# Stability of the Gaussian Isoperimetric Problem

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- Symmetry of minimizers of a Gaussian isoperimetric problem

  with Vesa Julin

- Sharp dimension free quantitative estimates for the gaussian isoperimetric inequality, *Ann. Probab.* (2017)

  with Vesa Julin and Alessio Brancolini

http://cvgmt.sns.it/people/barchiesi/

# Gaussian isoperimetric inequality

Gauss space is $\mathbb{R}^n$ with measure

$$\gamma(E) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\frac{|x|^2}{2}} dx$$

for every $E \subset \mathbb{R}^n$. It is a probability measure, $\gamma(\mathbb{R}^n) = 1$.

Gaussian surface measure or Gaussian perimeter

$$P_\gamma(E) := \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x)$$

when $E$ sufficiently regular. We will use the notation

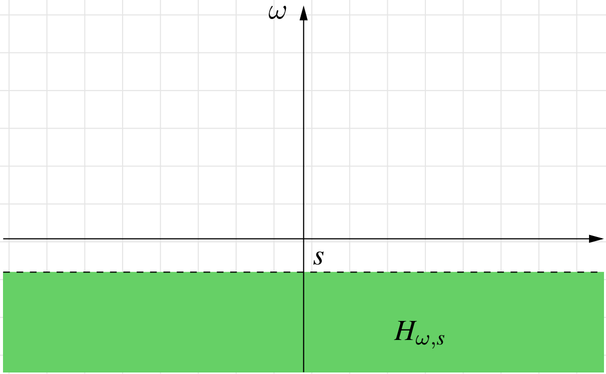
$$\mathcal{H}^{n-1}_\gamma = e^{-\frac{|x|^2}{2}} \mathcal{H}^{n-1}.$$

# Gaussian isoperimetric inequality

*Among all sets with given Gaussian measure, the half-space has the smallest Gaussian perimeter.*

## Some notation

- $H_{\omega,s} := \{x \in \mathbb{R}^n : x \cdot \omega < s\}, \quad \omega \in \mathbb{S}^{n-1}$
- $\phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}} dt.$



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Observe that in every dimension

$$\gamma(H_{\omega,s}) = \phi(s) \qquad \text{and} \qquad P_\gamma(H_{\omega,s}) = e^{-s^2/2}.$$

**Theorem (Gaussian isoperimetric inequality)**

*For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ it holds*

$$P_\gamma(E) \geq e^{-s^2/2},$$

*and the equality holds if and only if $E = H_{\omega,s}$ for some $\omega \in \mathbb{S}^{n-1}$.*

A lot of proofs... Sudakov-Tsirelson (1974), Borell (1975), Carlen-Kerce (2001). The latter characterizes the extemals.

# Symmetric case

The half-space is not symmetric. So a natural question is this.

**Question**: “Among all sets with given Gaussian measure, what is the symmetric set with the smallest Gaussian perimeter?”

Easy question, hard answer: at the moment we have no a precise idea about the possible shape of the solution. One of the main difficulties is that symmetrization techniques fail (I mean, we failed in using them).

We go along a different path...

# Stability

**Question**: How much is positive the following quantity?

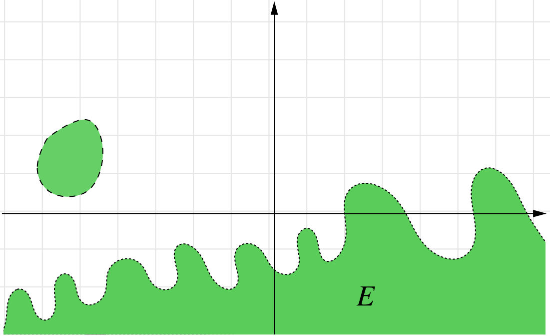
$$P_\gamma(E) - e^{-s^2/2}$$

### Theorem (Cianchi-Fusco-Maggi-Pratelli (2011))

*For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ it holds*

$$P_\gamma(E) - e^{-s^2/2} \geq c_{n,s}\, \alpha(E)^2$$

*for some constant $c_{n,s}$ depending both on the dimension $n$ and the volume $\phi(s)$. Here $\alpha(E) := \min_{|\omega|=1} \gamma(E \Delta H_{\omega,s})$*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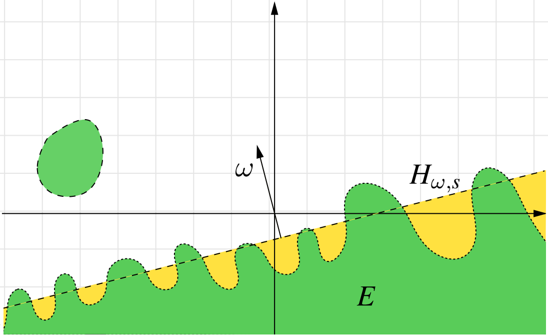

ω
H_{ω,s}
E

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+ The decay rate is sharp.
- The constant should not depend on the dimension (here $\sim 2^n$).

**Theorem (Mossel-Neeman (2013))**

*For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ holds*

$$P_\gamma(E) - e^{-s^2/2} \geq c_s\, \alpha(E)^{4+\varepsilon}$$

*for some constant $c_s$ depending only on the volume $\phi(s)$.*

- \- The decay rate is not sharp.
- \+ The constant does not depend on the dimension.

## Theorem (Eldan (2015))

*For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ holds*

$$P_\gamma(E) - e^{-s^2/2} \geq c_s\, \beta(E) \big| \log \beta(E) \big|^{-1}$$

*for some constant $c_s$ depending only on the volume $\phi(s)$. Here $\beta(E) := \min_{|\omega|=1} \big| b(E) - b(H_{\omega,s}) \big|$ and $b(E) := \int_E x\, d\gamma$.*

The asymmetry $\beta$ is stronger since it controls the standard $\alpha$ as

$$\beta(E) \geq \frac{e^{\frac{s^2}{2}}}{4}\, \alpha(E)^2.$$

**Theorem (B-Brancolini-Julin (2017))**

*For every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ holds*

$$P_\gamma(E) - e^{-s^2/2} \geq \frac{c}{1+s^2}\,\beta(E)$$

*for some absolute constant $c$.*

+ The decay rate is sharp.
+ The constant does not depend on the dimension.
+ The dependence on the volume is optimal.
- The constant $c$ is not sharp.

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# The barycenter

The half-space maximizes the lenght of the barycenter: if $\gamma(E) = \phi(s)$, then

$$b_s := |b(H_{\omega,s})| \geq |b(E)|.$$

Moreover the asymmetry $\beta(E)$ is not obtained via a minimum problem.

$$\beta(E) := b_s - |b(E)|.$$

# A new approach

We consider the functional

$$\mathcal{F}(E) = P_\gamma(E) + \varepsilon\sqrt{\pi/2}|b(E)|^2, \qquad \gamma(E) = \phi(s)$$

**Remark**

*In minimizing $\mathcal{F}$ the two terms $P_\gamma(E)$ and $|b(E)|$ are in competition. Minimizing $P_\gamma(E)$ means to push the set $E$ at infinity in one direction, so that it becomes closer to a half-space. On the other hand, minimizing $|b(E)|$ means to balance the volume of $E$ with respect to the origin. For $\varepsilon$ small enough the perimeter term overcomes the barycenter, and the only minimizers of $\mathcal{F}$ are the half-spaces $H_{\omega,s}$.*

# Old result

**Theorem (B-Brancolini-Julin (2017))**

*The only minimizers of the functional $\mathcal{F}$ are the half-spaces when when $\varepsilon > 0$ is small.*

**Question**: "What does it happen when $\varepsilon$ is not longer small? Does the barycenter term win?"

# Old result

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**Question**: “What does it happen when $\varepsilon$ is not longer small? Does the barycenter term win?”

Some other notation

- $D_{\omega,s} := \{x \in \mathbb{R}^n : |x \cdot \omega| < a(s)\}, \quad \omega \in \mathbb{S}^{n-1}$, where $a(s)$ is chosen such that $\gamma(D_{\omega,s}) = \phi(s)$

The asymptotic behavior for $s$ going to $+\infty$

$$a(s) = s + \frac{\ln 2}{s} + o(1/s).$$

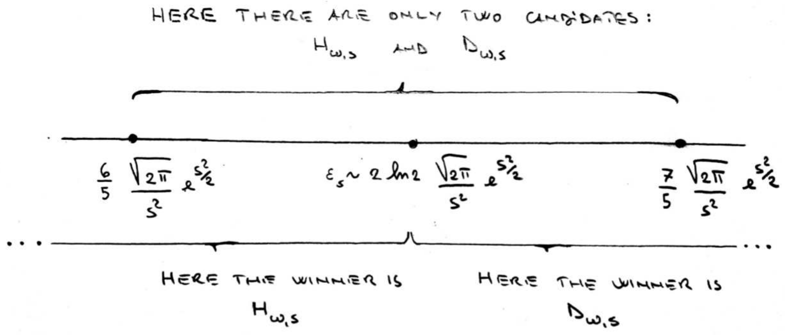
# New result

**Theorem (B-Julin (2018))**

*There exists $s_0 > 0$ such that the following holds: when $s \geq s_0$ there is a threshold $\varepsilon_s$ such that for $\varepsilon \in [0, \varepsilon_s)$ the minimizer of $\mathcal{F}$ under volume constraint $\gamma(E) = \phi(s)$ is the half-space $H_{\omega,s}$, while for $\varepsilon \in (\varepsilon_s, \infty)$ the minimizer is the symmetric strip $D_{\omega,s}$.*

$\varepsilon_s$ is the unique value of $\varepsilon$ for which $\mathcal{F}(H_{\omega,s}) = \mathcal{F}(D_{\omega,s})$. The asymptotic behavior is

$$\varepsilon_s = 2 \ln 2 \, \frac{\sqrt{2\pi}}{s^2} e^{\frac{s^2}{2}} \left(1 + o(1)\right) .$$


HERE THERE ARE ONLY TWO CANDIDATES:
$H_{\omega,s}$ AND $D_{\omega,s}$
$\frac{6}{5}\frac{\sqrt{2\pi}}{s^2}e^{s^2/2}$
$\varepsilon_s \sim 2\ln 2 \frac{\sqrt{2\pi}}{s^2}e^{s^2/2}$
$\frac{7}{5}\frac{\sqrt{2\pi}}{s^2}e^{s^2/2}$
HERE THE WINNER IS $H_{\omega,s}$
HERE THE WINNER IS $D_{\omega,s}$

# The first answer

Since symmetric sets have barycenter zero, we have the solution for the symmetric Gaussian problem (when the volume is close to one).

**Corollary**

*There exists $s_0 > 0$ such that for $s \geq s_0$ it holds*

$$P_\gamma(E) \geq 2e^{-\frac{a(s)^2}{2}} = \Big(1 + \frac{\ln 2}{s^2} + o(1/s^2)\Big)e^{-\frac{s^2}{2}},$$

*for any symmetric set $E$ with volume $\gamma(E) = \phi(s)$, and the equality holds if and only if $E = D_{\omega,s}$ for some $\omega \in \mathbb{S}^{n-1}$.*

# The second answer

We have also the optimal constant in the quantitative Gaussian isoperimetric inequality (when the volume is close to one).

**Corollary**

*There exists $s_0 > 0$ such that for $s \geq s_0$ it holds*

$$P_\gamma(E) - e^{-s^2/2} \geq c_s \beta(E),$$

*for every set $E$ with volume $\gamma(E) = \phi(s)$. The optimal constant is given by*

$$c_s = \sqrt{2\pi}\, e^{s^2/2} \left(P_\gamma(D_{\omega,s}) - P_\gamma(H_{\omega,s})\right) = \sqrt{2\pi}\, \frac{\ln 2}{s^2} + o(1/s^2).$$

# The proof:

The proof is based on a dimensional reduction.

When the vector $\omega$ is orthogonal to the barycenter, then the function $\nu_\omega$ has zero average and the second variation of $\mathcal{F}$ provides the inequality

$$\int_{\partial^* E} -\nu_\omega^2 \, d\mathcal{H}_\gamma^{n-1} + \frac{\varepsilon}{\sqrt{2\pi}} \left| \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}_\gamma^{n-1} \right|^2 \geq 0.$$

If the second term is small enough, then $\nu_\omega \equiv 0$ and $E$ is constant in that direction. But “is it small enough?”

By using Cauchy-Schwarz inequality, we may estimate the second term by

$$\Big|\int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}_\gamma^{n-1}\Big|^2 \le \left(\int_{\partial^* E} x_\upsilon^2 \, d\mathcal{H}_\gamma^{n-1}\right)\left(\int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}_\gamma^{n-1}\right)$$

and then, by the Eulero equation,

$$\frac{\varepsilon}{\sqrt{2\pi}} \int_{\partial^* E} x_\upsilon^2 \, d\mathcal{H}_\gamma^{n-1} \le \frac{8}{5} \fint_{\partial^* E} \nu_\upsilon^2 \, d\mathcal{H}_\gamma^{n-1}$$

Ops, it is larger than one! :(
And we cannot shrink $\varepsilon$.

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# From $n$ to 2

No panic: all fine for $n-2$ directions!

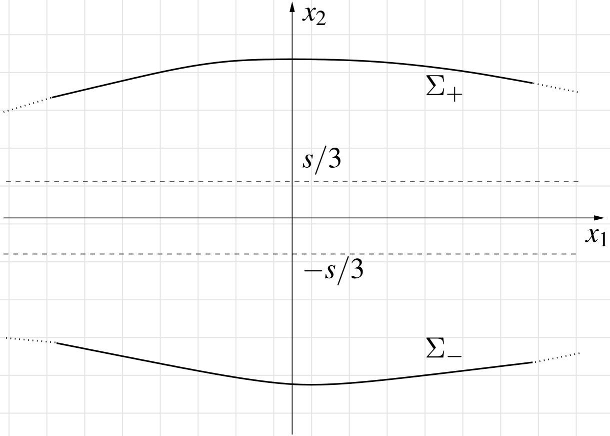
For these directions

$$\frac{\varepsilon}{\sqrt{2\pi}}\left|\int_{\partial^* E}\nu_\omega\, x\, d\mathcal{H}^{n-1}_\gamma\right|^2 \le \frac{63}{65}\int_{\partial^* E}\nu_\omega^2\, d\mathcal{H}^{n-1}_\gamma .$$

So the problem is 2-dimensional.

# From 2 to 1

A bit painfull, however...



# Future, maybe...

## Conjecture (1)

*The solution of the symmetric problem is a cylinder* $B_r^k \times \mathbb{R}^{n-k}$*, or its complement, for some* $k$ *depending on the volume and on the dimension. Here* $B_r^k$ *denotes the* $k$*-dimensional ball with radius* $r$*.*

## Conjecture (2)

*The minimizers of* $\mathcal{F}$ *are symmetric for any volume (tuning* $\varepsilon$*). Moreover, they should be finite-dimensional (with the dimension depending on the volume).*