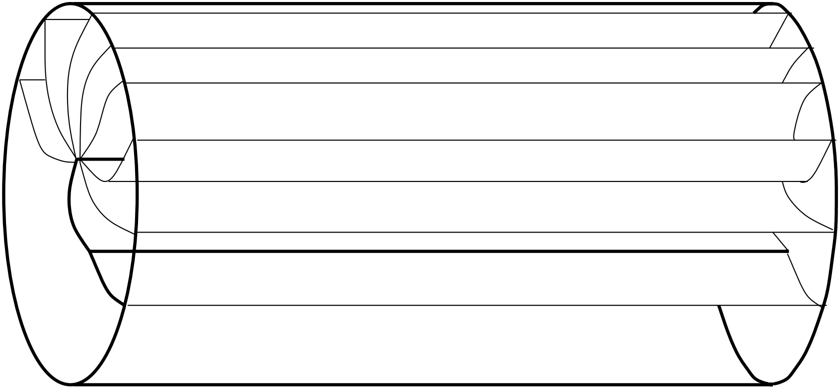


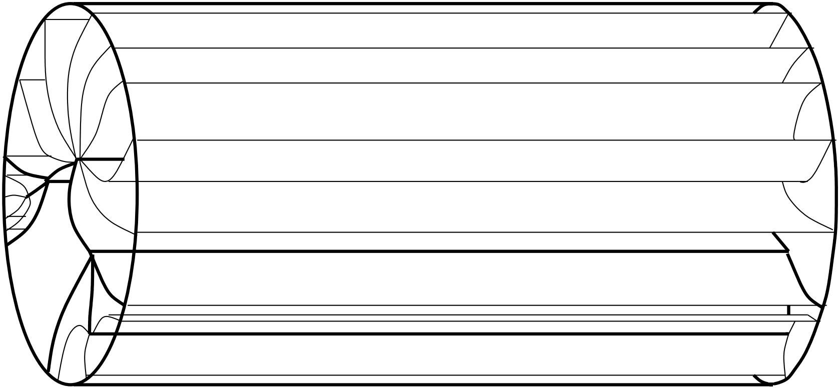


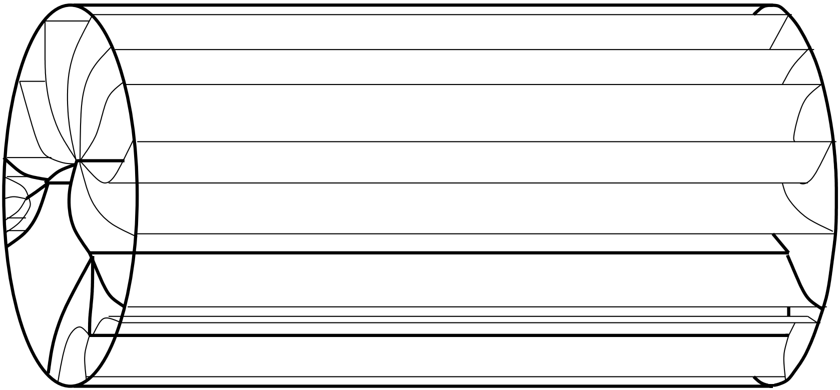


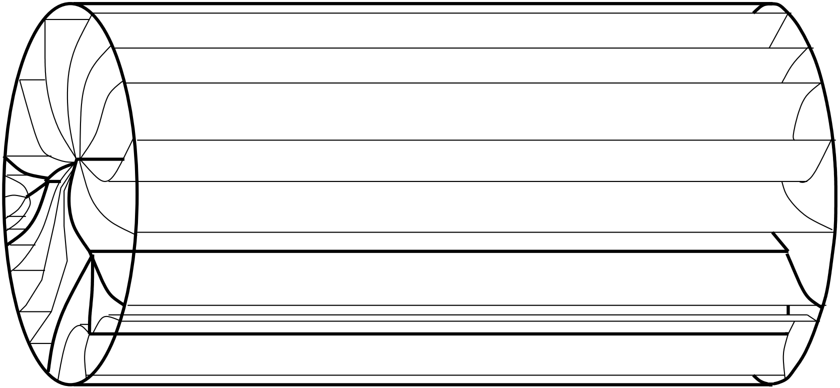


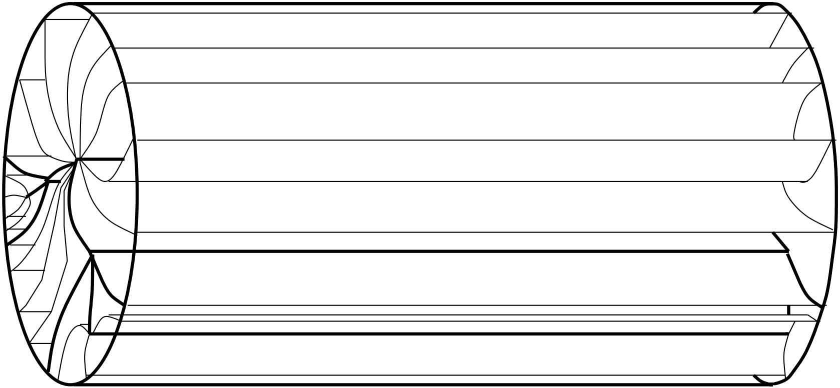




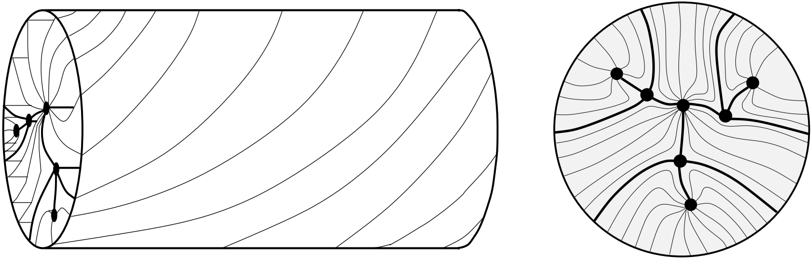








Projected finite energy foliation and cross-section.



A variety of boundary conditions