Viscosity Solutions of Fully Nonlinear Path-Dependent PDEs

Nizar TOUZI

Ecole Polytechnique, France

HAPPY BIRTHDAY IVAR

June 18, 2014

Outline

Paths space and non-anticipative process

- $\Omega = \{\omega \in C^0([0,T],\mathbb{R}^d), \omega_0 = 0\}$, $\|\omega\| = \sup_{t\le T}|\omega_t|$
- B canonical process, i.e. $B_t(\omega) = \omega(t)$
- $\mathbb{F} = \{\mathcal{F}_t\}$ the corresponding filtration, i.e. $\mathcal{F}_t = \sigma(B_s, s \le t)$
- $\Lambda = [0,T]\times\Omega$, $d\big[(t,\omega),(t',\omega')\big] = |t-t'| + \|\omega_{.\wedge t} - \omega'_{.\wedge t'}\|$
- $u : \Lambda \longrightarrow \mathbb{R}$ non-anticipative if $u(t,\omega) = u\big(t,(\omega_s)_{s\le t}\big)$

In particular, $u \in C^0(\Lambda) \Longrightarrow u$ non-anticipative

Probability measures on the paths space

- $\mathbb{P}_0$: Wiener measure on Ω, so that B is a $\mathbb{P}_0-$Brownian motion

- $\mathcal{P}_L$: collection of all $\mathbb{P} = \mathbb{P}^{\alpha,\beta}$ such that

$$B_t = \int_0^s \alpha_s^{\mathbb{P}} ds + \int_0^s \beta_s^{\mathbb{P}} dW_t^{\mathbb{P}}, \quad \mathbb{P} - \text{a.s.} \quad \text{for some}$$

  - adapted processes $\alpha^{\mathbb{P}}$, $\beta^{\mathbb{P}}$, with $|\alpha^{\mathbb{P}}| \leq L$ and $\frac{1}{2}|\beta^{\mathbb{P}}|^2 \leq L$
  - and $\mathbb{P}-$Brownian motion $W^{\mathbb{P}}$

In particular,

- $\mathcal{P}_0 = \{\mathbb{P}_0\}$
- Quandratic variation : $\langle B\rangle_t = \int_0^t \left(\beta_s^{\mathbb{P}}\right)^2 ds$, $\mathbb{P}^{\alpha,\beta}-$a.s.

Our objective : nonlinear path-dependent PDEs

Find non-anticipative process $u(t,\omega)$ satisfying :

$$-\partial_t u - G(., u, \partial_\omega u, \partial^2_{\omega\omega} u) = 0, \quad \text{on } [0,T) \times \Omega,$$
$$u(T,\omega) = \xi(\omega)$$

where $\xi(\omega) = \xi\big((\omega_s)_{s\le T}\big)$ and $G(t,\omega,y,z,\gamma)$ is non-anticipative

$$G : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \longrightarrow \mathbb{R}$$

$$G(t,\omega,y,z,\gamma) = G\big(t, (\omega_s)_{s\le t}, y, z, \gamma\big)$$

Differentiability of processes

- For $\varphi \in C^0(\Lambda)$, the right time-derivative is defined by Dupire :

$$\partial_t \varphi(t,\omega) := \lim_{h \to 0+} \frac{1}{h}\Big[\varphi\big(t+h, \omega_{\cdot \wedge t}\big) - \varphi\big(t,\omega\big)\Big], \quad \text{if exists}$$

Definition $\varphi \in C^{1,2}(\Lambda)$ if

- $\varphi, \partial_t \varphi \in C^0(\Lambda)$,
- and there exist $Z \in C^0(\Lambda, \mathbb{R}^d)$, $\Gamma \in C^0(\Lambda, \mathbb{S}^d)$ s.t.

$$d\varphi_t = \partial_t \varphi_t dt + Z_t dB_t + \frac{1}{2}\Gamma_t d\langle B\rangle_t, \ \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \cup_{L>0}\mathcal{P}_L$$

Denote $\partial_\omega \varphi := Z$ and $\partial^2_{\omega\omega}\varphi := \Gamma$

Differentiability of processes

- For $\varphi \in C^0(\Lambda)$, the right time-derivative is defined by Dupire :

$$\partial_t\varphi(t,\omega) := \lim_{h\to 0+} \frac{1}{h}\Big[\varphi\big(t+h,\omega_{\cdot\wedge t}\big) - \varphi\big(t,\omega\big)\Big], \quad \text{if exists}$$

Definition $\varphi \in C^{1,2}(\Lambda)$ if

- $\varphi, \partial_t\varphi \in C^0(\Lambda)$,
- and there exist $Z \in C^0(\Lambda, \mathbb{R}^d)$, $\Gamma \in C^0(\Lambda, \mathbb{S}^d)$ s.t.

$$d\varphi_t = \partial_t\varphi_t dt + Z_t dB_t + \tfrac{1}{2}\Gamma_t d\left\langle B\right\rangle_t, \ \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \cup_{L>0}\mathcal{P}_L$$

Denote $\partial_\omega\varphi := Z$ and $\partial^2_{\omega\omega}\varphi := \Gamma$

Relationship with pathwise derivatives

Dupire 2006 introduced

- Time derivative :

$$\partial_t\varphi(t,\omega) := \lim_{h\to 0+}\frac{1}{h}\Big[\varphi\big(t+h,\omega_{\cdot\wedge t}\big)-\varphi\big(t,\omega\big)\Big], \quad \text{if exists}$$

- Space derivative :

$$\partial_\omega\varphi(t,\omega) := \lim_{h\to 0}\frac{1}{h}\Big[\varphi\big(t,\omega+h\delta_{\{t\}}\big)-\varphi\big(t,\omega\big)\Big], \quad \text{if exists}$$

and proved : if φ is $C^{1,2}$ in this sense,

$$d\varphi_t = \partial_t\varphi_t dt + \partial_\omega\varphi_t dB_t + \frac{1}{2}\partial^2_{\omega\omega}\varphi_t d\langle B\rangle_t, \quad \mathbb{P}-\text{a.s.}$$

for all semimartingale measure $\mathbb{P}$

Path-dependent heat equation : the smooth case

- By using the r.c.p.d. define for $\xi \in \mathbb{L}^1(\mathbb{P}_0)$:

$$u(t,\omega) := \mathbb{E}^{\mathbb{P}_0^{t,\omega}}\left[\xi\right] \quad \text{for all} \quad t \le T,\ \omega \in \Omega$$

- Assume that $u \in C^{1,2}$, then (only $\mathbb{P}_0$ needed) :

$$du_t \;=\; \left(\partial_t u_t + \frac{1}{2}\partial^2_{\omega\omega} u_t\right)dt + \partial_\omega u_t dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

Since u is a $\mathbb{P}_0-$martingale, we obtain the heat equation :

$$\partial_t u + \frac{1}{2}\partial^2_{\omega\omega} u = 0 \quad \text{and} \quad u_T = \xi$$

- Note $u_t(\omega) := \mathbb{E}^{\mathbb{P}_0^{t,\omega}}\left[B_{\frac{T}{2}}\right]$ is not $C^{1,2}$

Path-dependent heat equation : the smooth case

- By using the r.c.p.d. define for $\xi \in \mathbb{L}^1(\mathbb{P}_0)$:

$$u(t,\omega) := \mathbb{E}^{\mathbb{P}_0^{t,\omega}}\left[\xi\right] \quad \text{for all} \quad t \le T,\ \omega \in \Omega$$

- Assume that $u \in C^{1,2}$, then (only $\mathbb{P}_0$ needed) :

$$du_t \;=\; \left(\partial_t u_t + \frac{1}{2}\partial^2_{\omega\omega} u_t\right) dt + \partial_\omega u_t dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

Since u is a $\mathbb{P}_0-$martingale, we obtain the heat equation :

$$\partial_t u + \frac{1}{2}\partial^2_{\omega\omega} u = 0 \quad \text{and} \quad u_T = \xi$$

- Note $u_t(\omega) := \mathbb{E}^{\mathbb{P}_0^{t,\omega}}\left[B_{\frac{T}{2}}\right]$ is not $C^{1,2}$

Path-dependent heat equation : the smooth case

- By using the r.c.p.d. define for $\xi \in \mathbb{L}^1(\mathbb{P}_0)$:

$$u(t,\omega) := \mathbb{E}^{\mathbb{P}_0^{t,\omega}}\left[\xi\right] \quad \text{for all} \quad t \leq T,\ \omega \in \Omega$$

- Assume that $u \in C^{1,2}$, then (only $\mathbb{P}_0$ needed) :

$$du_t \quad = \quad \left(\partial_t u_t + \frac{1}{2}\partial^2_{\omega\omega} u_t\right) dt + \partial_\omega u_t dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

Since u is a $\mathbb{P}_0-$martingale, we obtain the heat equation :

$$\partial_t u + \frac{1}{2}\partial^2_{\omega\omega} u = 0 \quad \text{and} \quad u_T = \xi$$

- Note $u_t(\omega) := \mathbb{E}^{\mathbb{P}_0^{t,\omega}}\left[B_{\frac{T}{2}}\right]$ is not $C^{1,2}$

Example 2 : Backward SDEs

- Backward SDE (Pardoux & Peng '91...) :

$$dY_t = -F_t(\omega, Y_t, Z_t)dt + Z_t dB_t, \quad Y_T = \xi, \quad \mathbb{P}_0 - \text{a.s.}$$

if $u(t, \omega) := Y_t(\omega)$ is $C^{1,2}$, i.e.

$$du_t = \Big(\partial_t u_t + \frac{1}{2}\partial^2_{\omega\omega} u_t\Big)dt + \partial_\omega u_t dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

Then, u solves the semilinear P-PDE

$$-\partial_t u - \frac{1}{2}\partial^2_{\omega\omega} u - F(., u, \partial_\omega u) = 0, \qquad u_T = \xi$$

Note : Existing literature establishes wellposedness of the backward SDE in the space

$$\mathbb{E}^{\mathbb{P}_0}\Big[\int_0^T \big(|Y_t|^2 + |Z_t|^2\big)dt\Big] < \infty$$

i.e. Sobolev solutions of P-PDE (Barles & Lesigne '94)

Example 3 : Stochastic control of path-dependent diffusions

- Stochastic control of non-Markov systems :

$$dX_t^\alpha = b(t, X^\alpha, \alpha_t)dt + \sigma(t, X^\alpha, \alpha_t)dB_t, \quad \mathbb{P}_0 - \text{a.s.}$$

for some $b : \Lambda \times A \longrightarrow \mathbb{R}^d$, $\sigma : \Lambda \times A \longrightarrow \mathbb{S}^d$, and

$$u(t, x_.) := \inf_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0}\Big[\int_t^T L(s, X^\alpha, \alpha_s)ds + \xi\big((X^\alpha{}_s)_{s \le T}\big)\Big]$$

$\Longrightarrow$ Path-dependent HJB equation

$$-\partial_t u - \inf_{a \in A}\Big\{b(., a)\partial_\omega u + \frac{1}{2}\sigma^2(., a)\partial^2_{\omega\omega} u + L(., a)\Big\} = 0, \qquad u_T = \xi$$

- Alternative approach to control of hereditary systems...

OBJECTIVE

Inspired by fully nonlinear PDEs in finite dimensional spaces,

we want to develop a theory of viscosity solutions for path-dependent parabolic fully nonlinear equations

- Existence
- Uniqueness implied by a comparison result (maximum principle)
- Powerful stability result

- Numerical implications : branching diffusion representation $\Longrightarrow$ Monte Carlo approximation (Henry-Labordère, Tan & NT)
- Extension of Barles-Souganidis Monotone schemes (Zhang & Zhuo)
- Regular and singular perturbation (Ma, Ren, Zhang & NT)

Outline

Standard viscosity solutions [M. Crandall & P.-L. Lions '83]

$g(x, y, z, \gamma)$ nondecreasing in γ. Consider the PDE :

$$\text{(E)} \qquad -g(., v, Dv, D^2v)(x) \;=\; 0, \quad x \in \mathcal{O} \quad \text{(open subset of } \mathbb{R}^d\text{)}$$

- v subsolution if $-g(., v, Dv, D^2v) \leq 0$ on $\mathcal{O}$
- v supersolution if $-g(., v, Dv, D^2v) \geq 0$ on $\mathcal{O}$

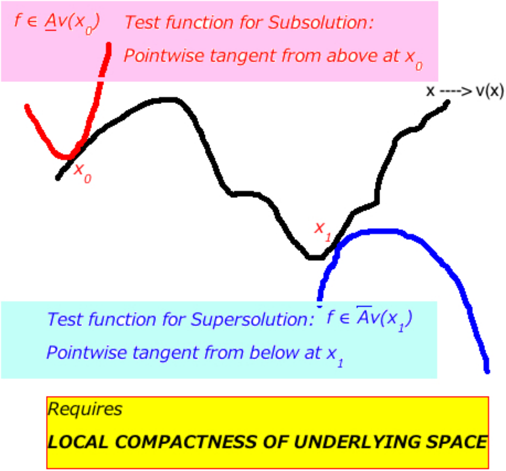

Figure : M.Crandall & PL.Lions test functions

Standard definition of viscosity solutions

Let

$$\underline{A}v(x) \;:=\; \{\varphi \in C^2(\mathcal{O}) : (\varphi - v)(x) = \min_{\mathcal{O}}(\varphi - v)\}$$
$$\overline{A}v(x) \;:=\; \{\varphi \in C^2(\mathcal{O}) : (\varphi - v)(x) = \max_{\mathcal{O}}(\varphi - v)\}$$

Definition $v \in \mathrm{LSC}(\mathcal{O})$ (resp. $\mathrm{USC}(\mathcal{O})$) is a viscosity subsolution (resp. supersolution) of (E) if :

$$-g\big(x, v(x), D\varphi(x), D^2\varphi(x)\big) \leq 0 \text{ (resp. } \geq 0)$$
$$\text{for all } x \in \mathcal{O} \text{ and } \varphi \in \underline{A}v(x) \text{ (resp. } \overline{A}v(x))$$

$v \in C^0(\mathcal{O})$ viscosity solution if viscosity subsol. and supersol.

Intuition from the heat equation

$$v(t,x) := \mathbb{E}^{\mathbb{P}_0}\big[g(B_T)\big|B_t = x\big] \text{ solution of } -\partial_t v - \tfrac{1}{2}v_{xx} = 0.$$

- Tower property :

$$\begin{aligned} v(t,x) &= \mathbb{E}^{\mathbb{P}_0}\Big[\mathbb{E}^{\mathbb{P}_0}[g(B_T)|B_{t+h}]\ \Big|B_t = x\Big] \\ &= \mathbb{E}^{\mathbb{P}_0}\Big[v(t+h, B_{t+h})\Big|B_t = x\Big] \end{aligned}$$

- Viscosity subsol : for $\varphi \in \underline{A}v(t,x)$, $v(t,x) = \varphi(t,x)$ and $v \leq \varphi$

$$\Longrightarrow \quad \varphi(t,x) \leq \mathbb{E}^{\mathbb{P}_0}\Big[\varphi(t+h, B_{t+h})\Big|B_t = x\Big]$$

$$\Longrightarrow \quad \Big(-\partial_t\varphi - \frac{1}{2}\varphi_{xx}\Big)(t,x) \ \leq\ 0$$

Main observation : only need $\mathbb{E}^{\mathbb{P}_0}[v(.,.)] \ \leq\ \mathbb{E}^{\mathbb{P}_0}[\varphi(.,.)]$

Smooth test processes

Nonlinear expectation $\overline{\mathcal{E}}_L := \sup_{\mathbb{P}\in\mathcal{P}_L} \mathbb{E}^{\mathbb{P}}$ and $\underline{\mathcal{E}}_L := \inf_{\mathbb{P}\in\mathcal{P}_L} \mathbb{E}^{\mathbb{P}}$

$\mathcal{T}$: collection of all stopping times τ (i.e. $\{\tau \le t\} \in \mathcal{F}_t$)

Test processes for subsolution and supersolution

$$\underline{\mathcal{A}}^L u_t(\omega) := \Big\{\varphi \in C^{1,2}(\Lambda) : \ (\varphi - u^{t,\omega})_0 = \min_{\tau\in\mathcal{T}} \underline{\mathcal{E}}_L\big[(\varphi - u^{t,\omega})_{.\wedge \mathrm{H}}\big] \text{ for some } \mathrm{H} \in \mathcal{H}\Big\}$$

$$\overline{\mathcal{A}}^L u_t(\omega) := \Big\{\varphi \in C^{1,2}(\Lambda) : \ (\varphi - u^{t,\omega})_0 = \max_{\tau\in\mathcal{T}} \overline{\mathcal{E}}_L\big[(\varphi - u^{t,\omega})_{.\wedge \mathrm{H}}\big] \text{ for some } \mathrm{H} \in \mathcal{H}\Big\}$$

TEST FUNCTIONS TANGENT IN MEAN

Definition (Ekren, NT & Zhang 2012)

$u \in C^0(\Lambda)$ is a viscosity...

- subsolution of PPDE if there exists L :

$$-\partial_t \varphi_0 - G\big(t, \omega, u_t(\omega), \partial_\omega \varphi_0, \partial^2_{\omega\omega} \varphi_0\big) \leq 0$$
$$\text{for all } \ (t, \omega) \in [0, T) \times \Omega \text{ and } \varphi \in \underline{\mathcal{A}}^L u(t, \omega)$$

- supersolution of PPDE if there exists L :

$$-\partial_t \varphi_0 - G\big(t, \omega, u_t(\omega), \partial_\omega \varphi_0, \partial^2_{\omega\omega} \varphi_0\big) \geq 0$$
$$\text{for all } \ (t, \omega) \in [0, T) \times \Omega \text{ and } \varphi \in \overline{\mathcal{A}}^L u(t, \omega)$$

- solution of PPDE if it is viscosity subsolution and supersolution

Comparison with Crandall-P.-L. Lions Definition

The parabolic PDE for $v(t,x)$

$$-\partial_t v - g(t,x,v,Dv,D^2v) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d$$

can be viewed as the path-dependent PDE for $u(t,\omega) := v(t,\omega_t)$:

$$-\partial_t u - G\big(t,\omega,u,\partial_\omega u,\partial^2_{\omega\omega} u\big) = 0, \quad (t,\omega) \in [0,T) \times \Omega$$

where $G(t,\omega,.) = g(t,\omega_t,.)$.Notice that :

$$\varphi \in \underline{A}v(t^*,x^*) \quad \xrightarrow{\phi(t,\omega):=\varphi(t,\omega_t)} \quad \phi \in \underline{\mathcal{A}}u(t^*,\omega^*) \text{ whenever } \omega^*_t = x^*$$

Hence, Our definition involves a larger class of test functions

$\Longrightarrow$ **Helps for uniqueness, existence is more restricted**

On local compactness

- In Crandall-PL.Lions definition, finds point of pointwise tangency

$$(\varphi - u)(x^*) = \min_{\text{closed ball}} (\varphi - u)$$

$\varphi - u$ is LSC, the minimizer exists, need to ensure interior min

- In the context of our definition, find a point of tangency in mean

$$\min_{\tau:\text{stop.time}} \mathbb{E}\big[(\varphi - u)_{\tau\wedge \mathrm{H}}\big]$$

Optimal stopping theory : $\tau^* := \inf\big\{t \geq 0 : Y_t = (\varphi - u)_t\big\}$ is an optimal stopping time, where

$$Y_t \;:=\; \min_{\tau:\text{stop.time}\geq t} \mathbb{E}_t\big[(\varphi - u)_{\tau\wedge \mathrm{H}}\big]$$

and we still need to ensure that $\tau^*(\omega^*) < \mathrm{H}(\omega^*)$ at some point ω^*

Equivalent semijets definition

For $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^d$, $\gamma \in \mathbb{S}_d$, denote the paraboloid :

$$Q^{\alpha,\beta,\gamma}(t,\omega) \quad := \quad \alpha t + \beta \cdot \omega_t + \frac{1}{2}\gamma \omega_t \cdot \omega_t$$

- subjet : $\underline{\mathcal{J}}^L u_t(\omega) := \big\{(\alpha,\beta,\gamma) : Q^{\alpha,\beta,\gamma} \in \underline{\mathcal{A}}^L u_t(\omega)\big\}$,
- superjet : $\overline{\mathcal{J}}^L u_t(\omega) := \big\{(\alpha,\beta,\gamma) : Q^{\alpha,\beta,\gamma} \in \overline{\mathcal{A}}^L u_t(\omega)\big\}$

Theorem

Viscosity subsolution and supersolution can be reduced to paraboloids

Outline

Consistency with classical solutions

Assumption G1 $G(t, \omega, y, z, \gamma)$ nondecreasing in γ and satisfies :
(i) G is uniformly continuous in (t, ω), and $\|G(\cdot, 0, 0, 0)\|_\infty < \infty$.
(ii) G is uniformly Lipschitz in (y, z, γ)

Theorem (Ekren, NT & Zhang 2012a)

Let Assumption G1 hold and $u \in C_b^{1,2}(\Lambda)$. Then the following assertions are equivalent :

- *u classical solution (resp. subsolution, supersolution) of PPDE*
- *u viscosity solution (resp. subsolution, supersolution) of PPDE*

Stability

Consider the perturbed PPDEs

$$\text{PPDE}^\varepsilon: \quad G^\varepsilon\left(., u, \partial_\omega u, \partial^2_{\omega\omega} u\right) = 0 \text{ on } \Lambda$$

Theorem (Ekren, NT & Zhang 2012a)

Let u^ε viscosity $L-$subsolution (resp. $L-$supersolution) of PPDE(G^ε), for some fixed $L > 0$. Assume

$$\left(G^\varepsilon, u^\varepsilon\right) \longrightarrow (G, u) \quad \text{as} \quad \varepsilon \to 0, \quad \text{loc. unif. in} \quad \Lambda.$$

Then u is a viscosity $L-$subsolution (resp. supersolution) of PPDE with coefficient G.

Proofs are easy application of optimal stopping theory

Additional assumptions

Assumption G2 Either one of the following conditions :
(i) G convex in γ and uniformly elliptic,
(ii) or, G is convex in (y, z, γ)
(iii) or, $d \leq 2$

Allows to apply standard PDE theory to a conveniently defined path-frozen equation...

[In next section : avoid relying on PDE results...]

Existence and uniqueness results

Theorem (Ekren, NT & Zhang 2012b)

Under Assumptions G1-G2, let $u^1, u^2 \in UCB(\Lambda)$, $\xi \in UCB(\Omega)$ s.t.

- u^1 *is a bounded viscosity subsolution of PPDE*
- u^2 *is a bounded viscosity supersolution of PPDE*
- $u^1(T,\cdot) \le \xi \le u^2(T,\cdot)$

Then $u^1 \le u^2$ on Λ.

Theorem (Ekren, NT & Zhang 2012b)

Under Assumptions G1, G2, for any $\xi \in \mathrm{UCB}(\Omega)$, the PPDE with terminal condition ξ has a unique bounded viscosity solution $u \in \mathrm{UCB}(\Lambda)$.

Comparison for Semilinear Path-dependent PDEs

Semilinear path-dependent PDEs

(PPDE) $$-\partial_t u - \frac{1}{2}\mathrm{Tr}\big[\partial^2_{\omega\omega} u\big] - F(t,\omega,u,\partial_\omega u) \;=\; 0, \quad t < T$$
$$u_T \;=\; \xi$$

In this case, we only need a subset of probability measures on Ω :

$$\mathcal{P}_L^0 \;:=\; \big\{\mathbb{P} \in \mathcal{P}_L : \;\langle B\rangle = tI_d\big\}$$

Rk We may also add a diffusion $\sigma_t(\omega)$ (positive, Lipschitz in ω)...

Main result

Assumption $F(t, \omega, y, z)$ Lipschitz in (y, z), uniformly in (t, ω), and $F(t, \omega, 0, 0)$ bounded

Denote :

$$C^0_{2,\mathcal{P}^0_L} := \left\{ u \in C^0(\Lambda) : \sup_{\mathbb{P}\in\mathcal{P}^0_L} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} |u_t|^2 \Big] < \infty \right\}$$

Theorem (Ren, NT & Zhang 2014)

Let $u, v \in C^0_{2,\mathcal{P}^0_L}$ be viscosity subsolution and super solution, respectively, of PPDE. Then

$$u_T \le v_T \implies u \le v \text{ on } [0, T] \times \Omega$$

Purely probabilistic proof adapting ideas from Caffarelli and Cabre...

The Linear Case

Theorem (Ren, NT & Zhang 2014)

For $u \in C^0_{2,\mathbb{P}_0}$, the following are equivalent :

- *u is a viscosity subsolution of $-\partial_t u - \frac{1}{2}\partial^2_{\omega\omega} u \leq 0$*
- *u is a submartingale.*

A similar statement holds for supersolutions.

Consequence : comparison for the path-dependent heat equation follows immediately

Punctual Derivatives

- $(\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$: $Q^{\alpha,\beta,\gamma}(t,\omega) := \alpha t + \beta\omega_t + \frac{1}{2}\gamma\omega_t \cdot \omega_t$

- The subject and the superjet of u are defined by

$$\underline{\mathcal{J}}u(t,\omega) := \Big\{(\alpha,\beta,\gamma) : Q^{\alpha,\beta,\gamma} \in \underline{\mathcal{A}}u(t,\omega)\Big\}$$
$$\overline{\mathcal{J}}u(t,\omega) := \Big\{(\alpha,\beta,\gamma) : Q^{\alpha,\beta,\gamma} \in \overline{\mathcal{A}}u(t,\omega)\Big\}$$

Definition u is punctually $C^{1,2}$ at (t,ω) if

$$\mathrm{cl}\big[\underline{\mathcal{J}}\varphi(t,\omega)\big] \cap \mathrm{cl}\big[\overline{\mathcal{J}}\varphi(t,\omega)\big] \neq \emptyset$$

Punctual smoothness of semimartingales

Theorem (Ren, NT & Zhang 2014)

Let u be a pathwise continuous $\mathbb{P}-$ submartingale for some $\mathbb{P} \in \mathcal{P}_L^0$. Then, u is punctually $C^{1,2}$, $Leb \otimes \mathbb{P}_0 - a.e.$

compare to convex functions...

THANK YOU FOR YOUR ATTENTION