Hypocoercivity in Φ-entropy for the linear relaxation Boltzmann Equation.

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March 12, 2018

Φ-entropies

If Φ is a function $\mathbb{R}_+ \to \mathbb{R}_+$ which satisfies $\Phi(1) = 0$ and $\Phi''(z) > 0$ then we define the Φ-entropy relative to μ of a probability density f by

$$H_\mu^\Phi(f) = \int \Phi\left(\frac{f}{\mu}\right) \mathrm{d}\mu.$$

We define the Φ-Fisher information by

$$I_\mu^\Phi(f) = \int \Phi''\left(\frac{f}{\mu}\right) \left|\nabla\left(\frac{f}{\mu}\right)\right|^2 \mathrm{d}\mu.$$

See Chafaï 2004.

Φ-entropies

In particular we can look at

$$\Phi_1(x) = x \log(x) - x + 1$$

which gives relative entropy. Also

$$\Phi_p(x) = \frac{1}{p-1}\left(x^p - 1 - p(x-1)\right), \quad p \in (1, 2].$$

These p entropies interpolate between Boltzmann entropy and L^2.

Φ-entropy inequalities

For relative entropy the inequality μ satisfies a log-Sobolev inequality for some constant C if

$$\int h \log(h) \mathrm{d}\mu \leq C \int \frac{|\nabla h|^2}{h} \mathrm{d}\mu$$

For general Φ a Φ-sobolev inequality is

$$\int \Phi(h) \mathrm{d}\mu \leq C \int \Phi''(h) |\nabla h|^2 \mathrm{d}\mu.$$

When $\Phi = \Phi_p$ these are due to Beckner, when $p = 2$ it is a Poincaré inequality.

Convergence in Φ-entropy

So if we have a non-degenerate diffusion

$$\partial_t f + \sum_i A_i^* A_i f = 0.$$

Where Ai^* is the conjugate in $L^2(\mu)$ then write $h_t = f_t/\mu$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \Phi\left(h_t\right) \mathrm{d}\mu = -\int \Phi''\left(h_t\right) \left|A_i h_t\right|^2 \mathrm{d}\mu \leq -c \int \Phi''(h_t) |\nabla h_t|^2 \mathrm{d}\mu.$$

So showing convergence in Φ-entropy comes down to showing a Φ-sobolev inequality or an entropy-entropy production inequality. In hypocoercive situations we usually already know this inequality and have to deal with the degeneracy in the elliptic part.

Hypocoercivity

Hypocoercivity was introduced in Villani's memoire *Hypocoercivity*. It means convergence of the form

$$\|f_t\| \leq Ce^{-\lambda t}\|f_0\|.$$

More specifically, it refers normally to constructive, quantifiable methods for proving a convergence result of this kind when f_t is the solution to a degenerate equation. It is often applied to kinetic equations

$$\partial_t f + v \cdot \nabla_x - \nabla_x V \cdot \nabla_v f = Q_v(f),$$

Where Q_v acts only on the velocity variable so in these cases the degeneracy is the missing x directions.

Hypocoercivity in Φ-entropy

Suppose f_t is a solution to the equation

$$\partial_t f + Lf = 0, \quad f|_{t=0} = f_0$$

with equilibrium solution μ. Then we can say this equation is hypocoercive in Φ entropy if

$$\int \Phi\left(\frac{f_t}{\mu}\right) \mathrm{d}\mu \leq Ce^{-\lambda t} \int \Phi\left(\frac{f_0}{\mu}\right) \mathrm{d}\mu$$

or

$$\int \Phi\left(\frac{f_t}{\mu}\right) \mathrm{d}\mu \leq Ce^{-\lambda t} \left(\int \Phi\left(\frac{f_0}{\mu}\right) \mathrm{d}\mu + \int \Phi''\left(\frac{f_t}{\mu}\right) \left|\nabla\left(\frac{f_t}{\mu}\right)\right|^2 \mathrm{d}\mu \right)$$

Hypocoercivity in Φ-entropy

Hypocoercivity in Boltzmann entropy was introduced by Villani in the memoire *Hypocoercivity*. He showed hypocoercivity for operators of the form

$$L = \sum_i A_i^* A_i + B.$$

Here the A_i are first order derivations, A_i^* is the conjugate in $L^2(\mu)$ where μ is the equilibrium and $B^* = -B$. The main example of an equation of this type is the kinetic Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \nabla_v \cdot (\nabla_v f + vf).$$

Works on Hypocoercivity in Φ-entropy

There are several works studying hypocoercivity in Φ-entropies

- Arnold & Erb (2014) followed by Arnold, Einav, Wöhrer study hypocoercivity with sharp rates in Φ-entropy for degenerate diffusions with linear forces i.e.

$$\partial_t f = \nabla \cdot (\Sigma \nabla f + Czf)\,.$$

- Letizia & Olla (2016) Study diffusive limits for anharmonic chain of oscillators using entropic hypocoercivity techniques.
- Baudoin ('17) (+ other papers) looks at local gradient bounds for the kinetic Fokker-Planck equation combining Bakry-Emery style Γ- calculus. This integrated gives convergence in relative entropy and H^1.

Works on Hypocoercivity in Φ-entropy

- Monmarché (2018) Also looks at Γ calculus and local estimates this time with a general Φ. This recovers the sharp rates of Arnold and Erb and applications to interacting particles on a graph.
- Cattiaux, Guillin, Monmarché, Zhang (2018) Extend Villani's proof of hypocoercivity in relative entropy for kinetic Fokker-Planck equation to potentials with unbounded Hessians.
- Dolbeault & Li (2018) Show improved rates for the kinetic Fokker-Planck equation in p-entropies for $p \in (1, 2)$

Entropic hypocoercivity for diffusion equations

Villani showed hypocoercivity for diffusions in the Hörmander sum of squares form. We try and explain the strategy of this proof particularising to the case of the kinetic Fokker-Planck equation on the torus.

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + vf).$$

In this case we have that

$$\mu(x, v) = \mathcal{M}(v) \times 1(x) = (2\pi)^{-d/2} e^{-|v|^2/2}.$$

$$A = \nabla_v + v, B = v \cdot \nabla_x.$$

Entropic hypocoercivity for diffusion equations

The Q_v part of this equation is a Fokker-Planck operator just in the v variable and pushes towards local equilibria. i.e. functions of the form

$$\rho(x)\mathcal{M}(v).$$

The convergence is transferred to the x-variable due to the interaction between Q_v and the transport operator. In particular crucially that

$$[T, \nabla_v] = \nabla_x.$$

This motivates the introduction of a 'twisted' Fisher information

$$\int \frac{\nabla h^T S \nabla h}{h} \mathrm{d}\mu.$$

Here again $h = f/\mu$ and S will be a positive definite matrix that we choose.

Entropic hypocoercivity for diffusion equations

The reason this twisted Fisher information is so useful is that due to the commutation between A and B we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_T \int \frac{\nabla_x h \cdot \nabla_v h}{h} \mathrm{d}\mu = -\int \frac{|\nabla_x h|^2}{h} \mathrm{d}\mu.$$

This means that if S is not symmetric when we differentiate the twisted Fisher information we will generate terms acting in the missing directions. So we look at differentiating a functional of the form

$$\mathscr{G}(f) = H_\mu(f) + \alpha \int \frac{|\nabla_v h|^2}{h} \mathrm{d}\mu + 2\beta \int \frac{\nabla_v h \cdot \nabla_x h}{h} \mathrm{d}\mu + \gamma \int \frac{|\nabla_x h|^2}{h} \mathrm{d}\mu.$$

For α, β, γ well chosen satisfying $\beta^2 < \alpha\gamma$.

So the broad strategy is to choose α, β, γ so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{G}(f_t) \leq -CI_\mu(f_t)$$

for some constant C. Then thanks to our restrictions on α, β, γ we have

$$\mathscr{G}(f_t) \leq c(H_\mu(f_t) + I_\mu(f_t)).$$

By the log-Sobolev inequality for μ we have that

$$-CI_\mu(f_t) \leq -C'(I_\mu(f_t) + H_\mu(f_t)) \leq -C''\mathscr{G}(f_t).$$

Therefore we can close a Grönwall estimate on $\mathscr{G}(f_t)$. We also have $H_\mu(f_t) \leq \mathscr{G}(f_t) \leq A(H_\mu(f_t) + I_\mu(f_t))$ so we combine these to get

$$H_\mu(f_t) \leq Ce^{-\lambda t}(H_\mu(f_0) + I_\mu(f_0)).$$

Hypocoercivity for diffusion equations

So to choose α, β, γ we look at

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H_\mu(f) &= -\int \frac{|\nabla_v h|^2}{h} \mathrm{d}\mu, \\
\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{|\nabla_x h|^2}{h} \mathrm{d}\mu &= -2 \int \frac{|\nabla_x \nabla_v h|^2}{h} \mathrm{d}\mu, \\
\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{|\nabla_v h|^2}{h} \mathrm{d}\mu &= -2 \int \frac{|\nabla_v \nabla_v h|^2}{h} \mathrm{d}\mu - 2 \int \frac{|\nabla_v h|^2}{h} \mathrm{d}\mu \\
&\quad -2 \int \frac{\nabla_x h \cdot \nabla_v h}{h} \mathrm{d}\mu, \\
\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{\nabla_x h \cdot \nabla_v h}{h} \mathrm{d}\mu &= -\int \frac{|\nabla_x h|^2}{h} \mathrm{d}\mu - 2 \int \frac{\nabla_x \nabla_v h : \nabla_v \nabla_v h}{h} \mathrm{d}\mu \\
&\quad - \int \frac{\nabla_x h \cdot \nabla_v h}{h} \mathrm{d}\mu.
\end{aligned}$$

The linear relaxation Boltzmann equation

Now instead we look at the equation

$$\partial_t f + v \cdot \nabla_x f = \lambda(\Pi_{\mathcal{M}} - I)f \quad f = f(t,x,v), (x,v) \in \mathbb{T}^d \times \mathbb{R}^d.$$

Here

$$\Pi_{\mathcal{M}} f = \int f(x,u) \mathrm{d}u \mathcal{M}(v), \quad \mathcal{M}(v) := (2\pi)^{-d/2} e^{-|v|^2/2}.$$

In the previous calculations the diffusion structure is very important. In particular $\frac{\mathrm{d}}{\mathrm{d}t} H_\mu(f)$ is comparable to terms in $I_\mu(f)$ is very helpful in the proof.

If instead we have $Q_v(f) = \lambda(\Pi_{\mathcal{M}} f - f)$ then we wont have this behaviour. In fact we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} H_\mu(f) = -\lambda \int (\Pi_{\mathcal{M}} f - f) \log(h) \mathrm{d}x \mathrm{d}v \leq 0.$$

The linear relaxation Boltzmann equation

If we try and repeat calculation from the kinetic Fokker-Planck equation we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\nabla_x h|^2}{h}\mathrm{d}\mu =& -\lambda \int \frac{|\nabla_x(\Pi h/h)|^2}{(\Pi h/h)} h \mathrm{d}\mu \\
& - \lambda \left(\int \frac{|\nabla_x h|^2}{h}\mathrm{d}\mu - \int \frac{|\nabla_x \Pi h|^2}{\Pi h}\mathrm{d}\mu \right), \\
\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\nabla_v h|^2}{h}\mathrm{d}\mu =& -\lambda \int \frac{|\nabla_v h|^2}{h}\mathrm{d}\mu - \lambda \int \frac{\Pi h}{h}\frac{|\nabla_v h|^2}{h}\mathrm{d}\mu \\
& -2 \int \frac{\nabla_x h \cdot \nabla_v h}{h}\mathrm{d}\mu, \\
\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\nabla_x h \cdot \nabla_v h}{h}\mathrm{d}\mu \leq & - \int \frac{|\nabla_x h|^2}{h}\mathrm{d}\mu + \lambda\epsilon \int \frac{\Pi h}{h}\frac{|\nabla_v h|^2}{h}\mathrm{d}\mu \\
& + \frac{\lambda}{\epsilon}\int \frac{|\nabla_x(\Pi h/h)|^2}{(\Pi h/h)} h\mathrm{d}\mu - \lambda \int \frac{\nabla_x h \cdot \nabla_v h}{h}\mathrm{d}\mu.
\end{aligned}$$

The mixed term again

So now the term $\int (\nabla_x h \cdot \nabla_v h)/h \mathrm{d}\mu$ becomes a problem in our estimates (because there is too much). We need to use something more complicated than Cauchy-Schwartz to control it on the right hand side of our equation.

Lemma

For any positive η we have

$$
\begin{aligned}
&-\int \frac{\nabla_x h \cdot \nabla_v h}{h} \mathrm{d}\mu \\
\leq &\frac{\eta}{2} \int \frac{|\nabla_v h|^2}{h} \mathrm{d}\mu + \frac{1}{2\eta}\left(\int \frac{|\nabla_x h|^2}{h} \mathrm{d}\mu - \int \frac{|\nabla_x \Pi h|^2}{\Pi h} \mathrm{d}\mu\right) \\
&- \frac{\mathrm{d}}{\mathrm{d}t} \int \Pi h \log(\Pi h) \mathrm{d}\mu.
\end{aligned}
$$

The mixed term again

This lemma allows us to add $\int \Pi h \log(\Pi h) \mathrm{d}\mu$ to our functional and then control the mixed term by

$$\int \frac{|\nabla_x h|^2}{h} \mathrm{d}\mu - \int \frac{|\nabla_x \Pi h|^2}{\Pi h} \mathrm{d}\mu \text{ and } \int \frac{|\nabla_v h|^2}{h} \mathrm{d}\mu,$$

instead of

$$\int \frac{|\nabla_x h|^2}{h} \mathrm{d}\mu \text{ and } \int \frac{|\nabla_v h|^2}{h} \mathrm{d}\mu.$$

This means we can increase the relative amount of $\int |\nabla_x h|^2 / h \mathrm{d}\mu$ in our functional in order to be able to close a Gronwall estimate.

The linear Boltzmann equation

If we try and repeat calculation from the kinetic Fokker-Planck equation we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\nabla_x h|^2}{h}\mathrm{d}\mu = -\lambda \int \frac{|\nabla_x(\Pi h/h)|^2}{(\Pi h/h)} h \mathrm{d}\mu$$

$$-\lambda \left(\int \frac{|\nabla_x h|^2}{h}\mathrm{d}\mu - \int \frac{|\nabla_x \Pi h|^2}{\Pi h}\mathrm{d}\mu \right),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{|\nabla_v h|^2}{h}\mathrm{d}\mu = -\lambda \int \frac{|\nabla_v h|^2}{h}\mathrm{d}\mu - \lambda \int \frac{\Pi h}{h}\frac{|\nabla_v h|^2}{h}\mathrm{d}\mu$$

$$+2\int \frac{\nabla_x h \cdot \nabla_v h}{h}\mathrm{d}\mu,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\nabla_x h \cdot \nabla_v h}{h}\mathrm{d}\mu \leq - \int \frac{|\nabla_x h|^2}{h}\mathrm{d}\mu + \lambda\epsilon \int \frac{\Pi h}{h}\frac{|\nabla_v h|^2}{h}\mathrm{d}\mu$$

$$+\frac{\lambda}{\epsilon}\int \frac{|\nabla_x(\Pi h/h)|^2}{(\Pi h/h)} h\mathrm{d}\mu - \lambda \int \frac{\nabla_x h \cdot \nabla_v h}{h}\mathrm{d}\mu.$$

Hypocoercivity for the Linear Boltzmann Equation

Theorem (E '17)

If f is a solution to

$$\partial_t f + v \cdot \nabla_x f = \lambda(\Pi_{\mathcal{M}} f - f),$$

with initial data f_0 then if

$$\int_{\mathbb{R}^d \times \mathbb{T}^d} \frac{|\nabla h_0|^2}{h_0} \mathrm{d}\mu < \infty, f_0 \in W^{1,1}(\mu).$$

there exist constants B, Λ and A depending on λ such that

$$I_\mu(f) + BH_\Pi(f) \leq \exp(-\Lambda t)\left(AI_\mu(f_0) + 2BH_\Pi(f_0)\right).$$

Where

$$H_\Pi(f) = \int \Pi h \log(\Pi h) \mathrm{d}\mu.$$

Confining Potential Case

Arnold and Erb show that in the case of linear forces for the diffusion case e.g. here a quadratic confining potential a nice cancellation occurs between the mixed terms appearing in

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\nabla h^T S \nabla h}{h}\mathrm{d}\mu.$$

Which means you can close a Grönwall estimate on the twisted Fisher information. This also works for close to quadratic confining potentials.

Using this nice cancellation Pierre Monmarché also showed convergence for the linear Boltzmann equation with close to quadratic potentials. He shows

Theorem (Monmarché '17)

If f_t is a solution to

$$\partial_t f + v \cdot \nabla_x f - \rho x \cdot \nabla_v f - \delta \nabla_x U \cdot \nabla_v f = \lambda (\Pi_{\mathcal{M}} - I) f$$

with $|\nabla^2 U| \leq 1$ then if δ is small enough in terms of ρ and λ there exists Λ, C such that

$$I_\mu(f_t) \leq C e^{-\Lambda t} I_\mu(f_0).$$

Next steps...

- See if this/something similar works for other non-diffusion operators (e.g. other scattering operators, spatially inhomogeneous scattering rate, linear Boltzmann).
- Move away from close to quadratic in the confining potential case?