Diffusion and Hamiltonian equations, back and forth

Yann Brenier
CNRS Orsay Paris-Saclay et équipe CNRS-INRIA "PARMA"

Journées en l'honneur de Christian Léonard
1-2 juillet 2025, Université Paris-Cité

PART I
FROM DIFFUSION TO HAMILTONIAN EQUATIONS: THE EXAMPLE OF CLASSICAL GRAVITATION

The trajectory $t \in \mathbb{R}_+ \to X_t(a) \in \mathbb{R}^3$ of each "particle" labelled by $a \in \mathbb{R}^3$ (mod $\mathbb{Z}^3$ for simplicity) is driven by

$$\boxed{\frac{d^2X_t}{dt^2} + (\nabla\varphi)(t, X_t) = 0, \quad 1 + \triangle\varphi = \rho = \int_{\mathbb{T}^3} \delta(x - X_t(a))da}$$

where $\rho(t, x)$ and $\varphi(t, x)$, $x \in \mathbb{T}^3$, respectively denote the density field (supposed to be of unit average) and the gravitational potential.

MONGE-AMPERE GRAVITATION (MAG)

(Y.B., G. Loeper GAFA ’04, Y.B.Confl. Math ’11, B. Lévy, Y.B., R. Mohayahee arXiv 24)

$$\boxed{\rho(t,x) = \det(I + D^2\varphi(t,x)) \ \text{ instead of } \ \rho(t,x) = 1 + \triangle\varphi(t,x)}$$

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iii) might be as good as the Poisson equation as an approximation to the Einstein equations (conjecture), based on the "vague" analogy

$$\frac{\text{Einstein} \quad \text{equation}}{\text{Ricci} \quad \text{curvature}} \sim \frac{\text{Monge}-\text{Ampere} \quad \text{equation}}{\text{Gauss} \quad \text{curvature}}$$

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iv) has a computational complexity similar to Poisson thanks to the Monge-Ampère solver by Quentin Mérigot (2D) and Bruno Lévy (3D).

MONGE-AMPERE vs NEWTON (B. Lévy, Y.B., R. Mohayaee arxiv 2404.07697v2)

Présentation roya-bruno-p12 1 page Réduire/agrandir Partager Surbrillance Pivoter Annoter Rechercher Recherche

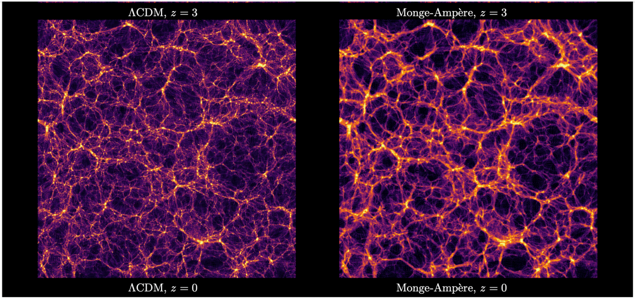

FIG. 8. Comparison between 3D simulations of ΛCDM (using an adaptive-mesh algorithm similar to [21]) and Monge-Ampère in a cube of 300 Mpc/h, 512^3 particles, z=5, 3 and 0. Projected integrated density in a 15 Mpc/h thick slab, using a logarithmic color scale. Large-scale similarity between the two models is striking, however MAG creates more abundant and diffuse filaments, whereas ΛCDM creates highly-clustered small haloes. There is weaker clustering because MAG does not diverge and is screened at short distances.

PURELY STOCHASTIC ORIGIN OF DISCRETE MONGE-AMPERE GRAVITATION FROM THE MODEL OF A BROWNIAN CLOUD

Ambrosio, Baradat, B., Analysis and PDEs '22, Léonard, Mohayaee arXiv 24
(picture taken from B. Lévy, Y.B., R. Mohayaee arxiv 2404.07697v2)

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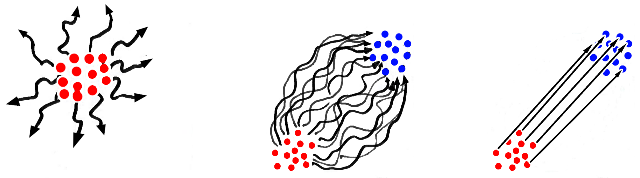

FIG. 1. Left panel: unconditioned motion of M independent Brownian particles; Center panel: motion of independent Brownian particles conditioned by their initial and final positions (in red and blue respectively); Right: conditioned Brownian motion with vanishing noise, all trajectories tend to geodesics.

BROWNIAN CLOUDS

We define a brownian cloud to be a finite set of N indistinguishable points in the euclidean space, i.e. as a point X in the quotient space $(\mathbb{R}^d)^N/\mathfrak{S}_N$, initially located on a finite cubic lattice $\{A(\alpha) \in \mathbb{R}^d,\ \alpha = 1, \cdots, N\}$ and subject to N independent Brownian motions in $\mathbb{R}^d$, with uniform noise ν.

HEAT EQUATION AND BROWNIAN CLOUDS

In PDE terms, this corresponds to the heat equation in $\mathbb{R}^{Nd}$:

$$\boxed{\frac{\partial \rho}{\partial t}(t,X) = \frac{\nu}{2} \triangle \rho(t,X), \quad \rho(t=0,X) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{\alpha=1}^{N} \delta(X(\alpha) - A(\sigma(\alpha)))}$$

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where the initial data takes the relabeling symmetry into account so that $\rho(t,X)$ is the probability density of finding the brownian cloud at position X (up to a permutation of the labels) at time t

$$\boxed{\rho(t,X) = \frac{1}{N!}(2\pi\nu t)^{-Nd/2}\sum_{\sigma\in\mathfrak{S}_N}\prod_{\alpha=1}^{N}\exp(-\frac{|X(\alpha) - A(\sigma(\alpha)|^2}{2\nu t})}$$

"L'ONDE PILOTE"

After solving the heat equation in the space of "clouds" $X \in \mathbb{R}^{Nd}$

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we may solve the companion ODE in the same space $\mathbb{R}^{Nd}$

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This is an adaptation of de Broglie's "onde pilote" idea. As a matter of fact, a similar calculation also works for the free Schrödinger equation:
$(i\partial_t + \triangle)\psi = 0, \quad \psi(0, X) = \sum_\sigma \exp(-||X - A_\sigma||^2/a^2), \quad v = \nabla \mathcal{I}m \log \psi$

"ONDE PILOTE" AND ZERO NOISE LIMIT

Setting $t = \exp(2\theta)$, we more explicitly get

(with abuse of notation $X_t \to X_\theta$):

$$\boxed{\frac{dX_\theta}{d\theta} = -\nabla_X \Phi_{\nu,\theta}(X_\theta)\ , \quad \Phi_{\nu,\theta}(X) = \nu \exp(2\theta) \log \sum_{\sigma \in \mathcal{S}_N} \exp\left(\frac{-||X - A_\sigma||^2}{2\nu \exp(2\theta)}\right)}$$

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Surprisingly enough, we may easily pass to the limit $\nu \to 0$
in the class of maximal monotone operators (cf. Brezis' book)

$$\boxed{\frac{d_+ X_\theta}{d\theta} = -\overline{\nabla}_X \Phi(X_\theta), \quad \Phi(X) = \lim_{\nu \to 0} \Phi_{\nu,\theta}(X) = -\inf_{\sigma \in \mathcal{S}_N} \ ||X - A_\sigma||^2/2}$$

Indeed, $\Phi_{\nu,\theta}(X)$ reads $-\frac{||X||^2 + ||A||^2}{2} \ + \ $ a convex function of X.

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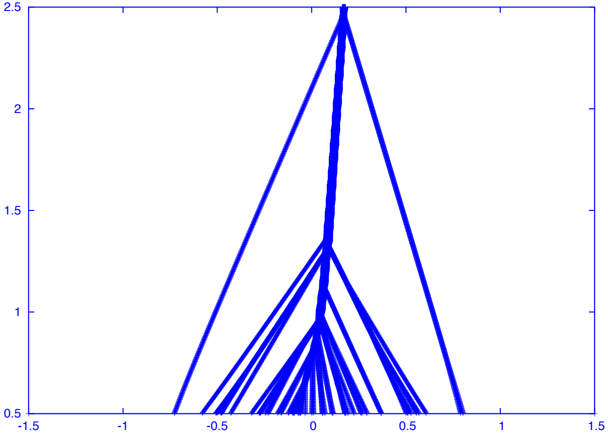
Indeed, $\Phi_{\nu,\theta}(X)$ reads $-\frac{||X||^2 + ||A||^2}{2} \ +$ a convex function of X.
N.B. Through $\overline{\nabla}_X \Phi$, this equation includes sticky collisions in 1D.

1D sticky collisions

horizontal : 51 *grid points in x* / *vertical* : 60 *grid points in t*



LARGE DEVIATIONS FOR THE "ONDE PILOTE"

$$\boxed{\frac{dX_\theta}{d\theta} = -\nabla_X \Phi_{\nu,\theta}(X_\theta) + \sqrt{\eta}\frac{dB_\theta}{d\theta}} \quad \Phi_{\nu,\theta}(X) = \nu e^{2\theta} \log \sum_{\sigma \in \mathfrak{S}_N} \exp\left(\frac{-||X - A_\sigma||^2}{2\nu e^{2\theta}}\right),$$

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Then, we may pass to the limit $\nu \to 0$ and obtain as "$\Gamma-$limit"

$$\boxed{\int_{\theta_0}^{\theta_1} ||\frac{d_+ X_\theta}{d\theta} + \overline{\nabla}_X \Phi(X_\theta)||^2 d\theta, \quad \Phi(X) = -\inf_{\sigma \in \mathfrak{S}_N} ||X - A_\sigma||^2/2}$$

which (at least in 1D) handles sticky collisions thanks to $\overline{\nabla}_X \Phi$!

L. Ambrosio, A. Baradat, Y.B. Analysis and PDE 2023

BACK TO MONGE-AMPERE GRAVITATION!

Using the least action principle, we obtain

$$\boxed{\frac{d^2 X_\theta(\alpha)}{d\theta^2} = X_\theta(\alpha) - A(\sigma_{opt}(\alpha)) \,, \quad X_\theta(\alpha) \in \mathbb{R}^d, \;\; \alpha = 1, \cdots, N}$$

$$\boxed{\sigma_{opt} = \mathrm{Arginf}_{\,\sigma \in \mathfrak{S}_N} \;\; \sum_{\alpha=1}^{N} |X_\theta(\alpha) - A(\sigma(\alpha))|^2}$$

(but not in the sense of Bouchut/Ambrosio!)

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and finally, using Optimal Transport tools, the continuous version

$$\boxed{\frac{d^2 X_\theta}{d\theta^2} + (\nabla\varphi)(\theta, X_\theta) = 0, \quad \det(\mathbb{I}_d + D^2\varphi) = \rho = \int \delta(x - X_\theta(a))da}$$

From Hamiltonian equations to diffusions

Y.B. Ann. Toulouse '17, Y.B., Xianglong Duan ARMA '18,
Lecture notes "Hidden Convexity" Y.B. '20, hal-02928398.

First example: from Euler to Fourier.

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First example: from Euler to Fourier.

The Euler model for fluids (1755/57) reads

$$\boxed{\partial_t\rho + \nabla\cdot(\rho v) = 0, \quad \partial_t(\rho v) + \nabla\cdot(\rho v\otimes v) = -\nabla p}$$

where $(\rho, p, v) \in \mathbb{R}^{1+1+3}$ are the density, pressure and velocity fields of a fluid and p is assumed to be a given function of ρ.

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Let us now perform the quadratic change of time

$$\boxed{\rho(t,x) = \tilde\rho(\theta,x), \quad \theta = t^2/2, \quad v(t,x) = \tilde v(\theta,x)\frac{d\theta}{dt}}$$

from Euler to Fourier and diffusion in porous media

After the quadratic change of time, the Euler equations become

$$\boxed{\partial_\theta \rho + \nabla \cdot (\rho v) = 0, \;\; \rho v + 2\theta[\partial_\theta(\rho v) + \nabla \cdot (\rho v \otimes v)] = -\nabla(p(\rho))}$$

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leading as $\theta << 1$ to the Fourier law and the porous media equation

$$\boxed{\rho v = -\nabla p, \quad \partial_\theta \rho = \triangle(p(\rho))}$$

This can be done in many other ways (for ex. Lattanzio-Tzavaras did it some years ago with the relative entropy method, by adding suitable friction terms).

From Schrödinger to Derrida-Lebowitz-Speer-Spohn

In a similar way the quantum diffusion equation
(aka DLSS equation)

$$\boxed{\partial_t \rho + \Delta^2 \rho - \nabla \otimes \nabla : \frac{\nabla \rho \otimes \nabla \rho}{\rho} = 0}$$

can be obtained after the same quadratic change of time from the Schrödinger equation (written in Madelung's form).

From Born-Infeld's Electromagnetism to the diffusion of divergence-free vector fields

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We will concentrate on the 3+1 Minkowski space of special relativity (as Max Born and Leopold Infeld did in 1934).

Max Born (1882-1970)

Through quadratic change of time, a diffusion equation for divergence-free vector fields à la Born-Infeld!

(Xianglong Duan arXiv:1706.01661,
Y.B. Ann. Toulouse, Y.B. and Xianglong Duan ARMA '18)

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An "incompressible" version ($\rho = cst$) reads

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which is a quite decent diffusion equation from a MHD viewpoint

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with fascinating metastability issues!

MERCI CHRISTIAN ET BONNE CONTINUATION !