Filling the gap between evolutionary individual-based models and Hamilton-Jacobi equations

Sylvie Méléard, dite Sylvie Claquettes

Ecole Polytechnique, Centre de Mathématiques Appliquées

Journées en l'honneur de Christian Léonard

Université Paris-Cité, 2025

Joint work with N. Champagnat (Nancy), S. Mirrahimi (Toulouse) and V.C. Tran (Lille).

Funded by the European Union

Christian

Evolutionary biology

Individuals are characterized by genetic or phenotypic information (trait) that influences their ability to reproduce and their probability of survival.

The evolution of the trait distribution results from the following mechanisms:

- Heredity. (Vertical) transmission of the ancestral trait to the offsprings.
- Mutation. Generates variability in the trait values.
- Selection. Individuals with traits increasing their survival probability or their reproduction ability will spread through the population over time (genetical selection).

Asexual populations (cells, bacteria).

Usual biological assumptions:

- large populations
- rare mutations
- small mutation steps
- long (evolutionary) time scale.

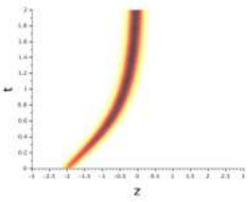
The main goal:

- predict the long term evolutionary dynamics.
- model and quantify the successive invasions of successful mutants: by mutation-selection, the population concentrates on advantageous mutants.

That is a multi-scale question : different mathematical approaches using different analytical tools.

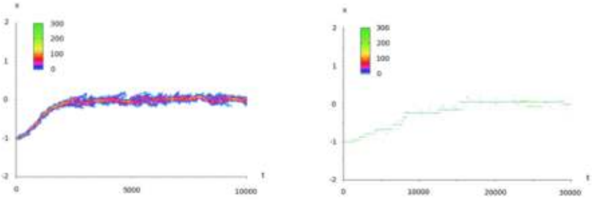
- Game Theory - Dynamical Systems:
  Maynard-Smith 1974, Hofbauer-Sigmund 1990, Marrow-Law-Cannings 1992, Metz-Geritz-Meszéna et al. 1992, 1996, Dieckmann-Law 1996, Diekmann 2004.

- Partial or integro-differential and Hamilton-Jacobi equations (Hopf-Cole transformation):
  Perthame-Barles-Mirrahimi 07-10, Jabin, Desvillettes, Raoul, Mischler 08-10.
  Concentration phenomenon on advantageous mutants but evolution seems too fast.



- **Stochastic individual-based processes** (birth and death processes with mutation and selection) :
  (Bolker-Pacala 97, Kisdi 99, Dieckmann-Law 00, Fournier-M. 04, Ferrière-Champagnat-M. 06, Champagnat 06, Champagnat-M. 10).

  Concentration phenomenon on advantageous mutants but evolution seems too slow (time scale separation between competition phases and mutation arrivals).



- How are the previous mathematical approaches related?
- Stochastic approach: mutations rare but not so rare.
- Deterministic approach : how to keep track of small subpopulations (and then possible local extinctions) in large population approximations?
- Main question : How to interpret the results of the Hamilton-Jacobi approach with regard to the initial individual based model.

The classical framework for structured population dynamics with mutation and selection

- Large population: the model is parametrized by a carrying capacity parameter K, $K \to +\infty$.
- The stochastic (Markov) population process $(Z_t^K)_{t\geq 0}$ is a point measure valued process taking into account all birth and death events and scaled by K : $Z_0^K \approx n_0(x)dx$ and

$$Z_t^K = \frac{1}{K}\sum_{i\in V_t^K} \delta_{X_t^{i,K}} \; ; \; V_t^K = \{\text{individuals alive at time } t\}.$$

- Each individual is characterized by a trait $x \in \mathcal{X} \subset \mathbb{R}$. For an individual with trait x, the birth rate is $b(x)$, the mutation rate is $p(x)$ and the mutation kernel is $G(y-x)dy$, the death rate is $d(x)$.
- When K tends to infinity, the stochastic population process $(Z_t^K)_{t\geq 0}$ converges in probability to the solution of the integro-differential equation

$$\partial_t n(t,x) = \Big(b(x) - d(x)\Big) n(t,x) + \int_{\mathcal{X}} p(y) G(x-y) n(t,y) dy \; ; \; n(0,x) = n_0(x).$$

Hopf-Cole transformation

To take into account the biological scales, the analysts (cf. Barles, Mirrahimi, Perthame 2009) assume that mutations are of order ε and the evolutionary time scale in $1/\varepsilon$.
For G a centered Gaussian kernel and

$$G_\varepsilon(z)dz = G\left(\frac{z}{\varepsilon}\right)\frac{dz}{\varepsilon} \quad ; \quad t \rightsquigarrow \frac{t}{\varepsilon},$$

the equation becomes

$$\varepsilon\, \partial_t n^\varepsilon(t,x) = \Big(b(x) - d(x)\Big) n_\varepsilon(t,x) + \int p(x+\varepsilon z)\, G(z) n_\varepsilon(t, x+\varepsilon z) dz.$$

Considering the Hopf-Cole transformation

$$n^\varepsilon(t,x) = \exp\left(\frac{u^\varepsilon(t,x)}{\varepsilon}\right),$$

one obtains by straitgthforward computation that

$$\partial_t u_\varepsilon(t,x) = b(x) - d(x) + \int p(x+\varepsilon z)\, e^{\frac{u_\varepsilon(t,x+\varepsilon z) - u_\varepsilon(t,x)}{\varepsilon}} G(z) dz.$$

Hamilton-Jacobi Equation

Theorem (cf. Barles-Mirrahimi-Perthame 2009) :
Assume that $(u^0_\varepsilon)_\varepsilon$ *is a sequence of uniformly Lipschitz continuous functions which converges uniformly to* u^0. *As* $\varepsilon \to 0$, *the sequence* (u_ε) *converges uniformly to the unique viscosity solution* u *of*

$$\begin{cases} \partial_t u(t,x) = b(x) - d(x) + \int p(x)\, e^{\partial_x u(t,x).z}\, G(z) dz, \\ u(0,x) = u_0(x). \end{cases}$$

But long term evolutionary dynamics may be strongly influenced by small subpopulations : local extinction.

How to keep track of small populations in large population models?

We wish to derive directly such HJ equation from the stochastic framework to find the good limiting object. We will see two approaches:

(i) A direct convergence proof which necessitates strong assumptions in a discretized setting (Champagnat, M., Mirrahimi, Tran, JEP 2023).

(ii) A variational approach using Large Deviation results. In progress.

The first (direct) approach - the torus trait space

- Let $(\delta_K)_K$ such that as $K \to \infty$,

$$h^K := \delta_K \log K \ll 1.$$

For fixed K, the trait space is trait space is the torus $\mathbb{T}$ and $\mathcal{X}_K = \{i\delta_K : i \in \mathbb{Z}\}$.

- We consider the multitype birth-death process

$$(N_i^K(t), t \geq 0, i \in \{0, 1, \cdots, \frac{1}{\delta_K} - 1\})$$

$N_i^K(t)$: number of individuals at time t with trait $x = i\delta_K$.

- For an individual with trait $x \in \mathcal{X}_K$:
  - birth rate : $b(x)$; $b(i\delta_K) = b_i^K$,
  - death rate : $d(x)$; $d(i\delta_K) = d_i^K$,
  - Mutation rate: an individual with trait $i\delta_K \in \mathcal{X}_K$ gives birth to a mutant with trait $j\delta_K \in \mathcal{X}_K$ at rate

$$\mu_{ij}^K = p(i\delta_K) \int_{(j-i-1/2)h_K}^{(j-i+1/2)h_K} G(y)\, dy.$$

  Mutation size $\approx 1/\log K$.

Total asymptotic mutation rate from an individual with trait $x_K = [x/\delta_K]\delta_K$:

$$\lim_{K\to+\infty} p(x_K) \sum_j \mu^K_{[x/\delta_K]j} = p(x) \int G(y)\,dy = p(x).$$

We wish to capture the sub-populations of size of order K^β.

We assume that for all i and K, $\mathbb{E}(N_i^K(0)) = K^{\beta_i^K(0)}$.

Let us introduce the exponent processes

$$\beta_i^K(t) = \frac{\log(1 + N_i^K(t \log K))}{\log(K)}, \text{ with } \beta_i^K(t) = 0 \text{ if } N_i^K(t \log K) = 0.$$

For all $x \in \mathbb{T}$ and $K \geq 1$, and i such that $x \in [i\delta_K, (i+1)\delta_K)$, we define

$$\widetilde{\beta}^K(t,x) = \beta_i^K(t)(1 - \frac{x}{\delta_K} + i) + \beta_{i+1}^K(t)(\frac{x}{\delta_K} - i),$$

with the convention $\beta^K_{1/\delta_K}(t) = \beta_0^K(t)$.

Main question : convergence, as $K \to +\infty$, of the stochastic process $(\widetilde{\beta}^K(t,.))$?

Assumptions :

• b , d and p are non negative Lipschitz continuous functions and b and p are bounded.

• A super-critical branching process : $b(x) > d(x)$ and $p(x) > 0$.

• There exists $a_1 > 0$ such that for all K and $\forall i \in \{0, 1, \cdots, \frac{1}{\delta_K} - 1\}$,

$$N_i^K(0) \geq K^{a_1}.$$

• There exists $a_2 < a_1$ such that as $K \to \infty$,

$$K^{-a_2/4} \ll \delta_K \ll \frac{1}{\log K}.$$

• Lipschitz assumption on the initial condition $\beta_i^K(0)$: there exists $A > 0$, such that

$$\lim_{K\to\infty} \mathbb{P}\left(\sup_{i\neq j} \frac{|\beta_i^K(0) - \beta_j^K(0)|}{\rho(j\delta_K, i\delta_K)} > A\right) = 0.$$

The main result

The processes $\widetilde{\beta}^K$ belong to $\mathbb{D}([0,T], C(\mathbb{T},\mathbb{R}))$, endowed with the Skorokhod topology.

Theorem

Assume that $(\widetilde{\beta}^K(0,\cdot))_K$ *converges to a deterministic function* $\beta_0(\cdot)$ *and the assumptions above.*
Then the processes $\widetilde{\beta}^K$ *converge in probability in* $\mathbb{D}([0,T], C(\mathbb{T},\mathbb{R}))$ *to the unique Lipschitz viscosity solution of the Hamilton-Jacobi equation (HJ):*

$$\begin{cases} \frac{\partial}{\partial t}\beta(t,x) = b(x) - d(x) + p(x)\int_{\mathbb{R}} G(h) e^{h\partial_x \beta(t,x)} dh, & (t,x) \in [0,T]\times\mathbb{T} \\ \beta(0,x) = \beta_0(x), & x \in \mathbb{T}. \end{cases}$$

Proof : we prove that the laws of the processes $\widetilde{\beta}^K$ are relatively compact in $\mathcal{P}(\mathbb{D}([0,T], C(\mathbb{T},\mathbb{R})))$ and we identify any limiting value as unique Lipschitz viscosity solution of (HJ).

A variational approach - A more general case

- The state space is $\mathbb{R}$.
- The functions b, d, p are continuous on $\mathbb{R}$ and b, p are bounded.
- At rate $p(x)$, the mutant trait is given by $x + \frac{Y}{\log K}$ with Y distributed as $G(y)dy$.
- Z_0^K is a Poisson point measure on $\mathbb{R}$ with intensity measure $K^{\beta_0^K(x)}dx$, where β_0^K is continuous on $\mathbb{R}$ and converges uniformly to a continuous function β_0 s.t. $\beta_0(x) \to_{|x|\to+\infty} -\infty$.

We introduce the historical process $(H_t^K, t \geq 0)$, a point measure-valued process taking values in $\mathcal{M}_P(\mathbb{D}([0,t],\mathbb{R}))$:

$$Z_t^K = \frac{1}{K}\sum_{i\in V_t^K} \delta_{X_t^{i,K}} \quad ; \quad H_t^K = \sum_{i\in V_t^K} \delta_{X_{.\wedge t}^{i,K}},$$

where $(X_{s\wedge t}^{i,K}, s \in \mathbb{R}_+)$ is the lineage of the individual $i \in V_t^K$. Note that

$$\mathbb{E}(\langle H_0^K, 1\rangle) = \mathbb{E}(\langle Z_0^K, 1\rangle) = \int K^{\beta_0^K(x)}dx.$$

The main results

For $T > 0$ and a measurable $A \subset \mathbb{D}([0,T],\mathbb{R})$, we define for $t \in [0,T]$ the set $A_t := \{\varphi_{|[0,t]},\ \varphi \in A\}$ and

$$N_t^{K,A} = \langle H^K_{t\log K}, \mathbf{1}_{A_t}\rangle,$$

which counts the lineages belonging to A_t.

Our main results are upper and lower bounds on $\log N_t^{K,A}$.

For $\varphi \in \mathbb{D}([0,t],\mathbb{R})$, we define

$$F_t(\varphi) = \beta_0(\varphi(0)) + \int_0^t (b+p-d)(\varphi_s)ds - I_t(\varphi),$$

with

$$I_t(\varphi) = \int_0^t L(\varphi_s,\dot\varphi_s)ds \text{ if } \varphi \in AC([0,t],\mathbb{R})\,;\ = +\infty \text{ otherwise.}$$

$$L(x,\beta) = \sup_{\alpha\in\mathbb{R}} \left(\alpha\beta - H(x,\alpha)\right)\ ;\ H(x,\alpha) = p(x)\int_{\mathbb{R}} (e^{\alpha y}-1)G(y)\,dy.$$

We define

$$u_a(t,x) = \sup\{F_t(\varphi);\, \varphi \in \mathrm{AC}[0,t],\ \varphi(t)=x,\ \forall s \in [0,t],\ F_s(\varphi) \geq a\};$$

$$\widetilde{\Omega}_a = \{(t,x) \in [0,+\infty) \times \mathbb{R};\, \exists \varphi \in AC[0,t],\ \varphi(t) = x,\ \forall s \in [0,t],\ F_s(\varphi) \geq a\}.$$

Notice that

$$u_a(t,x) \geq a, \quad \text{for all } (t,x) \in \widetilde{\Omega}_a.$$

We next define

$$\Omega_a = \{(t,x) \in \widetilde{\Omega}_a \,|\, u_a(t,x) > a\}.$$

Proposition : (cf Barles, Mirrahimi, Perthame, Souganidis 2012)
The function u_a belongs to $C(\overline{\Omega}_a)$ and it is the unique locally Lipschitz continuous and bounded above viscosity solution of the following (state constrained) Hamilton-Jacobi equation

$$\begin{cases} \partial_t u_a = H(x, \partial_x u_a) + b(x) + p(x) - d(x), & (t,x) \in \Omega_a \\ u_a(t,x) = 0, & (t,x) \in \partial\Omega_a,\ t > 0, \\ u_a(0,x) = \beta_0(x), & \text{for all } x \text{ s.t } (0,x) \in \Omega_a. \end{cases}$$

Moreover, the function $a \to u_a(t,x)$ is almost everywhere continuous with respect to a.

Defining for all $t \geq 0$, $x \in \mathbb{R}$ and $\delta > 0$

$$A_t^{x,\delta} = \{\varphi \in \mathbb{D}[0,t],\ \varphi(t) \in [x-\delta, x+\delta]\},$$

we have the

Theorem :

For all $t \geq 0$ and $x \in \mathbb{R}$, almost surely,

$$\limsup_{a \searrow 0} u_a(t,x) \leq \lim_{\delta \to 0} \liminf_{K \to \infty} \frac{1}{\log K} \log N_t^{K, A_t^{x,\delta}} \leq \lim_{\delta \to 0} \limsup_{K \to \infty} \frac{1}{\log K} \log N_t^{K, A_t^{x,\delta}} = u_0(t,x).$$

Moreover, if u_a is continuous with respect to a at $a = 0$, we have

$$\lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{\log K} \log N_t^{K, A_t^{x,\delta}} = u_0(t,x).$$

In fact we will prove that

Theorem:

(i) For any $t > 0$ and any closed set $A \subset AC([0,t],\mathbb{R})$, we have, almost surely,

$$\limsup_{K\to+\infty} \frac{1}{\log K} \log N_t^{K,A} \leq \sup\{F_t(\varphi); \varphi \in A,\ \forall s \in [0,t],\ F_s(\varphi) \geq 0\}.$$

(ii) For any $t > 0$ and any open set $G \subset AC([0,t],\mathbb{R})$, we have, almost surely,

$$\liminf_{K\to+\infty} \frac{1}{\log K} \log N_t^{K,G} \geq \sup\{F_t(\varphi); \varphi \in G,\ \forall s \in [0,t],\ F_s(\varphi) > 0\}.$$

A Feynman-Kac formula for $\mathbb{E}(N_t^{K,A})$

Consider the auxiliary pure jump process $(X_t^K)_{t\in\mathbb{R}_+}$ based on the mutation dynamics and with generator

$$\mathcal{A}^K\varphi(x) = p(x)\int_{\mathbb{R}}\left[\varphi\left(x+\frac{y}{\log K}\right)-\varphi(x)\right]G(y)dy.$$

For all $t>0$ and $A\subset\mathbb{D}[0,t]$,

$$\mathbb{E}(N_t^{K,A}) = \int_{\mathbb{R}} K^{\beta_0^K(x)}\mathbb{E}_x\left[\exp\left(\int_0^{t\log K}\left(b(X_s^K)+p(X_s^K)-d(X_s^K)\right)ds\right)\mathbb{1}_{(X^K_{s\log K})_{s\in[0,t]}\in A}\right]dx.$$

Note that the processes $(X^K_{t\log K})_K$ converges to 0 in probability for the L^∞ norm on $[0,T]$ for all $T>0$.

Let $\mu^K_{x,T}$ be law of $(X^K_{t\log K}, t\in[0,T])$ given $X^K_0=x$.

Large deviation principle for $(\mu^K_{x,T})_{K\geq 1}$

For all $t > 0$ and $x \in \mathbb{R}$, we define for all $\varphi \in \mathbb{D}([0,t],\mathbb{R})$,

$$I_{t,x}(\varphi) = I_t(\varphi) \text{ if } \varphi(0) = x \; ; \; = +\infty \text{ otherwise.}$$

Theorem :

(i) *For all $T > 0$, the family of laws $(\mu^K_{x,T})_{K\geq 1}$ satisfies a large deviation principle on $\mathbb{D}[0,T]$ with rate $1/\log K$ and good rate function $I_{T,x}$.*

(ii) *For all $T > 0$, the family of measures $(\mu^K_{x,T})_{K,x}$ is exponentially tight, uniformly on compact sets: for $M < \infty$, there exists a compact subset A of $\mathbb{D}[0,T]$ such that*

$$\limsup_{K\to\infty} \sup_{x\in B \text{ compact}} \frac{1}{\log K} \log \mu^K_{x,T}(A^c) \leq -M.$$

We can use Léonard (2000), Dupuis-Ellis, Dembo-Zeitouni.

Large deviation estimates on $\mathbb{E}(N_t^{K,A})$

We can deduce by an adaptation of classical arguments that

For any $t > 0$ and for any closed $A \subset \mathbb{D}([0,t],\mathbb{R})$,

$$\limsup_{K\to+\infty} \frac{1}{\log K} \log\left(\mathbb{E}[N_t^{K,A}]\right) \leq \sup_{\varphi\in A} F_t(\varphi), \tag{1}$$

and for any open $G \subset \mathbb{D}([0,t],\mathbb{R})$

$$\liminf_{K\to+\infty} \frac{1}{\log K} \log\left(\mathbb{E}[N_t^{K,G}]\right) \geq \sup_{\varphi\in G} F_t(\varphi),$$

with the usual convention that $\sup\emptyset = -\infty$.

We use Markov inequality and (1) to show that for any $t > 0$ and closed subset A, for $\delta > 0$ and K large enough,

$$\mathbb{P}(N_t^{K,A} \geq K^{F_t(A)+\delta}) \leq K^{-\delta/2}.$$

Hence by Borel-Cantelli lemma, we obtain that almost surely,

$$\limsup_{K\to\infty} \frac{\log N_t^{K,A}}{\log K} \leq \sup\{F_t(\varphi); \varphi\in A\} := F_t(A). \tag{2}$$

Almost sure upper-bound

Proof of :

$$\limsup_{K\to+\infty} \frac{1}{\log K} \log N_t^{K,A} \leq \sup\{F_t(\varphi); \varphi \in A,\ \forall s \in [0,t],\ F_s(\varphi) \geq 0\}.$$

Note that if A s.t. $F_t(A) < 0$, since $N_t^{K,A}$ has integer values, it follows that almost surely

$$\limsup_{K\to\infty} \frac{\log N_t^{K,A}}{\log K} = -\infty.$$

One writes $A = A_0 \cup A_1$, with $A_0 = \{\varphi \in A, \exists s \in [0,t]\ s.t.\ F_s(\varphi) < 0\}$ and $A_1 = \{\varphi \in A, \forall s \in [0,t], F_s(\varphi) \geq 0\}$.

We prove that $\limsup_{K\to\infty} \frac{\log N_t^{K,A_0}}{\log K} = -\infty$.

As $I_{t,x}(\varphi) = +\infty$ if φ is not continuous, then A_1 is a closed subset of $C([0,t])$. One can apply (2).

Combining both completes the proof.

Almost sure lower bound

The most difficult part - Inspired by Berestycki-Brunet-Harris-Harris-Roberts (SPA 2015).

1 - A classical first step.
Define $T(\alpha) = \int_{\mathbb{R}} e^{\alpha y} G(y) dy$. We fix a function $f \in C(\mathbb{R}_+, \mathbb{R})$ s.t. $f(0) = x$ and define ψ_f as the unique C^1-function satisfying

$$T'(\psi_f(\cdot)) = \frac{f'(\cdot)}{p(f(\cdot))}.$$

We define a martingale $(L_t^{K,\psi_f})_{t\geq 0}$ and a change of measure $\mathbb{Q}^{K,\psi_f}$ for the process X^K defined for all $t \geq 0$ by

$$L_t^{K,\psi_f} = \exp\left[\log K \int_0^t \psi_f\left(\frac{s}{\log K}\right) dX_s^K - \int_0^t p(X_s^K)\Big(T\Big(\psi_f\left(\frac{s}{\log K}\right)\Big) - 1\Big) ds\right] ;$$

$$\frac{d\mathbb{Q}^{K,\psi_f}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = L_t^{K,\psi_f}.$$

Then under $\mathbb{Q}^{K,\psi_f}$, the sequence of processes $(X^K_{t \log K})_K$ converges to f in probability for the L^∞ norm on $[0, T]$ for all $T > 0$.

A martingale controlling the number of particles in $B(f, \varepsilon)$

(i) We find a time-dependent harmonic function under $\mathbb{Q}^{K,\psi_f}$ that vanishes out of $B(f, \varepsilon)$:

$\forall \varepsilon > 0$, for K large, there exists a bounded function $h^K_\varepsilon : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ s.t.

$$h^K_\varepsilon(t \log K, y) > 0 \ \text{ for all } \ |y - f(t)| < \varepsilon \ ,$$

$$h^K_\varepsilon(t \log K, y) = 0 \ \text{ for all } \ |y - f(t)| \geq \varepsilon \ , \ \ h^K_\varepsilon(s, f(s/\log K)) = 1 \text{ for all } s \geq 0,$$

and

$$h^K_\varepsilon(t \wedge \tau, X^K_{t\wedge\tau}) \ \text{ is a } \mathbb{Q}^{K,\psi_f} - \text{martingale.}$$

Proof: we use the approach of Champagnat-Villemonais 2018 (QSD for time-inhomogeneous Markov processes).

We deduce that

$$W^{K,f}_t = L^{K,\psi_f}_{t \log K} \, h^K_\varepsilon(t \log K, X^K_{t \log K}) \ \text{ is a } \mathbb{P} - \text{martingale.}$$

Consider a function f such that $F_t(f) > 0$.

(ii) Then

$$U_t^{K,f} = \sum_{u \in V^K_{t \log K}} W_t^{K,f,u} \exp\left(- \log K \int_0^t (b+p-d)(X^{K,u}_{s \log K}) ds \right)$$

is a martingale uniformly integrable.

The proof is based on the spinal decomposition of the branching process Z^K.

(iii) Let $\delta = \min_{s \in [0,t]} F_s(f) \wedge \varepsilon$.
Let us define

$$N_T^{K,\varepsilon,f} = N_T^{K,\mathcal{B}(f,\varepsilon)},$$

issued from a Poisson point measure with intensity measure
$1_{[f(0)-\varepsilon, f(0)+\varepsilon]} K^{\beta_0^K(x) - \delta/2} dx$.
Consider the associated (uniformly integrable) martingale $U^{K,\varepsilon,f}$, hence

$$\mathbb{E}(U_0^{K,\varepsilon,f}) = \int_{f(0)-\varepsilon}^{f(0)+\varepsilon} K^{\beta_0^K(x) - \delta/2} h_\varepsilon^K(0,x) dx.$$

By using an upperbound of $U^{K,\varepsilon,f}$, we prove that there exists $0 < \gamma < 1$ s.t. for all $t \in [0, T]$ and K large enough,

$$\mathbb{P}\left(N_t^{K,\varepsilon,f} < \|h_\varepsilon^K(t\log K,\cdot)\|_\infty^{-1}\,\mathbb{E}(U_0^{K,\varepsilon,f})\,K^{\int_0^t (b+p-d)(f_s)ds - I_t(f) - 2C\epsilon}\right) \leq \gamma.$$

We deduce that for all $t \in [0, T]$ and all $\varepsilon > 0$, almost surely,

$$\liminf_{K\to+\infty} \frac{1}{\log K} \log N_t^{K,\varepsilon,f} \geq F_t(f).$$

(iv) We conclude easily that for any G open subset of $\mathbb{D}([0,t],\mathbb{R})$, almost surely,

$$\liminf_{K\to\infty} \frac{1}{\log K} \log N_t^{K,G} \geq \sup\{F_t(g);\, g \in G,\ \forall s \in [0,t],\ F_s(g) > 0\}.$$

Que les aventures les plus belles continuent, cher Christian!

Merci pour ta joie, ta gentillesse et ton enthousiasme!