Introduction to non-commutative hypercontractivity and log-Sobolev inequality

Ivan Bardet

Institut des Hautes Études Scientifiques (IHES), France

Based on a joint work with Cambyse Rouzé (Cambridge)

Non-commutative functional inequalities

1. Gross 1975, *Hypercontractivity and logarithmic Sobolev inequalities for the Clifford-Dirichlet form*:
   $\rightarrow$ statement of the problem and proof of HC with **non-optimal constant**
2. Carlen and Lieb 1993, *Optimal hypercontractivity for Fermi fileds and related non-commutative integration inequalities*:
   $\rightarrow$ **optimal constant**
3. Olkiewicz and Zegarlinski 1998, *Hypercontractivity for non-commutative L_p spaces*:
   $\rightarrow$ Gross equivalence with LSI in the context of spin systems, but **no proof of the positivity of the LSI constant**
4. Kastoryano and Temme 2013, *Quantum logarithmic Sobolev inequalities and rapid mixing*:
   $\rightarrow$ Adaptation of the theory for **finite dimensional systems** (similar to Markov chains on finite set)
5. More recent developments: applications in quantum information theory and some progress for spin systems

In this talk:

How some properties of the (non-commutative) L_p norms are central in the study of hypercontractivity.

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Hypercontractivity for the Orstein-Uhlenbeck semigroup

The Orstein Uhlenbeck semigroup:

- functional space $L_\infty(\mathbb{R},\gamma)$ where γ is the Gaussian measure;
- Markov semigroup $(T_t = e^{tL})_{t\geq 0}$ where

$$-\gamma(\, f\, Lf\,) = \gamma(\,|\nabla f|^2\,)$$

Theorem (Nelson 73, Gross 75)

$$\|T_t f\|_{2,\gamma} \leq \|f\|_{p,\gamma} \qquad \forall t\geq 0\,,\quad p = 1+e^{-2t}\,.$$

The two points space:

- functional space $L_\infty(\{-1,1\}) = \mathbb{C}^2$ with uniform distribution μ;
- Markov semigroup $P_t f = e^{-t}\, f + (1-e^{-t})\,\mu(f)$;

The Bernoulli space

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HC of the OU semigroup can be proved by a TCL from HC of $(P_t^{\otimes n})_{t\geq 0}$.

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The role of uniform convexity of the L_p norms

Uniform convexity of the L_p norm

For all $g, h \in L_\infty(\Omega, \mathcal{F}, \nu)$ where $(\Omega, \mathcal{F}, \mu)$ is a probability space,

$$\left(\frac{\|g+h\|_p^p + \|g-h\|_p^p}{2}\right)^{2/p} \geq \|g\|_p^2 + (p-1)\ \|h\|_p^2 \ .$$

- Any $f \in L_\infty(\{-1,1\}^n, \mu^{\otimes n})$ can be writen:

$$f = g \otimes \mathbb{1} + h \otimes x_n \, ,$$

  where $g, h \in L_\infty(\{-1,1\}^{n-1}, \mu^{\otimes n-1})$ and x_n Bernoulli variable of parameter $1/2$;
- As $g \otimes \mathbb{1}$ and $h \otimes x_n$ are othogonal in $L^2(\{-1,1\}^n, \mu^{\otimes n})$, we have:

$$\begin{aligned}
\left\|P_t^{\otimes n} f\right\|_2^2 &= \left\|P_t^{\otimes n} g \otimes \mathbb{1}\right\|_2^2 + \left\|P_t^{\otimes n} h \otimes x_n\right\|_2^2 \\
&= \left\|P_t^{\otimes n-1} g\right\|_2^2 + (p-1)\ \left\|P_t^{\otimes n-1} h\right\|_2^2 \\
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Summary of the classical case

- The inequality:

$$\left(\frac{\|g+h\|_p^p + \|g-h\|_p^p}{2}\right)^{2/p} \geq \|g\|_p^2 + (p-1)\,\|h\|_p^2$$

implies HC of the Orstein Uhlenbeck semigroup with optimal constant

- Remark that a second related inequality implies the strong LSI for Markov chains on a finite set:

$$\|f\|_p^2 \geq (p-1)\,\|f-\nu(f)\|_p^2 + \nu(f)^2$$

In particular, it implies for $f \geq 0$:

$$\mu(f^2 \log f^2) - \mu(f)^2 \log \mu(f)^2 \leq \mu(\tilde{f}^2 \log \tilde{f}^2) - \mu(\tilde{f})^2 \log \mu(\tilde{f})^2 + 2\left\|\tilde{f}\right\|_2^2$$

where $\tilde{f} = |f - \mu(f)|$.

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where $\tilde{f}=|f-\mu(f)|$.

Non-commutative L_p spaces

We consider the **interpolating family** of L^p-norms $\|\cdot\|_p$ on $\mathcal{M}_d(\mathbb{C})$ defined by:

$$\|X\|_p = \left(\frac{\mathrm{Tr}}{d}\,|X|^p\right)^{\frac{1}{p}}, \qquad 1 \leq p \leq +\infty$$

(normalized Schatten norms). We will also need the (normalised) Hilbert-Schmidt inner product:

$$\langle X, Y\rangle = \frac{\mathrm{Tr}}{d}\,[X^* Y]$$

Theorem (Uniform convexity of the NC L_p norms)

For all $1 \leq p \leq 2$ and all $X, Y \in \mathcal{M}_d(\mathbb{C})$, we have (Carlen and Lieb 93)

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The fermionic algebra

The fermionic algebra is a non-commutative analogue of the Bernoulli space.
Let $X_1, ..., X_n$ be operators in $\mathcal{M}_2(C)^{\otimes n}$ such that:

- **Anti-commutation relation**:

$$X_i\, X_j + X_j\, X_i = 2\, \delta_{ij}\, I_{2^n}$$

- X_k is an element of $\mathcal{M}_2(\mathbb{C})^k \otimes I_{2^{n-k}}$

Lemma

The set:

$$\{I_{2^n}\,,\, X_{i_1}\, X_{i_2} \cdots X_{i_k} \quad \text{for } 1 \leq i_1 < i_2 < \cdots < i_k \leq n \quad \text{and } k = 1, ..., n\}$$

is a basis of $\mathcal{M}_2(C)^{\otimes n}$.

The Fermionic (OU) semigroup is define on $\mathcal{M}_2(C)^{\otimes n}$ by:

$$\mathcal{P}_t(X_{i_1}\, X_{i_2} \cdots X_{i_k}) = e^{-\,k\,t}\, X_{i_1}\, X_{i_2} \cdots X_{i_k}$$

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Fermionic hypercontractivity

Theorem (Carlen, Lieb 1993)

The fermionic semigroup is hypercontractive. More presicely, for all $X \in \mathcal{M}_2(C)^{\otimes n}$*, all* $t \geq 0$ *and* $p_t = 1 + e^{-2t}$*,*

$$\|\mathcal{P}_t(X)\|_2 \leq \|X\|_{p_t}$$

and the constant p_t *is optimal.*

Idea of the proof: write any $X \in \mathcal{M}_2(C)^{\otimes n}$ as:

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and apply the same proof as for the Bernoulli space.

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Quantum Markov semigroups

Evolution of open systems in the **Markovian** regime are modeled by **quantum Markov semigroups** (QMS) $(\mathcal{P}_t)_{t\geq 0}$ acting on $\mathcal{M}_d(\mathbb{C})$:

- $\mathcal{P}_t(I_d) = I_d$;
- $\mathcal{P}_{t+s} = \mathcal{P}_t\mathcal{P}_s$ for all $s, t \geq 0$;
- $\mathcal{P}_t(X) \geq 0$ if $X \geq 0$;
- $t \to \mathcal{P}_t(X)$ is continuous;

The generator $\mathcal{L}$ defined by $\mathcal{P}_t = \exp\, t\,\mathcal{L}$ is called a **Lindbladian**. The QMS gives the solution of the **quantum Master equation**:

$$\frac{d}{dt}X_t = \mathcal{L}(X_t)\,, \qquad X_0 = X \in \mathcal{M}_d(\mathbb{C})\,;$$

We will always assume that:

- $\frac{\text{Tr}}{d}$ is an **invariant state** for $\mathcal{P}$:

$$\frac{\text{Tr}}{d}\left[\mathcal{P}_t(X)\right] = \frac{\text{Tr}}{d}\left[X\right] \quad \forall X \in \mathcal{B}(\mathcal{H}), \forall t \geq 0\,,$$

- $\mathcal{P}$ is **reversible**:

$$\text{Tr}[\mathcal{P}_t(X^*)\,Y] = \text{Tr}[X^*\,\mathcal{P}_t(Y)] \qquad \forall t \geq 0\,.$$

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One example: the depolarizing QMS

Depolarizing QMS:

Define:

$$\mathcal{L}_{\mathrm{dep}}(X) = \frac{\mathrm{Tr}}{d}[X]\, I_d - X\,.$$

In this case:

- $\mathcal{P}_t(X) = e^{-\,t}\, X + (1 - e^{-\,t})\, \frac{\mathrm{Tr}}{d}[X]\, I_d$;
- $\frac{\mathrm{Tr}}{d}$ is indeed invariant;
- One has $\mathcal{P}_t(X) \underset{t \to +\infty}{\longrightarrow} \frac{\mathrm{Tr}}{d}[X]\, I_d$.

Mixing-time for primitive QMS:

When $\frac{\mathrm{Tr}}{d}$ is the unique invariant state,

$$\mathcal{P}_t(X) \underset{t \to +\infty}{\longrightarrow} \frac{\mathrm{Tr}}{d}[X]\, I_d\,;$$

One is then interested in the **mixing-time**

$$\tau(\varepsilon) = \inf\left\{ t \geq 0\,;\, \left\| X_t - \frac{\mathrm{Tr}}{d}[X] I_d \right\|_\infty \leq \varepsilon\, \|X\|_\infty \quad \forall X \in \mathcal{M}_d(\mathbb{C}) \right\}.$$

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In this case:

- $\mathcal{P}_t(X) = e^{-\,t}\, X + (1 - e^{-\,t})\, \frac{\mathrm{Tr}}{d}[X]\, I_d$;
- $\frac{\mathrm{Tr}}{d}$ is indeed invariant;
- One has $\mathcal{P}_t(X) \underset{t\to+\infty}{\longrightarrow} \frac{\mathrm{Tr}}{d}[X]\, I_d$.

Mixing-time for primitive QMS:

When $\frac{\mathrm{Tr}}{d}$ is the unique invariant state,

$$\mathcal{P}_t(X) \underset{t\to+\infty}{\longrightarrow} \frac{\mathrm{Tr}}{d}\,[X]\, I_d\,;$$

One is then interested in the **mixing-time**

$$\tau(\varepsilon) = \inf\left\{ t \geq 0\,;\, \left\| X_t - \frac{\mathrm{Tr}}{d}\,[X] I_d \right\|_\infty \leq \varepsilon\, \|X\|_\infty \quad \forall X \in \mathcal{M}_d(\mathbb{C}) \right\}.$$

Definition of hypercontractivity

Definition

We say that a ***primitive*** *QMS* $(\mathcal{P}_t)_{t\geq 0}$ *is* ***hypercontractive*** *with constants* $c, d \geq 0$ *if:*

$$\|\mathcal{P}_t\|_{2\to p} \leq \exp\left\{d\left(\frac{1}{2}-\frac{1}{p}\right)\right\} \qquad (\mathrm{HC}(c,d))$$

for all time $t \geq \frac{c}{2}\log(p-1)$*.*

Equivalently, if for all $t \geq 0$*,*

$$\|\mathcal{P}_t\|_{2\to p_t} \leq \exp\left\{d\left(\frac{1}{2}-\frac{1}{p_t}\right)\right\}$$

where $p_t = 1 + e^{2t/c}$*.*

Equivalence between HC and log-Sobolev inequality

Theorem (Olkiewicz, Zegarlinski 1999)

Let $\mathcal{P}$ be a reversible primitive QMS. The two following assertions are equivalent:

(i) $\mathrm{HC}(c,d)$ *holds;*

(ii) *The following* ***logarithmic Sobolev inequality*** *holds for all* $p \geq 2$*:*

$$\mathrm{Ent}_p(X) \leq c\,\mathcal{E}_{p,\mathcal{L}}(X) + d\,\|X\|_p^p\,. \qquad (\mathrm{LSI}_p(c,d))$$

where Ent_p, *is the p-relative entropy:*

$$\mathrm{Ent}_p(X) = \frac{Tr}{d}\left[X^p \log X\right]$$

and $\mathcal{E}_{p,\mathcal{L}}$ *is the p-Dirichlet form:*

$$\mathcal{E}_{p,\mathcal{L}}(X) = -\frac{p}{2(p-1)}\langle X^{p-1}, \mathcal{L}(X)\rangle\,.$$

(iii) $\mathrm{LSI}_2(c,d)$ *holds*

Sketch of the proof: put $p_t = 1 + (p-1)e^{2t/c}$ and differentiate the norm:

$$\frac{d}{dt}\|\mathcal{P}_t(X)\|_{p_t}\Big|_{t=0} = \frac{p'(0)}{p\,\|X\|_p^{p-1}}\left(\,\mathrm{Ent}_p(X) - c\,\mathcal{E}_{p,\mathcal{L}}(X)\,\right).$$

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Remarks

The L_p regularity

$$\mathcal{E}_2(X) \leq \frac{p}{2}\,\mathcal{E}_{p,\mathcal{L}}(X)$$

is less easy to prove than in the classical case. In particular, it is not known if it holds in the general case where the invariant state is not $\frac{\text{Tr}}{d}$;

The uniform convexity of the L_p norms

$$\|X\|_p^2 \geq (p-1)\left\|X - \frac{\text{Tr}}{d}[X]\right\|_p^2 + \left(\frac{\text{Tr}}{d}[X]\right)^2$$

allows to conclude that we can always take $d = 0$ with $c < +\infty$. In particular it implies that for all $X > 0$,

$$\text{Ent}_2(X) \leq \text{Ent}_2(|X - \frac{\text{Tr}}{d}[X]|) + 2\left\|X - \frac{\text{Tr}}{d}[X]\right\|_2^2 .$$

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Simple definition of the decoherence time

Decoherence is the idea that there exists a **preferred basis** such that the off-diagonal terms of any density matrix disappear in time:

$$X = \begin{pmatrix} X_1 & & \star \\ & \ddots & \\ \star & & X_d \end{pmatrix} \underset{t\to+\infty}{\longrightarrow} X_{\text{diag}} := \begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_d \end{pmatrix} .$$

Define an interpolating family of matrices between X and X_{diag} as:

$$X_t = e^{-t} X + (1-e^{-t}) X_{diag} : \qquad X_0 = X \in \mathcal{M}_d(\mathbb{C}) , \quad X_{+\infty} = X_{\text{diag}} ;$$

This defines a QMS with Lindbladian

$$\mathcal{L}_{\text{deco}}(X) = X_{\text{diag}} - X .$$

Question:

Can we adapt hypercontractivity to the study of the **decoherence time**:

$$\tau(\varepsilon) = \inf \left\{ t \geq 0 ; \left\| X_t - X_{\text{diag}} \right\|_\infty \leq \varepsilon \left\| X \right\|_\infty \quad \forall X \in \mathcal{M}_d(\mathbb{C}) \right\} .$$

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The fixed-points algebra

Proposition

Consider a QMS $(\mathcal{P}_t)_{t\geq 0}$ such as before (not necessarily primitive) and define

$$\mathcal{F}(\mathcal{P}) := \{X \in \mathcal{M}_d(\mathbb{C})\,;\, \mathcal{P}_t(X) = X \quad \forall t \geq 0\}\,.$$

Define $E_{\mathcal{F}}$ the orthogonal projection on $\mathcal{F}(\mathcal{P})$ for $\langle\cdot,\cdot\rangle$. Then

$$\mathcal{P}_t\,(X) \underset{t\to+\infty}{\longrightarrow} E_{\mathcal{F}}[X]\,.$$

Particular cases:

- $\mathcal{L}_{\text{dep}}$: $\mathcal{F}(\mathcal{P}) = \mathbb{C}\, I_d$;
- $\mathcal{L}_{\text{deco}}$: $\mathcal{F}(\mathcal{P}) =$ diagonal operators.

We are interested in the **decoherence-time**:

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Amalgamated $L_{p,q}$ norms (Junge and Parcet)

$(\mathcal{P}_t)_{t\geq 0}$ is hypercontractive if and only if it is primitive:

- HC(c, d) with $c > 0$ implies exponential convergence towards $\frac{\mathrm{Tr}}{d}$;
- Conversely, if $(\mathcal{P}_t)_{t\geq 0}$ is not primitive, there exists $X \in \mathcal{F}(\mathcal{P})$ such that $X \notin \mathbb{C}I_d$ and for $p > 2$:

$$\|\mathcal{P}_t(X)\|_p = \|X\|_p > \|X\|_2$$

Amalgamated $L_{p,q}$ norms

For $1 \leq q \leq p \leq \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$:

$$\|X\|_{(q,p),\,\mathcal{F}} := \inf_{\substack{A,B\in\mathcal{F}(\mathcal{P})\\ X=AYB}} \|A\|_{2r}\,\|B\|_{2r}\,\|Y\|_p$$

$$\|X\|_{(p,q),\,\mathcal{F}} := \sup_{A,B\in\mathcal{F}(\mathcal{P})} \frac{\|A\,X\,B\|_q}{\|A\|_{2r}\,\|B\|_{2r}}$$

- $\mathcal{F}(\mathcal{P}) = \mathbb{C}\mathbb{1}$:

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Extension of the quantum Gross lemma to non-primitive QMS

- **DF-q-relative entropy**:

$$\mathrm{Ent}_{q,\mathcal{F}}(X) := \frac{1}{q}\,\frac{\mathrm{Tr}}{d}\left[X^p(\log X^p - \log E_{\mathcal{F}}[X^p])\right];$$

- **q-DF logarithmic Sobolev inequality**: for any $X > 0$,

$$\mathrm{Ent}_{q,\mathcal{F}}(X) \le c\,\mathcal{E}_{q,\mathcal{L}}(X) + d\|X\|_q^q \qquad (\mathrm{LSI}_{q,\mathcal{F}}(c,d))$$

- **DF hypercontractivity**: for any $X \in \mathcal{M}_d(\mathbb{C})$, and $t \ge \frac{c}{2}\log(p-1)$:

$$\|\mathcal{P}_t(X)\|_{(2,p),\mathcal{F}} \le \exp\left\{d\left(\frac{1}{2}-\frac{1}{p}\right)\right\}\|X\|_2 \qquad (\mathrm{HC}_{\mathcal{F}}(c,d))$$

Lemma

For any positive definite $X \in \mathcal{M}_d(\mathbb{C})$ and $p_t = 1 + (q-1)\,e^{2t/c}$,

$$\frac{d}{dt}\|\mathcal{P}_t(X)\|_{(q,p(t)),\mathcal{F}}\Big|_{t=0} = \frac{p'(0)}{q\|X\|_q^{q-1}}\left(\mathrm{Ent}_{q,\mathcal{F}}(X) - \frac{2(q-1)}{p'(0)}\mathcal{E}_{q,\mathcal{L}}(X)\right).$$

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Almost uniform convexity of the amalgamated L_p norms

For $X > 0$, the amalgamated L_p norms take the form $(1 \le p \le 2)$:

$$\|X\|_{(2,p),\,\mathcal{F}} := \sup_{A \in \mathcal{F}(\mathcal{P})} \frac{\|A\,X\,A\|_p}{\|A\|_{2r}^2}\;;$$

Define:

$$\Phi(A, X, p) = \frac{\|A\,X\,A\|_q}{\|A\|_{2r}^2}\;;$$

Then one can prove:

Theorem (B., Rouzé 2018)

For all $X > 0$, one has

$$\Phi(A, X, p)^2 \ge (p-1)\,\Phi(A, |X - E_{\mathcal{F}}[X]|, p)^2 + \Phi(A, E_{\mathcal{F}}[X], p)^2$$

This implies

$$\mathrm{Ent}_{2,\mathcal{F}}(X) \le \mathrm{Ent}_{2,\mathcal{F}}(|X - E_{\mathcal{F}}[X]|) + 2\,\|X - E_{\mathcal{F}}[X]\|_2^2 + \sqrt{2}\,\|X\|_2^2$$

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Results

Theorem (B., Rouzé 2018)

Let $(\mathcal{P}_t)_{t\geq 0}$ be defined as before. Then,

(i) $\mathrm{HC}_{q,\mathcal{F}}(c,d) \Rightarrow \mathrm{LSI}_{q,\mathcal{F}}(c,d)$ *for all* $q \geq 1$;

(ii) $\mathrm{LSI}_{2,\mathcal{F}}(c,d) \Rightarrow \mathrm{HC}_{\mathcal{F}}(c, d + \log C_{\mathcal{F}})$, *where* $C_{\mathcal{F}}$ *is a parameter depending on* $\mathcal{F}(\mathcal{P})$;

(iii) $\mathrm{LSI}_{2,\mathcal{F}}(c, \log\sqrt{2})$ *holds with*

$$c \leq \frac{1+\log d}{\lambda(\mathcal{L})}\,;$$

(iv) *If* $\mathrm{LSI}_{2,\mathcal{F}}(c,d)$ *holds, then necessarily* $d > 0$;

(v)

$$\|\,\mathcal{P}_t(X) - E_{\mathcal{F}}[X]\,\|_\infty \leq d_{\mathcal{F}}\, e^{1+d-\kappa}\, \|X\|_\infty \qquad \text{for } t = \frac{c}{2}\ln\ln d + \frac{\kappa}{\lambda(\mathcal{L})}\,,\ \kappa > 0\,,$$

where the $d_{\mathcal{F}}$ *is again a parameter depending on* $\mathcal{F}(\mathcal{P})$.