From Doob's inequality to model-free super-hedging

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joint work with
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Doob's Maximal Inequality

To a martingale $(S_t)_{t=0}^T$ we associate the *maximal function* process

$$\bar{S}_t = \max_{0 \le u \le t} |S_u|, \qquad t = 0, \dots, T.$$

Doob's L^2-inequality:

For every square-integrable martingale S, we have

$$\mathbb{E}\left[\bar{S}_T^2\right] \le 4\mathbb{E}[S_T^2].$$

The factor 4 is sharp, but the inequality is not attained (except for $S \equiv 0$).

Pathwise L^2-inequality [ABPST 2012]:

For every martingale S, there is a predictable strategy H such that

$$\bar{S}_T^2 \le \left[\sum_{t=1}^{T} H_t \, \Delta S_t\right] + 4S_T^2, \qquad \text{almost surely.}$$

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Of course the pathwise inequality implies the classical inequality as

$$\mathbb{E}\left[\int_0^T H_t\ dS_t\right] = \mathbb{E}\left[\sum_{t=1}^T H_t\ \Delta S_t\right] = 0.$$

To show the pathwise inequality we need an easy result.

Elementary Fact:

Let $s_0, s_1, \ldots, s_T$ be non-negative numbers and $\bar{s}_t = \max_{0 \le u \le t} s_u$. Then

$$\bar{s}_T^2 \le \sum_{t=1}^T (-4\bar{s}_{t-1})[s_t - s_{t-1}] + 4s_T^2 - 2s_0^2. \quad (1)$$

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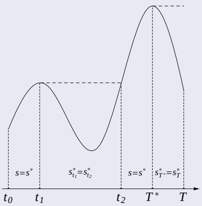
Sketch of Proof:



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4\int_0^T \bar{s}_t ds_t &= 4\int_0^{T^*} \bar{s}_t ds_t \quad + 4\int_{T^*}^T \bar{s}_t ds_t \\
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Equality holds in (1) if and only if $\bar{s}_T = 2s_T$ a.s.

Theorem (slightly sharpend) Doob inequality [ABPST 2012]:

$$\|\bar{S}_T\|_2 \leq \|S_T\|_2 + \|S_T - S_0\|_2$$

This inequality is attained for certain continuous Azema-Yor martingales.

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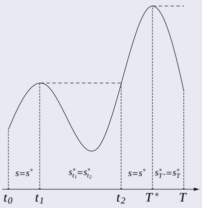

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Financial Interpretation of the pathwise inequality

Interpret $S = (S_t)_{0 \leq t \leq T}$ as a *stock price* process.

Exotic option: pays $\bar{S}_T^2$ at time T.
European option: pays S_T^2 at time T.

For each predictable H the random variable

$$(H \cdot S)_T = \int_0^T H_t dS_t$$

can be interpreted as the (random) gains/losses when applying the *trading strategy* H. These random variables must have price 0 (*no arbitrage*) which corresponds to the martingale property of S.

The pathwise inequality

$$\bar{S}_T^2 \leq \int_0^T H_t dS_t + 4S_T^2$$

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Classical (model-based) Mathematical Finance

Given is a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ and an adapted semi-martingale $(S_t)_{0 \leq t \leq T}$. Let $\mathcal{M}^e(S)$ denote the set of probability measures Q on $\mathcal{F}$, with $Q \sim \mathbb{P}$, and such that S is a (local) Q-martingale.
Basic assumption (no-arbitrage): $\mathcal{M}^e(S) \neq \emptyset$.

1. Complete case (Bachelier, Black-Scholes):

Suppose that $\mathcal{M}^e(S) = \{Q\}$. In this case the *martingale representation theorem* [Itô,...,Yor] gives that every *"contingent claim"* $S_T \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ can be *replicated* as

$$X_T = \mathbb{E}_Q[X_T] + \int_0^T H_t dS_t,$$

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Example:

The contingent claim $X_T = \bar{S}_T^2 - 4S_T^2$ can be *replicated* by

$$\bar{S}_T^2 - 4S_T^2 = X_0 + \int_0^T H_t dS_t$$

where $X_0 = \mathbb{E}_Q[X_T] < 0$ is a negative real number, and H some predictable strategy.

2. Incomplete case:

Suppose that $\mathcal{M}^e(S) \neq \emptyset$, but not a singleton.

Super-replication Theorem:

Every *contingent claim* $X_T \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ can be *super-replicated* by

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This theorem goes back to [El Karoui-Quenez] and, in greater generality, to [Delbaen-S.].

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In a (not necessarily complete) market model, the contingent claim $X_T = \bar{S}_T^2 - 4S_T^2$ can be *super-replicated* by

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Model-free Super-replication

This is a presently active field of research initiated some 15 years ago by D. Hobson:

D. Hobson, A. Cox, M. Davis, J. Obloj, B. Acciaio, M. Beiglböck, B. Bouchard, P. Henry-Labordère, M. Soner, N. Touzi, J. Zhang, Y. Dolinsky,...

We start with a simple setting: the time set of the process $S = (S_t)_{t=0}^2$ only ranges in $t \in \{0, 1, 2\}$. The real number S_0, as well as the *laws* μ_1 and μ_2 of the random variables S_1 and S_2 are *given*. This corresponds to assuming that one can trade all *European options*.

What is unknown, is the *joint law* of (S_1, S_2).

We also are given an *exotic option* $X_T = c(S_0, S_1, S_2)$, e.g. $X_T = \max(S_0, S_1, S_2)$.

Denote by $\mathcal{P}^{mart}$ the set of all probability measures π on $\mathbb{R}^3$ such that, under π, the coordinate process $S = (S_0, S_1, S_2)$ is a *martingale* and such that the one-dimensional marginals are given by μ_0, μ_1, and μ_2, where $\mu_0 = \delta_{S_0}$.

Theorem [Beiglböck, Henry-Labordère, Penkner 2013]:

Consider $(S_t)_{t=0}^2$ as above. Let c be bounded and upper semi-continuous, and consider the contingent claim $X_T = c(S_1, S_2)$. Then the *largest model independent martingale expectation price*, defined by

$$P = \sup_{\pi \in \mathcal{P}^{mart}} \{\mathbb{E}_\pi [c(S_1, S_2)]\}$$

equals the *smallest model independent arbitrage free price*, defined by

$$D = \inf_{h_1, h_2, H_1} \left\{ d : c(S_1, S_2) \le d + \underbrace{h_1(S_1) + h_2(S_2)}_{\text{European options}} + \underbrace{H_1(S_1)[S_2 - S_1]}_{\text{dynamic trading}} \right\}$$

where $h_1(\cdot), h_2(\cdot)$ are bounded, measurable functions with $\mathbb{E}_{\mu_1}[h_1] = \mathbb{E}_{\mu_2}[h_2] = 0$, and $H_1(\cdot)$ is bounded and measurable.

Sketch of proof:

$$
\begin{aligned}
P &= \sup_{\pi \in \mathcal{P}^{mart}} \left\{ \mathbb{E}_\pi \left[c(S_1, S_2) \right] \right\} \\
&= \sup_{\pi \in \mathcal{P}^{mart}} \inf_{H_1(\cdot)} \left\{ \mathbb{E}_\pi \left[c(S_1, S_2) - H_1(S_1)[S_2 - S_1] \right] \right\} \\
&= \sup_{\pi \in \mathcal{P}} \inf_{H_1(\cdot)} \left\{ \mathbb{E}_\pi \left[c(S_1, S_2) - H_1(S_1)[S_2 - S_1] \right] \right\}.
\end{aligned}
$$

Here $\mathcal{P}$ denotes *all* probability measures on $\mathbb{R}^3$ with the given marginals $\mu_0 = \delta_{S_0}, \mu_1, \mu_2$, but which are not necessarily martingale measures for the coordinate process.

The compactness of $\mathcal{P}$ now allows to interchange the sup and the inf, so that

$$
P = \inf_{H_1(\cdot)} \sup_{\pi \in \mathcal{P}} \left\{ \mathbb{E}_\pi \left[c(S_1, S_2) - H_1(S_1)[S_2 - S_1] \right] \right\}
$$

For fixed H_1 we finally may apply the duality theory of optimal transport [Kellerer 1984] to obtain $P = D$. ∎

The extension of the above theorem to finite discrete time is straight-forward [Beiglböck, Henry-Labordère, Penkner 2013].

Remarkable progress was recently made by [Dolinsky-Soner, 2013] who prove a version of the model-free super-replication theorem in *continuous time*, provided that the cost functional $c\left((S_t)_{0\leq t\leq T}\right)$ satisfies some continuity property with respect to the Skorohod topology.

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Towards a model-independent Fundamental Theorem of Asset Pricing

B. Bouchard, A. Cox, M. Davis, D. Hobson, M. Nutz...

We now assume that the financial market is given by some $\mathbb{R}_+$-valued discrete time process $S = (S_t)_{t=0}^T$, where S_0 and T are fixed. We do not specify further the model and/or a probabilistic base for S, except that we prescribe the *law of the terminal value* S_T, denoted by μ_T, which we assume to have finite first moment and barycenter S_0.
Let $(\varphi_n)_{n=0}^N$ be exotic options, given by continuous functions $\varphi_n = \varphi_n(s_0, s_1, \dots, s_N)$ which are at most of linear growth. We suppose that all φ_n can be traded at time 0 and assume w.l.g. that their price equals zero.

Definition

S allows for *model-independentent arbitrage* if there are real scalars $a_1, \ldots, a_N$, and continuous bounded functions $\Delta_t(s_1, \ldots, s_{t-1})$ such that

$$\sum_{n=1}^{N} a_n \varphi_n(s_1, \ldots, s_T) + \sum_{t=1}^{T} \Delta_t(s_1, \ldots, s_{t-1})[s_{t-1} - s_t] > 0,$$

for *all* $(s_1, \ldots, s_T) \in \mathbb{R}^T$.

Definition

For given S, a market compatible *martingale measure* is a measure π on $\mathbb{R}^T$ such that the coordinate process is a π-martingale, the law of the last coordinate S_T equals μ_T, and such that $\mathbb{E}_\pi(\varphi_n) = 0$ for $n = 1, \ldots, N$.

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Theorem [Acciaio, Beiglböck, Penkner, S. 12]

Under the above assumptions the following statements are equivalent.

(i) S does not allow for model-independent arbitrage

(ii) There exists a market compatible martingale measure on $\mathbb{R}^T$. ■

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Joyeux anniversaire,
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