De l'inégalité de Prekopa-Leindler à la régularité globale du transport optimal

Journées en l'honneur de Christian Léonard

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Outline

Works in collaboration with M. Fathi and M. Prod'Homme (2020) and M. Sylvestre (2025)

I - Introduction : W_2, Brenier and Caffarelli

II - Two ingredients : entropy regularized transport and Prekopa-Leindler inequality

III - Convexity/smoothness properties of entropic Legendre transforms

IV - Generalizations of Caffarelli's theorem, new proofs and results.

I - Introduction : W_2, Brenier and Caffarelli Theorems

Optimal Transport - classical definition

Let $c : E \times E \to \mathbb{R}^+$ be a measurable function on a Polish space (E, d).

Definition

The optimal transport cost between two probability measures μ and ν is given by

$$\mathcal{T}_c(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \iint_{E \times E} c(x, y) d\pi(x, y),$$

where $\Pi(\mu, \nu)$ denotes the set of probability measures π on $E \times E$ having μ and ν as marginals (called 'transport plans between μ and ν').

Equivalently

$$\mathcal{T}_c(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[c(X, Y)]$$

Classical Examples : Kantorovich distances of order $p \geq 1$

$$W_p^p(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[d^p(X, Y)].$$

Optimal transport plans

A transport plan π° is said optimal if

$$\mathcal{T}_c(\mu,\nu) = \iint c(x,y)\, d\pi^\circ(x,y).$$

Theorem

If c is lower-semicontinuous then there always exists at least one optimal transport plans.

Questions :

- How to characterize optimal transport plans ?
- Are they given by a transport map T : $\pi^\circ = \mathrm{Law}(X, T(X))$, $X \sim \mu$, and $T_\# \mu = \nu$? When it is the case, it holds

$$\mathcal{T}_c(\mu,\nu) = \int c(x, T(x))\, \mu(dx)$$

  and we say that T is a solution in the sense of Monge.
- Is T regular ? Main motivation of this talk : global Lipschitz continuity.
- ...

Brenier Theorem

Let $|\cdot|$ denote the standard Euclidean norm on $E = \mathbb{R}^n$.
The following result characterizes optimal transport plans for the cost function

$$c(x,y) = |y-x|^2, \qquad x, y \in \mathbb{R}^n.$$

Theorem (Brenier (1991))

If μ is absolutely continuous w.r.t. Lebesgue and if $\int |x|^2 d\mu(x) < +\infty$ and $\int |y|^2 d\nu(y) < +\infty$, then there exists a unique optimal transport plan π°, such that

$$W_2^2(\mu,\nu) = \iint |y-x|^2\, d\pi^\circ(x,y).$$

Moreover π° has the following structure : there exists a *convex* function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $\pi^\circ = \mathrm{Law}(X, \nabla\phi(X))$, with $X \sim \mu$ and so

$$W_2^2(\mu,\nu) = \int |\nabla\phi(x) - x|^2\, d\mu(x).$$

Remark :When the support of μ is $\mathbb{R}^n$ (which will always be the case in this talk), then ϕ is unique up to constant.

Motivation of this talk : understand when $\nabla\phi$ is globally Lipschitz.

Caffarelli contraction theorem

Theorem (Caffarelli (2000))

If γ is the standard Gaussian measure on $\mathbb{R}^n$ and $d\nu(y) = e^{-W(y)}\,dy$ is a probability measure associated to a $\mathcal{C}^2$ smooth function W on $\mathbb{R}^n$ such that $\mathrm{Hess}\,W \geq \mathrm{Id}$, then there exists a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^1$ such that $\nu = \nabla\phi_{\#}\gamma$ and such that $\nabla\phi$ is 1-Lipschitz.

In other words, the Brenier map from γ to ν is a contraction.

Original proof based on the Monge-Ampère equation satisfied by ϕ.

Generalizations by Kolesnikov ('10), Kim-Milman ('12), Colombo-Figalli-Jhaveri ('17), Carlier-Figalli-Santambrogio ('24), De Philippis-Shenfeld ('24).

Caffarelli contraction theorem

Numerous consequences in the field of functional inequalities.

Example : the standard Gaussian measure γ satisfies the log-Sobolev inequality (Gross (1975)) :

$$\text{(LSI)} \qquad \mathrm{Ent}_\gamma(f^2) \leq 2 \int |\nabla f|^2 \, d\gamma, \qquad \forall f : \mathbb{R}^n \to \mathbb{R} \ \mathcal{C}^1$$

If $d\nu(y) = e^{-W(y)}\, dy$ with $\mathrm{Hess}\, W \geq \mathrm{Id}$, then according to Caffarelli Theorem $\nu = \nabla\phi_{\#}\gamma$ with $\nabla\phi$ 1-Lipschitz.

Therefore, applying (LSI) to $f = g \circ \nabla\phi$ yields to

$$\begin{aligned} \mathrm{Ent}_\nu(g^2) &\leq 2 \int |\mathrm{Hess}\, \phi(x) \cdot \nabla g(\nabla\phi(x))|^2 \, d\gamma(x), \qquad \forall f : \mathbb{R}^n \to \mathbb{R} \ \mathcal{C}^1 \\ &\leq 2 \int |\nabla g(y)|^2 \, d\nu(y). \end{aligned}$$

So ν satisfies (LSI) : one recovers the Bakry-Emery criterion (with the good constant)

The proof of Caffarelli theorem relies on

- Caffarelli's results on the regularity of solutions of Monge-Ampère equations granting that ϕ is twice continuously differentiable.
- The Monge-Ampère equation satisfied by ϕ and a maximum principle argument.

Motivation : Obtain a new proof of Caffarelli contraction theorem based on probability or information theory arguments.

What I will present today comes from :

- M. Fathi, N. Gozlan, M. Prod'homme, *A proof of the Caffarelli contraction theorem via entropic regularization.* Calc. Var. Partial Differ. Equ. (2020).
- S. Chewi, A. Pooladian, *An entropic generalization of Caffarelli's contraction theorem via covariance inequalities.* C. R., Math., Acad. Sci. Paris (2023).
- N. Gozlan, M. Sylvestre, *Global Regularity Estimates for Optimal Transport via Entropic Regularisation.* Preprint (2025).

Two main ingredients : Entropy regularized optimal transport and Prekopa-Leindler inequality

II - Two ingredients : Entropy regularized optimal transport and Prekopa-Leindler inequality

Entropy regularized optimal transport

For $\epsilon > 0$, the entropic optimal transport problem between μ and ν is the following

$$\mathcal{C}_\epsilon(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int \frac{1}{2}|x-y|^2\, d\pi + \epsilon H(\pi|\mu\otimes\nu)$$

where

$$H(\pi|\mu\otimes\nu) = \int \log \frac{d\pi}{d(\mu\otimes\nu)}\, d\pi$$

is the relative entropy of π w.r.t the probability measure $\mu\otimes\nu$.

References :

- Christian Léonard, *A survey of the Schroedinger problem and some of its connections with optimal transport* (2014)
- Marcel Nutz, *Introduction to Entropic Optimal Transport* (2022).

It is now well known, that

$$\mathcal{C}_\epsilon(\mu,\nu) \to \frac{1}{2}W_2^2(\mu,\nu),$$

as $\epsilon \to 0$.

Entropy regularized optimal transport

The minimizer of the entropic regularized transport problem is of the form

$$\pi_\epsilon(dxdy) = e^{\frac{\langle x,y\rangle - \phi_\epsilon(x) - \psi_\epsilon(y)}{\epsilon}}\,\mu(dx)\nu(dy),$$

with $(\phi_\epsilon, \psi_\epsilon)$ a couple of convex functions solution of the following system

$$\phi_\epsilon(x) = \mathcal{L}_{\epsilon,\nu}(\psi_\epsilon)(x) = \epsilon \log\left(\int e^{\frac{\langle x,y\rangle - \psi_\epsilon(y)}{\epsilon}}\,\nu(dy)\right), \qquad \forall x \in \mathbb{R}^n$$

$$\psi_\epsilon(y) = \mathcal{L}_{\epsilon,\mu}(\phi_\epsilon)(y) = \epsilon \log\left(\int e^{\frac{\langle x,y\rangle - \phi_\epsilon(x)}{\epsilon}}\,\mu(dx)\right), \qquad \forall y \in \mathbb{R}^n.$$

Moreover,

Theorem (Nutz-Wiesel (2022))

Assume μ is absolutely continuous with full support. The entropic Kantorovich potential ϕ_ϵ converges μ-almost everywhere (along to some sequence ϵ_k) to the up to constant unique Kantorovich potential ϕ such that $T = \nabla\phi$ is the Brenier map transporting μ onto ν.

Brunn-Minkowski inequality

Theorem (Brunn-Minkowski)

For all compact sets $A, B \subset \mathbb{R}^n$,

$$\mathrm{Vol}(A+B)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n}$$

or, equivalently, for all $t \in [0,1]$,

$$\mathrm{Vol}((1-t)A + tB) \geq \mathrm{Vol}(A)^{1-t}\mathrm{Vol}(B)^{t}$$

Application :Contains the isoperimetric inequality on $\mathbb{R}^n$.

Functional form of Brunn-Minkowski inequality

Theorem (Prekopa-Leindler inequality)

Let $f_0, f_1, h : \mathbb{R}^n \to \mathbb{R}_+$ be measurable functions such that, for some $t \in]0,1[$, it holds

$$h((1-t)y_0 + ty_1) \geq f_0^{1-t}(y_0) f_1^t(y_1), \qquad \forall y_0, y_1 \in \mathbb{R}^n$$

then

$$\int h \geq \left(\int f_0\right)^{1-t} \left(\int f_1\right)^t.$$

Taking

$$h = 1_{(1-t)A+tB}, \quad f_0 = 1_A \quad f_1 = 1_B$$

one recovers the Brunn-Minkowski inequality.

Reference : S. Bobkov, M. Ledoux, *From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities.* Geom. Funct. Anal. (2000).

Applications :

1. Implies many functional inequalities for strongly log-concave probability measures (Log-Sobolev, Talagrand inequalities)
2. Implies the Brascamp-Lieb Poincaré type inequality for strictly log-concave probability measures.

III - Convexity/smoothness properties of entropic Legendre transforms.

Regularity of entropic potentials is transferred to Kantorovich potential

Let $\mu = \gamma$ (standard Gaussian) and $d\nu = e^{-W}\,dx$ with $\mathrm{Hess}\,W \geq Id$.
We want to prove that $\nabla\phi$ is 1-Lipschitz.

This is equivalent to show that $\phi - \frac{|\cdot|^2}{2}$ is concave, that is

$$\phi((1-t)x_0 + tx_1) \geq (1-t)\phi(x_0) + t\phi(x_1) - t(1-t)\frac{|x_1 - x_0|^2}{2}.$$

According to the convergence result for entropic potentials, it is sufficient to prove that ϕ_ϵ satisfies

$$\phi_\epsilon((1-t)x_0 + tx_1) \geq (1-t)\phi_\epsilon(x_0) + t\phi_\epsilon(x_1) - t(1-t)\frac{|x_1 - x_0|^2}{2}.$$

and then let $\epsilon \to 0$.

α-convex and β-smooth functions

Definition

A function ϕ such that

$$\phi((1-t)x_0 + tx_1) + \alpha t(1-t)\frac{|x_1 - x_0|^2}{2} \geq (1-t)\phi(x_0) + t\phi(x_1)$$

for some $\alpha > 0$, will be called α-smooth.
A function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that

$$\psi((1-t)x_0 + tx_1) + \beta t(1-t)\frac{|x_1 - x_0|^2}{2} \leq (1-t)\psi(x_0) + t\psi(x_1)$$

for some $\beta \geq 0$, will be called β-convex

Recall the Legendre transform

$$\phi^*(y) = \sup_{x \in \mathbb{R}^n} \langle x, y \rangle - \phi(x), \qquad y \in \mathbb{R}^n.$$

It is well known in Convex Analysis that the Legendre transform dualizes these notions : if ϕ is convex and l.s.c

$$\phi \text{ is } \alpha\text{-smooth} \Leftrightarrow \phi^* \text{ is } 1/\alpha\text{-convex}$$

Generalization : Smoothness and convexity moduli

References : Vladimirov, Nesterov, Chekanov (1978), Zalinescu (1983), Azé, Penot (1995)

Definition (Smoothness and convexity moduli)

We say that $\phi : \mathbb{R}^n \to \mathbb{R}$ is S-smooth, or that S is a smoothness modulus for ϕ, if for any $x_0, x_1 \in \mathbb{R}^n$ and $t \in [0,1]$, we have

$$\phi((1-t)x_0 + tx_1) + t(1-t)S(x_1 - x_0) \geq (1-t)\phi(x_0) + t\phi(x_1).$$

We say that $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is R-convex, or that R is a convexity modulus for ψ, if for any $x_0, x_1 \in \mathbb{R}^n$ and $t \in [0,1]$, we have

$$\psi((1-t)x_0 + tx_1) + t(1-t)R(x_1 - x_0) \leq (1-t)\psi(x_0) + t\psi(x_1).$$

If $R = \frac{\beta}{2}|\cdot|^2$, $\beta > 0$, we recover the β-convexity. Same remark for α-smoothness.
These notions are stable under pointwise convergence.

Definition

The smallest function S such that ϕ is S-smooth is called the smoothness modulus of ϕ and is denoted S_ϕ. It is convex when ϕ is convex.
The smallest function R such that ψ is R-convex is called the convexity modulus of ψ and is denoted R_ψ.

Examples

Quadratic functions. Let A be a $n \times n$ matrix, then the function $f(x) = \langle x, Ax \rangle$ admits the following moduli

$$R_f(d) = S_f(d) = \langle d, Ad \rangle, \qquad d \in \mathbb{R}^n,$$

Bounded hessians. Let f be a twice continuously differentiable function on $\mathbb{R}^n$, then it admits the following moduli

$$R_f(d) \geq \frac{1}{2} \inf_x \langle d, \nabla^2 f(x) d \rangle, \quad S_f(d) \leq \frac{1}{2} \sup_x \langle d, \nabla^2 f(x) d \rangle, \qquad d \in \mathbb{R}^n.$$

Examples

Radial functions. Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that $\alpha(ct) \geq c\alpha(t)$ for all $t \geq 0$ and $c \geq 1$. Define $A(r) = \int_0^r \alpha(u)\,du$, $r \geq 0$, and $f_\alpha(x) = A(|x|)$, $x \in \mathbb{R}^n$. Then, the function f_α is R-convex, with $R(d) = 2A(|d|/2)$, $d \in \mathbb{R}^n$.
For example, $W(y) = |y|^p$ with $p \geq 2$ is R-convex with $R(d) = c_p|d|^p$.

Continuous gradients. Let f be a continuously differentiable function such that ∇f admits a non-decreasing modulus of continuity ω then

$$S(d) \leq 2|d|\omega(|d|), \qquad d \in \mathbb{R}^n,$$

is a smoothness modulus for f.
For example, $V(x) = |x|^p$ with $1 \leq p \leq 2$ is S-smooth with $S(d) = c'_p|d|^p$.

Smoothness and subgradient regularity

Proposition

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $S : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be even.
If ϕ is S-smooth, then for all $x_0, x_1 \in \mathbb{R}^n$ and $y_0 \in \partial\phi(x_0), y_1 \in \partial\phi(x_1)$, it holds

$$S^*(y_1 - y_0) \leq S(x_1 - x_0).$$

Examples :

- If ϕ is α-smooth and continuously differentiable, we have $S(d) \leq \frac{\alpha}{2}|d|^2$ and $S^*(u) \geq \frac{1}{2\alpha}|u|^2$, thus we get

$$|\nabla\phi(x_0) - \nabla\phi(x_1)| \leq \alpha|x_1 - x_0|$$

  which is the classical Lipschitz property of the gradient.

- If ϕ is S-smooth and continuously differentiable with $S(d) = \alpha\frac{|d|^p}{p}$ with $1 \leq p \leq 2$, we get

$$|\nabla\phi(x_0) - \nabla\phi(x_1)| \leq \alpha c_p|x_1 - x_0|^{p-1}$$

  that is the Hölder continuity of the gradient.

Legendre transform

Theorem (Azé-Pénot (1995))

Let $\phi, \psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be two convex functions and $S, R : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be even functions.

(i) If ϕ is S-smooth, then ϕ^* is S^*-convex.

(ii) If ψ is R-convex, then ψ^* is R^*-smooth.

Note that when ϕ is α-convex we recover that ϕ^* is α^{-1}-smooth, and conversely.

Entropic Legendre transform

Consider

$$\mathcal{L}_\epsilon(\psi) = \epsilon \log \left(\int e^{\frac{\langle x,y\rangle - \psi(y)}{\epsilon}} \, dy \right)$$

Theorem (G.-Sylvestre (2025))

(a) Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an R-convex function with convex domain with positive Lebesgue measure, then the function $\mathcal{L}_\epsilon(\psi)$ is R^*-smooth.

(b) Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be an S-smooth function, then the function $\mathcal{L}_\epsilon(\phi)$ is S^*-convex.

As $\epsilon \to 0$, the result of Azé-Pénot is recovered.

Proof sketch of (a)

Assume ψ is R-convex.
Let $x_0, x_1 \in \mathbb{R}^n$, set $x_t = (1-t)x_0 + tx_1$ and consider the three functions

$$h(y) = \exp\left(\frac{\langle x_t, y\rangle - \psi(y)}{\epsilon}\right), f_0(y) = \exp\left(\frac{\langle x_0, y\rangle - \psi(y)}{\epsilon}\right), f_1(y) = \exp\left(\frac{\langle x_1, y\rangle - \psi(y)}{\epsilon}\right.$$

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which satisfy

$$\begin{aligned} h(y_t) &\geq \exp\left(\frac{t(1-t)(R(y_1-y_0) - \langle x_1 - x_0, y_1 - y_0)}{\epsilon}\right) f_0^{1-t}(y_0) f_1^t(y_1) \\ &\geq \exp\left(\frac{-t(1-t)R^*(x_1-x_0)}{\epsilon}\right) f_0^{1-t}(y_0) f_1^t(y_1) \end{aligned}$$

because ψ is R-convex.

Proof sketch of (a)

Applying Prekopa-Leindler inequality to h, f_0, f_1 grants

$$\int h \geq \exp\left(\frac{-t(1-t)R^*(x_1 - x_0)}{\epsilon}\right)\left(\int f_0\right)^{1-t}\left(\int f_1\right)^t.$$

Finally taking the logarithm and multiplying by ϵ gives

$$\mathcal{L}_\epsilon(\psi)(x_t) \geq -t(1-t)R^*(x_1 - x_0) + (1-t)\mathcal{L}_\epsilon(\psi)(x_0) + t\mathcal{L}_\epsilon(\psi)(x_1),$$

which ensures that $\mathcal{L}_\epsilon(\psi)$ is R^*-smooth.

IV - Generalizations of Caffarelli's theorem, new proofs and results.

Entropic regularity theorem

Theorem (G.-Sylvestre (2025))

Let $\mu(dx) = e^{-V(x)}dx$ and $\nu(dy) = e^{-W(y)}dy$ be two measures on $\mathbb{R}^n$ with finite second moment such that V is S_V-smooth, $\mathrm{dom}\, V = \mathbb{R}^n$ and W is R_W-convex. Then for all $\epsilon > 0$ the entropic potentials ϕ_ϵ is S-smooth with S such that

$$S(d) = \int_0^1 \sup_{R_W^{**}(p) \leq S_V(td)} \langle p, d \rangle \, dt, \qquad \forall d \in \mathbb{R}^n.$$

Note that the result holds for $\epsilon = 0$ by pointwise convergence of ϕ_ϵ towards ϕ.

Remark on radial moduli

In the case $S_V = \sigma_V(|\cdot|)$ and $R_W = \rho_W(|\cdot|)$, we obtain a radial smoothness modulus S given by

$$S(d) = \int_0^{|d|} (\rho_W^{**})^{-1}(\sigma_V(s))\, ds.$$

Which gives the following regularity estimate on the gradient of ϕ_ϵ

$$|\nabla\phi_\epsilon(x) - \nabla\phi_\epsilon(y)| \leq \frac{2}{|x-y|}\int_0^{|x-y|} (\rho_W^{**})^{-1}(\sigma_V(s))\, ds.$$

The same inequality holds for $\nabla\phi$.

Proof of the entropic regularity theorem

The entropic potentials ϕ_ϵ and ψ_ϵ satisfy

$$\phi_\epsilon(x) = \mathcal{L}_{\epsilon,\nu}(\psi_\epsilon)(x) = \epsilon \log\left(\int e^{\frac{\langle x,y\rangle - \psi_\epsilon(y)}{\epsilon}}\,\nu(dy)\right), \qquad \forall x \in \mathbb{R}^n$$

$$\psi_\epsilon(y) = \mathcal{L}_{\epsilon,\mu}(\phi_\epsilon)(y) = \epsilon \log\left(\int e^{\frac{\langle x,y\rangle - \phi_\epsilon(x)}{\epsilon}}\,\mu(dx)\right), \qquad \forall y \in \mathbb{R}^n.$$

and so using the notation $\mathcal{L}_\epsilon$:

$$\phi_\epsilon = \mathcal{L}_\epsilon(\psi_\epsilon + \epsilon W) \qquad \psi_\epsilon = \mathcal{L}_\epsilon(\phi_\epsilon + \epsilon V).$$

Then by the entropic Legendre transform property, we have that the modulus of smoothness S_ϵ of ϕ_ϵ and the modulus of convexity R_ϵ of ψ_ϵ satisfy

$$S_\epsilon \leq (R_\epsilon + \epsilon R_W)^*\,, \quad R_\epsilon \geq (S_\epsilon + \epsilon S_V)^*\,.$$

Combining the two inequalities and using the inverse monotonicity of the Legendre transform we get

$$S_\epsilon \leq ((S_\epsilon + \epsilon S_V)^* + \epsilon R_W)^* \leq (S_\epsilon + \epsilon S_V)\Box(\epsilon R_W)^*.$$

Proof of the entropic regularity theorem

Applying the inequality above at $u + \epsilon v$ and using the convexity of S_ϵ grants

$$\langle \nabla S_\epsilon(u), v \rangle \leq \frac{S_\epsilon(u + \epsilon v) - S_\epsilon(u)}{\epsilon} \leq S_V(u) + R_W^*(v).$$

Finally optimizing over v we have $R_W^{**}(\nabla S_\epsilon(u)) \leq S_V(u)$. The conclusion follows by integration :

$$S_\epsilon(d) = \int_0^1 \langle \nabla S_\epsilon(td), d \rangle \, dt \leq \int_0^1 \sup_{R_W^{**}(p) \leq S_V(td)} \langle p, d \rangle \, dt.$$

Examples

$$\mu(dx) = e^{-V(x)}\,dx \qquad \nu(dy) = e^{-W(y)}\,dy$$

V	W	$S_V(.)$	$R_W(.)$	Smoothness of ϕ
$\nabla^2 V \leq \alpha_V Id$	$\nabla^2 W \geq \beta_V Id$	$\frac{\alpha_V}{2}\|.\|^2$	$\frac{\beta_W}{2}\|.\|^2$	Caffarelli (2000) $\|\nabla\phi(x) - \nabla\phi(y)\| \leq \sqrt{\frac{\alpha_V}{\beta_W}}\|x-y\|$
$V \in \mathcal{C}^1, 1 \leq p \leq 2$	$W \in \mathcal{C}^1, 2 \leq q$	$\alpha_V\|.\|^p$	$\beta_W\|.\|^q$	Kolesnikov (2010) $\|\nabla\phi(x) - \nabla\phi(y)\| \leq \frac{2q}{p+q}\left(\frac{\alpha_V}{\beta_W}\right)^{\frac{1}{q}}\|x-y\|^{\frac{p}{q}}$
$V \in \mathcal{C}^1, 1 < p \leq 2$	$W \in \mathcal{C}^1, q = p^*$	$\alpha_V\|.\|_p^p$	$\beta_W\|.\|_q^q$	New variant $\|\nabla\phi(x) - \nabla\phi(y)\|_q \leq \beta_p\left(\frac{\alpha_V}{\beta_W}\right)^{\frac{1}{q}}\|x-y\|_p^{\frac{p}{q}}$
$\nabla^2 V \leq A^{-1}$	$\nabla^2 W \geq B^{-1}$	$\langle d, A^{-1}d\rangle$	$\langle d, B^{-1}d\rangle$	Improvement of Chewi-Pooladian (2024) $D^2\phi \leq B^{1/2}\left(B^{-1/2}A^{-1}B^{-1/2}\right)^{1/2}B^{1/2}$ with equality for Gaussians

Examples

The recent result by De Philippis and Shenfeld ('24) on the divergence of the Brenier map between a log-Subharmonic probability measure μ and a strongly log-concave probability measure ν can also be recovered and improved using a variant of our method.

Transport between Log-Lipschitz perturbations of Gaussians

Conjecture (Fathi-Mikulincer-Shenfeld)

If ν is a log-Lipschitz perturbation of γ the standard Gaussian measure, then the optimal transport map sending γ onto ν is globally Lipschitz.

Theorem (Fathi-Mikulincer-Shenfeld))

There exists a Lipschitz transport from γ to ν with a Lipschitz constant depending exponentially on L^2 but not on the dimension.

Transport between Log-Lipschitz perturbations of Gaussians

In what follows, for any Lipschitz function $a : \mathbb{R}^n \to \mathbb{R}$, we will denote by

$$\gamma_a(dx) = \frac{1}{Z_a} e^{-a(x) - \frac{|x|^2}{2}}\, dx$$

with Z_a a normalizing constant. We will denote by $T_{a,b} : \mathbb{R}^n \to \mathbb{R}^n$ the Brenier transport map sending γ_a on γ_b, and by $T_a : \mathbb{R}^n \to \mathbb{R}^n$ the Brenier map from γ to γ_a.

Theorem (G-Sylvestre (2025))

For all Lipschitz functions $a, b : \mathbb{R}^n \to \mathbb{R}$, the Brenier map $T_{a,b}$ from γ_a to γ_b satisfies

$$-8L + |x-y| \leq |T_{a,b}(x) - T_{a,b}(y)| \leq 8L + |x-y|, \qquad \forall x, y \in \mathbb{R}^n.$$

where $L = \max(L_a, L_b)$ and L_a, L_b are the Lipschitz constants of a, b.
The Brenier map T_a between γ and γ_a satisfies

$$-8L_a + |x-y| \leq |T_a(x) - T_a(y)| \leq 8L_a + |x-y|, \qquad \forall x, y \in \mathbb{R}^n.$$

In particular, if a is even, then T_a satisfies the following growth estimate

$$-8L_a + |x| \leq |T_a(x)| \leq 8L_a + |x|, \qquad \forall x \in \mathbb{R}^n.$$

Transport between Log-Lipschitz perturbations of Gaussians

In particular, if a is even, it always holds

$$\frac{|T_a(x)|}{|x|} \to 1, \qquad \text{as } |x| \to \infty.$$

One dimensional example : Let

$$\nu(dy) = \frac{1}{Z} e^{L|y|} \gamma(dy),$$

with $L > 0$. In this case, it holds

$$|x - y| \leq |T_a(x) - T_a(y)| \leq \alpha L + |x - y|, \qquad \forall x, y \in \mathbb{R},$$

where $\alpha = 2$. The optimal value of α belongs to $[1, 2]$.
The Lipschitz constant of T_a is of order e^{L^2} as L is large.

Transport between Log-Lipschitz perturbations of Gaussians

Proof idea : if a is a L-Lipschitz function, then a satisfies

$$a(x_t) \leq (1-t)a(x_0) + ta(x_1) + 2Lt(1-t)|x_1 - x_0|, \qquad \forall x_0, x_1 \in \mathbb{R}^n.$$

and

$$a(x_t) \geq (1-t)a(x_0) + ta(x_1) - 2Lt(1-t)|x_1 - x_0|, \qquad \forall x_0, x_1 \in \mathbb{R}^n.$$

In other words, a is ρ-convex with $\rho(u) = -2L|u|$ and σ-smooth with $\sigma(u) = 2L|u|$.

Transport between Log-Lipschitz perturbations of Gaussians

Other remarks and results :

- Contrary to the Poincaré and Log-Sobolev constants, the concentration constant behaves well under Log-Lipschitz perturbation.

Proposition

Let $a : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant L; for any $A \subset \mathbb{R}^n$ such that $\gamma_a(A) \geq 1/2$, it holds

$$\gamma_a(A_r) \geq 1 - \exp\left(-\frac{1}{2}\left[r - 16L - \sqrt{2\log(2)}\right]_+^2\right), \qquad \forall r \geq 0, \tag{1}$$

where $A_r = \{y \in \mathbb{R}^n : \inf_{x \in A} |x - y| \leq r\}$, the r-enlargement of A.

This result can be deduced from the Gaussian concentration inequality using that T_a is an approximate isometry. Direct proof is also possible.

- Log-Lipschitz perturbations of general potentials V, W can be considered.
- Using concentration for Log-Lipschitz perturbations of uniformly log-concave measures, one can obtain bounds on the Hessian of the entropic potential φ_ϵ in the spirit of a recent result by Conforti (2024).

Thank you for your attention !