Concentration for Coulomb gases

and Coulomb transport inequalities

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Outline

Concentration of measure

$$\mathbb{P}(|F(Z) - \mathbb{E}F(Z)| \geqslant r) \leqslant 2\mathrm{e}^{-\frac{1}{2}r^2}$$

- Sub-Gaussian concentration (Z Gaussian, F Lipschitz)

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- Books: Steele, Ledoux, Boucheron-Lugosi-Massart

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- Determinantal (Pemantle-Peres, Breuer-Duits)

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- Open problem: concentration $\mathbb{P}(\mathrm{d}(\mu_{\mathbf{G}},\mu_\bullet) \geqslant r) \overset{?}{\leqslant} \mathrm{e}^{-cN^2r^2}$

Sub-Gaussian concentration of measure

- Gaussian Unitary Ensemble (GUE) $\mathbf{H} = (\mathbf{H}_{jk})_{1 \leqslant j,k \leqslant N}$

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- Hoffman-Wielandt inequality for $H, H' \in \mathrm{Herm}_{N\times N}$

$$\min_{\sigma\in\Sigma_N} \sum_{k=1}^{N} (\lambda_k(H) - \lambda_{\sigma(k)}(H'))^2 \leqslant \sum_{j,k=1}^{N} (H_{jk} - H'_{jk})^2.$$

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Coulomb kernel in mathematical physics

- *Coulomb kernel* in $\mathbb{R}^d$, $d \geqslant 2$,

$$x \in \mathbb{R}^d \mapsto g(x) := \begin{cases} \log \dfrac{1}{|x|} & \text{if } d = 2, \\ \dfrac{1}{|x|^{d-2}} & \text{if } d \geq 3. \end{cases}$$

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- Fundamental solution of Poisson's equation

$$\Delta g = -c_d \, \delta_0 \quad \text{where} \quad c_d := \begin{cases} 2\pi & \text{if } d = 2, \\ (d-2)|\mathbb{S}^{d-1}| & \text{if } d \geq 3. \end{cases}$$

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$$(\mu, \nu) \mapsto \sqrt{\mathcal{E}(\mu - \nu)}.$$

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- μ_V is compactly supported and has density

$$\frac{1}{2c_d}\Delta V$$

Examples of equilibrium measures

d	g	V	μ_V
1	2	$\infty\mathbf{1}_{\text{interval}^c}(x)$	arcsine
1	2	x^2	semicircle
2	2	$\lvert x\rvert^2$	uniform on a disc
$\geqslant 3$	d	$\lVert x\rVert^2$	uniform on a ball
$\geqslant 2$	d	radial	radial in a ring

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Coulomb gas or one component plasma

- Interaction energy of N Coulomb charges in $\mathbb{R}^d$:

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- Boltzmann–Gibbs probability measure on $(\mathbb{R}^d)^N$

$$\frac{\mathrm{d}\mathbb{P}^N_{V,\beta}(x_1,\ldots,x_N)}{\mathrm{d}x_1\cdots\mathrm{d}x_N} \propto \exp\Big(-\frac{\beta}{2}H_N(x_1,\ldots,x_N)\Big)$$

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- V must be strong enough at infinity to ensure integrability.

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- Large Deviation Principle (BAG, HP, BAZ, CGZ, S, B)

$$\frac{\log \mathbb{P}^N_{V,\beta}\Big(\mathrm{d}(\mu_N,\mu_V)\geq r\Big)}{N^2} \underset{N\to\infty}{\longrightarrow} -\frac{\beta}{2}\inf_{\mathrm{d}(\mu,\mu_V)\geq r}\big(\mathcal{E}_V(\mu)-\mathcal{E}_V(\mu_V)\big).$$

Quantitative or non asymptotic estimates

- The LDP gives for any $r > 0$ and any $N \geqslant N_0$,

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- Nothing known otherwise (nothing for Ginibre ensemble!)

Key observation

- Write $\mathbb{P}^N_{V,\beta}$ with μ_N:

$$\frac{\mathrm{d}\mathbb{P}^N_{V,\beta}(x_1,\dots,x_N)}{\mathrm{d}x_1\cdots\mathrm{d}x_N}=\frac{\exp\left(-\frac{\beta}{2}N^2\mathcal{E}^{\neq}_V(\mu_N)\right)}{Z^N_{V,\beta}}$$

where

$$\mathcal{E}^{\neq}_V(\mu_N):=\int V(x)\mu_N(\mathrm{d}x)+\iint_{x\neq y}g(x-y)\mu_N(\mathrm{d}x)\mu_N(\mathrm{d}y).$$

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- Alternative: compare $\mathcal{E}^{\neq}_V(\mu_N)-\mathcal{E}_V(\mu_V)$ with $\mathrm{W}_1(\mu_N,\mu_V)$.

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Motivation

Electrostatics

Coulomb gas model

Probability metrics and Coulomb transport inequality

Concentration of measure for Coulomb gases

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$$\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_V)$$

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- Topologies

Local Coulomb transport inequality

Theorem (Transport type inequality – CHM 2016)

$D \subset \mathbb{R}^d$ compact, $\mathrm{supp}(\mu+\nu) \subset D$, $\mathcal{E}(\mu) < \infty$ and $\mathcal{E}(\nu) < \infty$,

$$\mathrm{W}_1(\mu,\nu)^2 \leq C_D\,\mathcal{E}(\mu-\nu).$$

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- Extends Popescu local free transport inequality to any d

Coulomb transport inequality for equilibrium measures

Theorem (Transport type inequality – CHM 2016)

We have for any probability measure μ

$$\mathrm{d}_{\mathrm{BL}}(\mu,\mu_V)^2 \leq C_{\mathrm{BL}}\Big(\mathcal{E}_V(\mu)-\mathcal{E}_V(\mu_V)\Big).$$

Moreover if V *is superquadratic then*

$$\mathrm{W}_1(\mu,\mu_V)^2 \leq C_{\mathrm{W}_1}(\mathcal{E}_V(\mu)-\mathcal{E}_V(\mu_V)).$$

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- Growth condition is optimal for W_1

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If V does not grows too fast then

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- LDP shows that order in N is optimal
- Explicit constants a, b, c if V sub-quadratic
- Extends Maïda-Maurel-Segala bound to any dimension:

$$\mathbb{P}^N_{V,\beta}\Big(\mathrm{W}_1(\mu_N,\mu_V) \geq r\Big) \leqslant \mathrm{e}^{-cN^2 r^2}, \quad r \geqslant \begin{cases} \sqrt{\frac{\log N}{N}} & \text{if } d = 2, \\ N^{-1/d} & \text{if } d \geqslant 3. \end{cases}$$

Convergence in Wasserstein distance

Corollary (Wasserstein convergence – CHM 2016)

If V superquadratic and $\beta_N \geqslant \beta_V \frac{\log N}{N}$ then under $\mathbb{P}^N_{V,\beta_N}$ a.s.

$$\lim_{N\to\infty} \mathrm{W}_1(\mu_N, \mu_V) = 0.$$

Convergence at mesoscopic scale

Corollary (Mesoscopic convergence – CHM 2016)

- *If $d = 2$ then*

$$\mathbb{P}^N_{V,\beta}\left(\mathrm{d}_{\mathrm{BL}}\left(\tau^{N^s}_{x_0}\mu_N, \tau^{N^s}_{x_0}\mu_V\right) \geqslant CN^s\sqrt{\frac{\log N}{N}}\right) \leqslant \mathrm{e}^{-cN\log N},$$

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Concentration for spectrum of Ginibre matrices

Corollary (Concentration for Ginibre – CHM 2016)

If $\mathbf{G}$ *is* $N \times N$ *with iid Gaussian entries of variance* $\frac{1}{2N}$ *then*

$$\mathbb{P}\Big(\mathrm{W}_1(\mu_{\mathbf{G}}, \mu_\bullet) \geq r\Big) \leq \mathrm{e}^{-\frac{1}{4C}N^2 r^2 + \frac{1}{2}N \log N + N[\frac{1}{C} + \frac{3}{2} - \log \pi]}.$$

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- Provides W_1 convergence

Exponential tightness

Theorem (Tightness – CHM 2016)

For any $r \geq r_0$

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- Gives W_p versions of convergence and concentration

$$\mathrm{W}_p^p(\mu,\nu) \leqslant (2M)^{p-1}\mathrm{W}_1(\mu,\nu) \leqslant M(2M)^{p-1}\mathrm{d}_{\mathrm{BL}}(\mu,\nu).$$

For $p = 2$ we get $\mathbb{P}^N_{V,\beta}(\mathrm{W}_2(\mu_N,\mu_V) \geq r) \leqslant 2\mathrm{e}^{-cN^{3/2}r^2}$.

Notes and comments

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[covered by our work since g is superharmonic]

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- Crossover and Sanov regime (Allez-Bouchaud-Guionnet)

That’s all folks!

Thank you for your attention.

Idea of proof of Coulomb transport inequality

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- Electric field: $\nabla U^\mu(x)$. “Carré du champ”: $|\nabla U^\mu|^2$
- Integration by parts + Schwarz’s inequality in $\mathbb{R}^d$ and L^2

$$
\begin{aligned}
c_d \int f(x)(\mu-\nu)(\mathrm{d}x) &= -\int f(x)\Delta U^{\mu-\nu}(x)\mathrm{d}x \\
&\leq \int |\nabla f(x)||\nabla U^{\mu-\nu}(x)|\mathrm{d}x \\
&\leq \|f\|_{\mathrm{Lip}} \int_{D_+} |\nabla U^{\mu-\nu}(x)|\mathrm{d}x \\
&\leq \|f\|_{\mathrm{Lip}} \Big(|D_+| \int |\nabla U^{\mu-\nu}(x)|^2 \mathrm{d}x\Big)^{1/2}.
\end{aligned}
$$

Idea... Continued

Again by integration by parts

$$\begin{aligned}\int |\nabla U^{\mu-\nu}(x)|^2 \mathrm{d}x &= -\int U^{\mu-\nu}(x)\Delta U^{\mu-\nu}(x)\mathrm{d}x \\ &= c_d \int U^{\mu-\nu}(x)(\mu-\nu)(\mathrm{d}x) \\ &= c_d\,\mathcal{E}(\mu-\nu).\end{aligned}$$

Finally

$$\mathrm{W}_1(\mu,\nu)^2 \leqslant |D_+| c_d \mathcal{E}(\mu-\nu).$$

Idea of proof of concentration

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- Coulomb transport $-\mathcal{E}_V(\mu_N^{(\varepsilon)})+\mathcal{E}_V(\mu_V)\leqslant-\frac{1}{C}\mathrm{W}_1^2(\mu_N^{(\varepsilon)},\mu_V)$.