Stability for Faber-Krahn inequalities

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References

The main result presented in this talk is contained in

- B. - De Philippis - Velichkov, *Duke Math. J.*, **164** (2015), 1777-1832

A summary of the proof (which is quite long) and a detailed account of similar problems can be found in

- B. - De Philippis, book chapter (2017), contained in *"Shape Optimization and Spectral Theory"* (edited by A. Henrot)

Both available at `http://cvgmt.sns.it`

Plan of the talk

The Faber-Krahn inequality

Some pioneering quantitative versions

Quantitative and sharp

Drums

Take a vibrating membrane fixed at the boundary of a set $\Omega \subset \mathbb{R}^2$

This is a superposition of a discrete set of stationary vibrations

$$U(x,t) = \sum_{k\in\mathbb{N}} u_k(x) \Big(\alpha_k \cos(\sqrt{\lambda_k(\Omega)}\, t) + \beta_k \sin(\sqrt{\lambda_k(\Omega)}\, t)\Big)$$

The **eigenpair** $(u_k, \lambda_k(\Omega))$ solves

$$-\Delta u_k = \lambda_k(\Omega)\, u_k \text{ in } \Omega, \qquad u_k = 0 \text{ on } \partial\Omega$$

- $\lambda_k(\Omega)$ is $k-$**the eigenvalue of the Dirichlet-Laplacian**
- u_k is $k-$**th eigenfunction**
- $k \mapsto \sqrt{\lambda_k(\Omega)}$ **increasing** (it is the frequency of vibration)

The fundamental frequency or first eigenvalue

$\sqrt{\lambda_1(\Omega)}$ is the **fundamental frequency of the drum**

Variational characterization

$$\lambda_1(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^2 : \int_\Omega |u|^2 = 1 \right\}$$

i.e. this is the sharp constant in the Poincaré inequality

$$\lambda_1(\Omega) \int_\Omega |u|^2 \leq \int_\Omega |\nabla u|^2$$

Remark

These definitions make (mathematical) sense in every dimension N

A question raised by Lord Rayleigh

Among drums with given area, which one has the lowest fundamental frequency λ_1?

Faber (1923) and Krahn (1925) answer

The disk

Scaling law

We have that $\quad \lambda_1(t\,\Omega) = t^{-2}\,\lambda_1(\Omega) \quad$ thus

$$\lambda_1(\Omega)\,|\Omega| \quad \text{is invariant under dilations}$$

In other words, this quantity only depends on the **shape** of the set, not on its **size**

Faber-Krahn inequality in dimension N

$$\lambda_1(\Omega)\,|\Omega|^{2/N} \geq \lambda_1(\text{ball})\,|\text{ball}|^{2/N}$$

with equality **if and only if** Ω itself is a ball

Proof.

Use the variational characterization of $\lambda_1(\Omega)$, plus the properties of the **spherically symmetric decreasing rearrangement**

- let u be a first eigenfunction of Ω
- let u^* be its spherically symmetric decreasing rearrangement
- by construction $1 = \int |u|^2 = \int |u^*|^2$
- moreover, by using Pólya-Szegő principle we have

$$\lambda_1(\Omega) = \int_\Omega |\nabla u|^2 \geq \int_{\Omega^*} |\nabla u^*|^2 \geq \lambda_1(\Omega^*)$$

A glimpse of Pólya-Szegő principle

If we set $\mu(t) = \Big|\{x \in \Omega \,:\, u(x) > t\}\Big|$ (distribution function)

$$\int_\Omega |\nabla u|^2 \overset{Coarea}{=} \int_0^{+\infty} \left(\int_{\{u=t\}} |\nabla u|^2 \frac{d\sigma}{|\nabla u|} \right) dt$$

$$\overset{Jensen}{\geq} \int_0^{+\infty} \left(\int_{\{u=t\}} |\nabla u| \frac{d\sigma}{|\nabla u|} \right)^2 \frac{dt}{\displaystyle\int_{\{u=t\}} |\nabla u|^{-1} \, d\sigma}$$

$$= \int_0^{+\infty} \frac{\text{Perimeter}(\{u > t\})^2}{-\mu'(t)} \, dt$$

$$\overset{Isoperimetry}{\geq} \int_0^{+\infty} \frac{\text{Perimeter}(\{u^* > t\})^2}{-\mu'(t)} \, dt = \int_{\Omega^*} |\nabla u^*|^2$$

If $\lambda_1(\Omega) = \lambda_1(\Omega^*)$, the superlevel sets of u are balls (by using the **equality cases in the isoperimetric inequality**)

Application I: hearing the shape of a drum[1]

Let $\mathrm{spec}(\Omega) = \{\lambda_1(\Omega), \lambda_2(\Omega), \dots\}$ the collection of eigenvalues of the Dirichlet-Laplacian on Ω

Weyl's asymptotic

$$\lim_{t \to +\infty} \frac{\#\{\lambda_k(\Omega) \,:\, \lambda_k(\Omega) \le t\}}{t^{N/2}} = \frac{\omega_N}{(2\,\pi)^N}\,|\Omega| \qquad \text{(W)}$$

Spectral rigidity

- if $\mathrm{spec}(\Omega) = \mathrm{spec}(\text{ball})$, then $\quad |\Omega| = |\text{ball}| \quad$ by (W)
- ...and obviously $\lambda_1(\Omega) = \lambda_1(\text{ball})$
- thanks to equality cases in Faber-Krahn inequality, Ω **is a ball**

[1] M. Kac, "*Can one hear the shape of a drum?*", Amer. Math. Month. (1966)

Application II: nodal domains

Theorem [Courant]

Let

$$\begin{array}{rcl} u_n & = & n\text{-th eigenfunction of } \Omega \\ \mathcal{N}(n) & = & \text{number of nodal domains of } \varphi_n \end{array}$$

then we have

$$\boxed{\mathcal{N}(n) \leq n}$$

Theorem [Pleijel]

In dimension $N = 2$, we have

$$\lim_{n\to\infty} \frac{\mathcal{N}(n)}{n} \leq \left(\frac{2}{j_{0,1}}\right)^2 \simeq 0.691$$

Proof.

Denote by $\{\Omega_i\}$ the nodal domains

$$|\Omega|\,\lambda_n(\Omega) = \sum_{i=1}^{\mathcal{N}(n)} |\Omega_i|\,\lambda_1(\Omega_i) \overset{F-K}{\geq} \pi\,(j_{0,1})^2\,\mathcal{N}(n)$$

Then we use Weyl's asymptotic

Application III: conformal mappings

Theorem [Pólya-Szegő]

$\Omega \subset \mathbb{R}^2$ simply connected such that $|\Omega| = |D_1(0)| = \pi$. Let $x_0 \in \Omega$ and $f_{x_0} : \Omega \to D_1(0)$ the conformal mapping such that $f_{x_0}(x_0) = 0$. Then

$$|f'_{x_0}(x_0)| \geq 1$$

Equality holds if and only if Ω is a disc

Proof.

Conformal transplantation technique and **sub-harmonicity** of $|(f_{x_0}^{-1})'|^2$ give

$$\frac{\lambda_1(\Omega)}{|f'_{x_0}(x_0)|^2} \leq j_{0,1}^2 \qquad \text{i.e.} \qquad \frac{\lambda_1(\Omega)}{j_{0,1}^2} \leq |f'_{x_0}(x_0)|$$

Now use Faber-Krahn!

The Faber-Krahn inequality

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Quantitative stability of Faber-Krahn inequality

Question 1.

Add a **remainder term** in Faber-Krahn which measures **how much Ω is far from being a ball**?

In other words, one looks for

$$\lambda_1(\Omega)\,|\Omega|^{2/N} - \lambda_1(B)\,|B|^{2/N} \geq g(d(\Omega)) \qquad \text{(FKquant)}$$

where

- $t \mapsto g(t)$ is a modulus of continuity
- $\Omega \mapsto d(\Omega)$ is an asymmetry functional

Question 2. (harder)

Answer **Question 1.** in a sharp way? i.e. such that for a sequence $\{\Omega_n\}_n$ converging to a ball we have

$$\lambda_1(\Omega_n)\,|\Omega_n|^{2/N} - \lambda_1(B)\,|B|^{2/N} \sim g(d(\Omega_n)) \quad \text{for } n \to \infty$$

The pioneers: Melas and Hansen & Nadirashvili

Melas [J. Diff. Geom. (1992)]

For **convex sets** in every dimension, quantitative Faber-Krahn (FKquant) is valid with

$$g(t) = t^{2N}$$

$$d_{\mathcal{M}}(\Omega) = \min \left\{ \max \left\{ \frac{|B_2 \setminus \Omega|}{|B_2|}, \frac{|\Omega \setminus B_1|}{|\Omega|} \right\} \, : \, B_1 \subset \Omega \subset B_2 \text{ balls} \right\}$$

Hansen & Nadirashvili [Potential Anal. (1994)]

For **simply connected sets** in dimension $N = 2$ or **convex sets** for $N \geq 3$, quantitative Faber-Krahn (FKquant) is valid with

$$g(t) = \text{a power} \qquad\qquad d_{\mathcal{N}}(\Omega) = 1 - \frac{\text{inradius of } \Omega}{\text{radius of } B_\Omega}$$

where B_Ω is a ball such that $|B_\Omega| = |\Omega|$

Topological obstructions

The topological restrictions in the previous results **can not be removed**

Counter-example

Take B_ε a ball with a small hole of radius ε at the center. Then

$$\lambda_1(B_\varepsilon)\,|B_\varepsilon|^{2/N} \to \lambda_1(B)\,|B|^{2/N} \quad \text{but} \quad d_{\mathcal{M}} \geq d_{\mathcal{N}} \to \frac{1}{2}$$

Remark

The asymmetry functionals are **too rigid**. If we want to treat general open set, a weaker asymmetry functional is needed

Fraenkel asymmetry

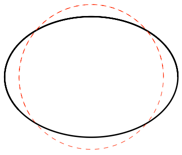
For a general open set, it is better to use

$$\mathcal{A}(\Omega) = \inf \left\{ \frac{|\Omega \Delta B|}{|\Omega|} \; : \; B \text{ ball with } |B| = |\Omega| \right\}$$

This is a L^1 distance from the "manifold" of balls

Remarks

- $0 \leq \mathcal{A} < 2$ and $\mathcal{A}(\Omega) = 0$ if and only if Ω is a ball (up to a set of measure zero)
- for a convex set with N orthogonal planes of symmetry, an optimal ball can be placed at the intersection of the planes



The Faber-Krahn inequality

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Towards sharpness

Nadirashvili – Bhattacharya & Weitsman conjecture

$$\lambda_1(\Omega)\,|\Omega|^{2/N} - \lambda_1(B)\,|B|^{2/N} \geq \frac{1}{C_N}\,\mathcal{A}(\Omega)^2$$

Exponent 2 is best possible

Take an ellipse E_ε, then

$$\lambda_1(E_\varepsilon) - \lambda_1(B_1) \sim \varepsilon^2 \qquad \mathcal{A}(E_\varepsilon) \sim \varepsilon$$

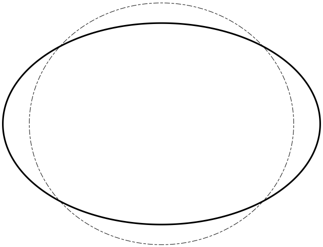


Figure: Ellipse E_ε with semi-axes $1+\varepsilon$ and $(1+\varepsilon)^{-1}$

Previous results

Many contributions by Bhattacharya, Sznitman, Povel, Fusco-Maggi-Pratelli...All of them, based on **boosted Pólya-Szegő principle**

With these methods, the best result up to now

The Hansen-Nadirashvili method (B. - De Philippis)

$$\lambda_1(\Omega)\,|\Omega|^{2/N} - \lambda_1(B)\,|B|^{2/N} \geq c_N\,\mathcal{A}(\Omega)^3$$

with $c_N > 0$ explicit dimensional constant

Proof.

Idea: go back to Pólya-Szegő inequality. In place of *isoperimetric inequality* for $\{u > t\}$, use the **the sharp quantitative isoperimetric inequality**

$$\text{Perimeter}\,(E) - \text{Perimeter}\,(B) \geq \beta_N\,\mathcal{A}(E)^2$$

Key point: link the asymmetry of $\{u > t\}$ to that of the zero-level set, i.e. Ω. □

Quantitative and sharp

The conjecture by Nadirashvili & Bhattacharya-Weitsman is true

More generally, at the same price, we get for free...

Main Theorem [B. - De Philippis - Velichkov]

For $1 \le q < 2^*$, *we define*

$$\lambda_{1,q}(\Omega) = \min_{u \in W_0^{1,2}(\Omega)} \left\{ \int_\Omega |\nabla u|^2 \, : \, \|u\|_{L^q(\Omega)} = 1 \right\}$$

we have

$$\lambda_{1,q}(\Omega)\,|\Omega|^\beta - \lambda_{1,q}(B)\,|B|^\beta \ge c_{N,q}\,\mathcal{A}(\Omega)^2$$

Remarks

- The exponent β is given by scale invariance
- The original conjecture is for $q = 2$
- $c_{N,q}$ is not explicit, we know his behaviour as $q \nearrow 2^*$

1st step : *"just prove the result for $q = 1$"*

Remark

For $q = 1$, the quantity $1/\lambda_{1,1}(\Omega)$ coincides with the **torsional rigidity** $T(\Omega)$

$$T(B) \geq T(\Omega) \qquad \text{if } |B| = |\Omega|$$

Kohler-Jobin inequality

"The ball minimizes $\lambda_{1,q}$ among sets with given torsional rigidity"

that is

$$\lambda_{1,q}(\Omega)\, T(\Omega)^{\vartheta} - \lambda_{1,q}(B)\, T(B)^{\vartheta} \geq 0$$

this implies

$$\frac{\lambda_{1,q}(\Omega)}{\lambda_{1,q}(B)} - 1 \geq \left(\frac{T(B)}{T(\Omega)}\right)^{\vartheta} - 1$$

1st step : "*just prove the result for $q = 1$*"

We thus have

Proposition [the Faber-Krahn hierarchy]

If one can prove

$$T(B_1) - T(\Omega) \geq c_N \, \mathcal{A}(\Omega)^2 \qquad \textit{for } |\Omega| = |B_1|$$

then the **Main Theorem is true for** $q > 1$, *with a constant* $\widetilde{c}_{N,q}$ *only depending on* q *and* c_N *above*

"*Why the torsional rigidity should be better?*"

Working with $T(\Omega)$ has the following advantage

$$-\frac{1}{2}\, T(\Omega) = \min_{u \in W_0^{1,2}(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega u \right\}$$

and this is a **convex problem without constraint**, with a linear Euler-Lagrange equation

2nd step: "*Main Theorem is true for almost spherical sets*"

We say that Ω is **almost spherical** if

$$\partial\Omega = \{x \,:\, x = (1+\varphi(y))\, y \quad \text{with } y \in \partial B_1\},$$

for a (smooth) funcion $\varphi : \partial B_1 \to [-\frac{1}{2}, \frac{1}{2}]$

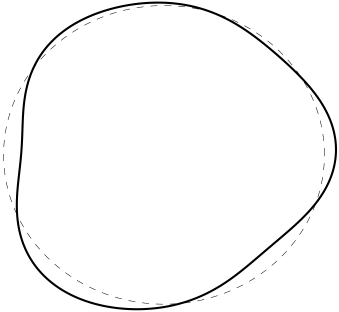


Figure: Here it is an almost spherical set!

Proposition

Let Ω *be* $C^{2,\gamma}$ *almost spherical with* $\|\varphi\|_{C^{2,\gamma}} \ll 1$ *such that*

$$|\Omega| = |B_1| \qquad \textit{and} \qquad x_\Omega := \textit{barycenter}(\Omega) = 0$$

then

$$T(B_1) - T(\Omega) \geq \tau_N \|\varphi\|^2_{H^{1/2}(\partial B_1)}$$

Proof.

We use the 2nd order Taylor expansion (Dambrine, 2002)

$$T(B_1) - T(\Omega) \geq \frac{1}{2} \partial^2 T(B_1)[\varphi, \varphi] - o(\|\varphi\|^2_{H^{1/2}})$$

The quadratic form $\partial^2 T(B_1)$ is **coercive** on a suitable subspace of $H^{1/2}$ □

Remark

$$\|\varphi\|^2_{H^{1/2}(\partial B_1)} \geq \int_{\partial B_1} \varphi^2 \gtrapprox \left(\int_{\partial B_1} |\varphi|\right)^2 \simeq |\Omega \Delta B_1|^2 \geq \mathcal{A}(\Omega)^2$$

3rd step : "*contradict the result!*"

1. We suppose that there can not exist a constant $c > 0$ such that

$$T(B_1) - T(\Omega) \geq c\,\mathcal{A}(\Omega)^2 \qquad \text{for every } |\Omega| = |B_1|$$

2. there is a **nasty sequence** $\{\Omega_n\}_{n\in\mathbb{N}}$ such that $|\Omega_n| = |B_1|$ and

$$\frac{T(B_1) - T(\Omega_n)}{\varepsilon_n^2} \to 0 \qquad \text{where } 0 < \varepsilon_n := \mathcal{A}(\Omega_n) \to 0$$

3. The sequence $\{\Omega_n\}$ converges to a ball...but for sets close to a ball, we know by the <u>2nd step</u> that

$$0 < \tau_N \leq \frac{T(B_1) - T(\Omega_n)}{\varepsilon_n^2}$$

contradiction! <u>**NOT AT ALL**</u>, of course there is a **problem of topology**

4th step : "*contradict in a smart way*"

Remark

The sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ **does not** converge in $C^{2,\gamma}$...but only in L^1!
For Ω_n, we can not use 2nd step

Idea

Replace the nasty sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ by a **smoother one**, still contradicting the stability, for example take U_n solving

$$\min\{T(B_1) - T(\Omega) \,:\, |\Omega| = |B_1| \quad \text{et} \quad \mathcal{A}(\Omega) = \varepsilon_n\}$$

By construction, we still have

$$\frac{T(B_1) - T(U_n)}{\varepsilon_n^2} \leq \frac{T(B_1) - T(\Omega_n)}{\varepsilon_n^2} \to 0 \qquad \text{and} \quad \varepsilon_n = \mathcal{A}(U_n) \to 0$$

U_n is the best way to contradict the stability

More precisely: penalized problem

We can suppose that

$$T(B_1) - T(\Omega_n) \leq \sigma\, \varepsilon_n^2$$

for a given $0 < \sigma \ll 1$

New problem

$$\min \Big(T(B_1) - T(\Omega) \Big) + \Lambda\, |\Omega| + \sqrt[3]{\sigma}\, \left(\mathcal{A}(\Omega) - \varepsilon_n\right)^2$$

1. Λ is a **Lagrange multiplier** such that for $\sigma = 0$ the ball B_1 is a solution

2. $0 < \sigma \ll 1$ is the parameter above, so that any solution U_n satisfies

- $|\mathcal{A}(U_n) - \varepsilon_n| \lesssim \sqrt[3]{\sigma}\, \varepsilon_n$
- $\Big| |U_n| - |B_1| \Big| \lesssim \sqrt[3]{\sigma}\, \varepsilon_n$
- $T(B_1) - T(U_n) \lesssim \sigma\, \varepsilon_n^2$

Free boundaries appear...

Memento

$-T(\Omega)$ as well is defined as a minimization problem over functions

Our penalized problem can be written as a **free boundary problem**

$$\min_u \frac{1}{2}\int |\nabla u|^2 - \int u + \Lambda\,|\{u>0\}| + \sqrt[3]{\sigma}\,(\mathcal{A}(\{u>0\}) - \varepsilon_n)^2$$

Relation with the previous problem

If u_n is a soluton, then $U_n = \{u_n > 0\}$

Then the **key point** is the **regularity of the free boundary** $\partial\{u_n > 0\}$

– We really hope for $C^{2,\gamma}$ regularity –

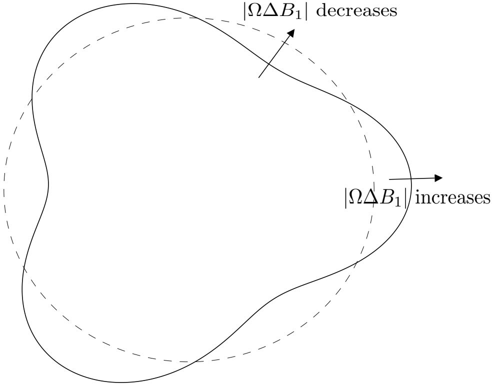
Optimality conditions and regularity

What is the best we can hope for?

Suppose that B_1 is optimal for $\mathcal{A}(\{u_n > 0\})$

The optimality condition for the free boundary problem is

$$\left|\frac{\partial u_n}{\partial \nu}\right|^2 = \Lambda + 2\sqrt[3]{\sigma}\,(\mathcal{A}(\{u_n > 0\}) - \varepsilon_n)\left(1_{\mathbb{R}^N\setminus B_1} - 1_{B_1}\right)$$



Problem

The term in red is not continuous

The free boundary ∂U_n is not even $C^{1,\gamma}$

Indeed by Elliptic Regularity, ∂U_n of class $C^{1,\gamma}$ forces the normal derivative of u_n to be continuous

We get stuck!

We needed to show that U_n converges $C^{2,\gamma}$ to a ball, but this is not possible

SO WHAT ?

A new asymmetry?

Of course, the problem is due to *lack of regularity of Fraenkel asymmetry*...what if we replace $\mathcal{A}(\Omega)$ by a "smoother" asymmetry?

Back to almost spherical sets

We have seen that

$$T(B_1) - T(\Omega) \geq \tau_N \|\varphi\|^2_{H^{1/2}(\partial B_1)}$$

then we said

$$\|\varphi\|^2_{H^{1/2}(\partial B_1)} \geq \|\varphi\|^2_{L^2(\partial B_1)} \geq c\, \|\varphi\|^2_{L^1(\partial B_1)} \simeq \mathcal{A}(\Omega)^2$$

Remark

The L^2 norm squared is **much more regular** than $\mathcal{A}(\Omega)^2$...And this is exactly what we are estimating for an almost spherical!

Yes! A new asymmetry

For a bounded set, we introduce

$$\alpha(\Omega) := \int_{\Omega \Delta B_1(x_\Omega)} \Big| 1 - |x - x_\Omega| \Big| \, dx$$

where $x_\Omega = \mathrm{barycenter}\,(\Omega)$

Properties of α

a. $|\Omega \Delta B_1(x_\Omega)|^2 \lesssim \alpha(\Omega)$

b. if Ω is almost spherical, then

$$\alpha(\Omega) \sim \int_{\partial B_1} \varphi^2 \lesssim T(B_1) - T(\Omega)$$

Conclusion: the Selection Principle

We use the scheme described above, with $\mathcal{A}$ in place of α

Selection Principle

There exist $\{E_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ *such that*

- $|E_n| = |B_1| \qquad$ *and* $\qquad x_{E_n} = 0$
- ∂E_n *converges to* ∂B_1 *in* C^k, *for every* k
- $T(B_1) - T(E_n) \leq C\,\sigma\,\alpha(E_n)$

Every E_n is a scaled and translated copy of $U_n = \{u_n > 0\}$, with u_n solution of the free boundary problem

$$\min_u \frac{1}{2}\int |\nabla u|^2 - \int u + \Lambda\,|\{u > 0\}| + \sqrt[3]{\sigma}\,(\alpha(\{u > 0\}) - \varepsilon_n)^2$$

The regularity is obtained with a careful adapation of Alt & Caffarelli [J. Reine Angew. Math. (1981)] and Kinderlehrer & Nirenberg [Ann. SNS. (1977)]

Further readings

Stability in Neumann case

- B., Pratelli, **GAFA** (2012)

Stability in Stekloff case

- B., De Philippis, Ruffini, **J. Funct. Anal.** (2012)

A "Selection Principle" for the isoperimetric inequality

- Cicalese, Leonardi, **ARMA** (2012)

Many thanks for your kind attention

"I knew it would take some time to get to that point.
And I worked hard to get there"
C. Schuldiner